Lec 42 Friday May 9

8.3 Continued

<u>Recall our goal</u>: For an $n \times n$ matrix A find fast way to compute the matrix

$$e^{At}$$

that comes up in the solution to $X' = AX$

Previously we did the case A diagonalizable

Now the general case.

<u>Motivation</u> Assume A is in Jordan Normal Form, say

$$A = \left(\begin{array}{ccc|cc|c} 2 & 1 & 0 & & & \\ 0 & 2 & 1 & \multicolumn{2}{c|}{0} & 0 \\ 0 & 0 & 2 & & & \\ \hline & 0 & & 3 & 1 & 0 \\ & & & 0 & 3 & \\ \hline & 0 & & \multicolumn{2}{c|}{0} & 5 \end{array}\right)$$

Eigenvalues of A are

$\lambda_1 = 2$ (mult 3)
$\lambda_2 = 3$ (mult 2)
$\lambda_3 = 5$ (mult 1)

Define

$$P_1 = \left(\begin{array}{ccc|cc|c} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ \hline & & & 0 & 0 & \\ & & & 0 & 0 & \\ \hline & & & & & 0 \end{array}\right)$$

$$P_2 = \left(\begin{array}{ccc|cc|c} 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ \hline & & & 1 & 0 & \\ & & & 0 & 1 & \\ \hline & & & & & 0 \end{array}\right)$$

$$P_3 = \left(\begin{array}{ccc|cc|c} 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ \hline & & & 0 & 0 & \\ & & & 0 & 0 & \\ \hline & & & & & 1 \end{array}\right)$$

<u>Observe</u>

$$P_1 + P_2 + P_3 = I$$

$$P_i P_j = \delta_{ij} P_i \qquad (1 \le i, j \le 3)$$

Define

$$N_1 = \left(\begin{array}{ccc|cc|c} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \\ \hline & & & 0 & 0 & \\ & & & 0 & 0 & \\ \hline & & & & & 0 \end{array}\right)$$

$$N_2 = \left(\begin{array}{ccc|cc|c} 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ \hline & & & 0 & 1 & \\ & & & 0 & 0 & \\ \hline & & & & & 0 \end{array}\right)$$

$$N_3 = 0 \qquad \left(\text{for notational convenience}\right)$$

Obs

$N_1, \quad N_2, \quad N_3$ nilpotent.

Indeed

$$N_i^{m_i} = \mathbb{O} \qquad m_i = \text{mult of } \lambda_i$$

Also

$$N_i N_j = \mathbb{O} \quad \text{if} \quad i \neq j \qquad (1 \le i, j \le 3)$$

$$N_i P_j = P_j N_i = 0 \quad \text{if } i \neq j \qquad (1 \le i, j \le 3)$$

$$N_i P_i = P_i N_i = N_i \qquad (1 \le i \le 3)$$

$$A = 2P_1 + N_1 + 3P_2 + N_2 + 5P_3 + N_3$$

$$= \sum_{i=1}^{3} (\lambda_i P_i + N_i)$$

claim for $1 \le i \le 3$,

$$N_i = P_i(A - \lambda_i I)$$

pf say $i=1$

$$A = \lambda_1 P_1 + N_1 + \lambda_2 P_2 + N_2 + \lambda_3 P_3 + N_3$$

$$I = P_1 + P_2 + P_3$$

$$A - \lambda_1 I = N_1 + (\lambda_2 - \lambda_1) P_2 + N_2 + (\lambda_3 - \lambda_1) P_3 + N_3$$

$$P_1(A - \lambda_1 I) = \underset{\substack{\| \\ N_1}}{P_1 N_1} + \underbrace{P_1\left(\left(\lambda_2 - \lambda_1\right) P_2 + N_2 + \left(\lambda_3 - \lambda_1\right) P_3 + N_3\right)}_{\substack{\| \\ 0}}$$

$$= N_1$$

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Find e^{At} in terms of P_i

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \cdots$$

For $n \geq 0$

$$A^n = \left(\lambda_1 P_1 + N_1 + \lambda_2 P_2 + N_2 + \lambda_3 P_3 + N_3\right)^n$$

$$= \left(\lambda_1 P_1 + \underset{P_1N_1}{\overset{\shortparallel}{N_1}}\right)^n + \left(\lambda_2 P_2 + \underset{P_2N_2}{\overset{\shortparallel}{N_2}}\right)^n + \left(\lambda_3 P_3 + N_3\right)^n$$

$$= P_1\left(\lambda_1 I + N_1\right)^n + P_2\left(\lambda_2 I + N_2\right)^n + P_3\left(\lambda_3 I + N_3\right)^n$$

So

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

$$= P_1 \sum_{n=0}^{\infty} \frac{(\lambda_1 I + N_1)^n t^n}{n!}$$

$$+ P_2 \sum_{n=0}^{\infty} \frac{(\lambda_2 I + N_2)^n t^n}{n!}$$

$$+ P_3 \sum_{n=0}^{\infty} \frac{(\lambda_3 I + N_3)^n t^n}{n!}$$

$$= P_1 e^{(\lambda_1 I + N_1)t} + P_2 e^{(\lambda_2 I + N_2)t} + P_3 e^{(\lambda_3 I + N_3)t}$$

$$= P_1 e^{\lambda_1 t} \underbrace{e^{N_1 t}}_{= I + N_1 t + \frac{N_1^2 t^2}{2}} + P_2 e^{\lambda_2 t} \underbrace{e^{N_2 t}}_{= I + N_2 t} + P_3 e^{\lambda_3 t} \underbrace{e^{N_3 t}}_{= I}$$

(since $N_1^3 = 0$) (since $N_2^2 = 0$) (since $N_3 = 0$)

$$= \sum_{i=1}^{3} P_i e^{\lambda_i t} \left(I + N_i t + \frac{N_i^2 t^2}{2} + \cdots + \frac{N_i^{m_i - 1} t^{m_i - 1}}{(m_i - 1)!} \right)$$

$N_i = P_i (A - \lambda_i I)$

Find P_1, P_2, P_3 in terms of A

Define polynomial

$$p(\lambda) = (\lambda - \lambda_1)^3 (\lambda - \lambda_2)^2 (\lambda - \lambda_3)$$

char poly of A

By Cayley - Hamilton

$$p(A) = 0$$

Consider partial fraction decomp

$$\frac{1}{p(\lambda)} = \frac{a_1(\lambda)}{(\lambda - \lambda_1)^3} + \frac{a_2(\lambda)}{(\lambda - \lambda_2)^2} + \frac{a_3(\lambda)}{\lambda - \lambda_3} \quad (*)$$

deg $a_1(\lambda) \le 2$
-- $a_2(\lambda) \le 1$
$a_3(\lambda) = 0$

Get

$$a_1(\lambda) = \frac{-103 + 115\lambda - 34\lambda^2}{27}$$

$$a_2(\lambda) = \frac{-17 + 5\lambda}{4}$$

$$a_3(\lambda) = \frac{1}{108}$$

In (*) mult both sides by $p(\lambda)$:

$$1 = a_1(\lambda)\underbrace{(\lambda-\lambda_2)^2(\lambda-\lambda_3)}_{=b_1(\lambda)} + a_2(\lambda)\underbrace{(\lambda-\lambda_1)^3(\lambda-\lambda_3)}_{=b_2(\lambda)}$$

$$+ a_3(\lambda)\underbrace{(\lambda-\lambda_1)^3(\lambda-\lambda_2)^2}_{=b_3(\lambda)}$$

$$= \sum_{i=1}^{3} a_i(\lambda)\, b_i(\lambda)$$

One checks

$$P_i = a_i(A)\, b_i(A) \qquad 1 \le i \le 3$$

Thm Given $n \times n$ matrix A

Consider char poly

$$|A - \lambda I| = (-1)^n \underbrace{(\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_q)^{m_q}}_{= p(\lambda)}$$

Here

$\lambda_1, \lambda_2, \ldots, \lambda_q$ are mut distinct

$m_1 + m_2 + \cdots + m_q = n$ $\qquad m_i \geq 1$

Find partial fraction decomp of

$$\frac{1}{p(\lambda)}$$

to get polynomials

$a_i(\lambda), \ b_i(\lambda)$ $\qquad 1 \leq i \leq q$

as above. Put

$$P_i = a_i(A)\, b_i(A) \qquad 1 \leq i \leq q$$

"projection matrix for λ_i"

Then

$$e^{At} =$$

$$\sum_{i=1}^{q} P_i \, e^{\lambda_i t} \left(I + (A-\lambda_i I)t + \frac{(A-\lambda_i I)^2 t^2}{2} + \cdots + \frac{(A-\lambda_i I)^{m_i-1} t^{m_i-1}}{(m_i-1)!} \right)$$

pf let J = Jordan Normal Form for A

there exists an invertible matrix Q such that

$$A = Q J Q^{-1}$$

We saw thm holds for J

In resulting equations, multiply each term on left by Q and right by Q^{-1}

Resulting equations show thm holds for A $\square$

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