Lec 41 Wednesday May 7 5/7/14

8.3 Spectral Decomposition Methods

<u>Goal</u> Given $n \times n$ matrix A, find fast way to compute the matrix

$$e^{At}$$

that comes up in the solution to $\mathbf{X}' = A\mathbf{X}$

<u>Motivation</u> $(n=3)$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

so

$$e^{At} = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}$$

Eigenvalues of A are

$\lambda_1 = 2$ $\lambda_2 = 3$ $\lambda_3 = 5$

Define

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>Observe</u>

$$P_1 + P_2 + P_3 = I \qquad (1)$$

$$P_i P_j = \mathbb{O} \quad \text{if} \quad i \neq j \qquad (1 \leq i, j \leq 3)$$

$$P_i^2 = P_i \qquad (1 \leq i \leq 3)$$

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 \qquad (3)$$

Define Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then

$$P_i P_j = \delta_{ij} P_i \qquad (1 \leq i, j \leq 3) \qquad (2)$$

We now compute e^{At} using only (1), (2), (3) Recall

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \cdots$$

Find A^2 in terms of P_1, P_2, P_3

$$A^2 = (\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3)(\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3)$$

use (2)

$$= \lambda_1^2 P_1 + \lambda_2^2 P_2 + \lambda_3^2 P_3$$

Similarly

$$A^3 = (\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3)^3$$

$$= \lambda_1^3 P_1 + \lambda_2^3 P_2 + \lambda_3^3 P_3$$

etc.

So

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \cdots$$

$$= P_1\left(1 + \lambda_1 t + \frac{\lambda_1^2 t^2}{2} + \frac{\lambda_1^3 t^3}{3!} + \cdots\right)$$

$$+ P_2\left(1 + \lambda_2 t + \frac{\lambda_2^2 t^2}{2} + \frac{\lambda_2^3 t^3}{3!} + \cdots\right)$$

$$+ P_3\left(1 + \lambda_3 t + \frac{\lambda_3^2 t^2}{2} + \frac{\lambda_3^3 t^3}{3!} + \cdots\right)$$

$$= P_1 e^{\lambda_1 t} + P_2 e^{\lambda_2 t} + P_3 e^{\lambda_3 t}$$

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Find P_1, P_2, P_3 in terms of A

<u>claim</u>

$$P_1 = \frac{(A-\lambda_2 I)(A-\lambda_3 I)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \qquad *$$

$$P_2 = \frac{(A-\lambda_1 I)(A-\lambda_3 I)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}$$

$$P_3 = \frac{(A-\lambda_1 I)(A-\lambda_2 I)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

check $*$:

$$RHS = \frac{(\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 - \lambda_2(P_1+P_2+P_3))(\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 - \lambda_3(P_1+P_2+P_3))}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$$

$$= \frac{((\lambda_1-\lambda_2)P_1 + (\lambda_3-\lambda_2)P_3)((\lambda_1-\lambda_3)P_1 + (\lambda_2-\lambda_3)P_2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$$

$$= \frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)P_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} = P_1 \qquad \checkmark$$

Above claim says

$$P_i = \prod_{\substack{1 \le j \le 3 \\ j \ne i}} \frac{A - \lambda_j I}{\lambda_i - \lambda_j}$$

Conclusion

$$e^{At} = e^{\lambda_1 t} P_1 + e^{\lambda_2 t} P_2 + e^{\lambda_3 t} P_3$$

where
$$P_i = \prod_{\substack{1 \le j \le 3 \\ j \ne i}} \frac{A - \lambda_j I}{\lambda_i - \lambda_j} \qquad 1 \le i \le 3$$

More generally ...

LEM Let A = any 3×3 matrix with 3 dist eigenvalues $\lambda_1, \lambda_2, \lambda_3$

Define

$$P_i = \prod_{\substack{1 \le j \le 3 \\ j \ne i}} \frac{A - \lambda_j I}{\lambda_i - \lambda_j} \qquad 1 \le i \le 3$$

then

$$P_1 + P_2 + P_3 = I$$

$$P_i P_j = \delta_{ij} P_i \qquad (1 \le i, j \le 3)$$

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$$

Moreover

$$e^{At} = e^{\lambda_1 t} P_1 + e^{\lambda_2 t} P_2 + e^{\lambda_3 t} P_3$$

pf Define

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

there exists an invertible matrix P

such that

$$A = P D P^{-1}$$

The present theorem is true for D

In resulting equations, mult each term on left

by P and right by P^{-1}.

Resulting equations show present thm holds for A as well. $\square$

More generally ...

<u>thm</u> Given $n \times n$ matrix A with dist eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n.$$

Define

$$P_i = \prod_{\substack{1 \le j \le n \\ j \ne i}} \frac{A - \lambda_j I}{\lambda_i - \lambda_j} \qquad (1 \le i \le n)$$

" Projection matrix for λ_i "

then

$$P_1 + P_2 + \cdots + P_n = I$$

$$P_i P_j = \delta_{ij} P_i \qquad (1 \le i, j \le n)$$

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n$$

Moreover

$$e^{At} = e^{\lambda_1 t} P_1 + e^{\lambda_2 t} P_2 + \cdots + e^{\lambda_n t} P_n$$

pf Sim to above LEM $\square$

Ex Apply above thm to

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

Sol Find eigenvalues of A

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(-1-\lambda) - 6$$

$$= \lambda^2 - 3\lambda - 10$$

$$= (\lambda - 5)(\lambda + 2)$$

Eigenvalues of A are

$\lambda_1 = 5$ $\quad\quad$ $\lambda_2 = -2$

Find P_1, P_2

$$P_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}$$

$$= \frac{A + 2I}{7}$$

$$= \frac{1}{7}\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$$

$$P_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}$$

$$= \frac{A - 5I}{-7}$$

$$= \frac{1}{7}\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

Check

$$P_1 + P_2 \stackrel{?}{=} I \qquad \checkmark$$

check

$$P_1^2 \stackrel{?}{=} P_1 \qquad P_1 P_2 \stackrel{?}{=} \mathbb{O}$$

$$P_2 P_1 \stackrel{?}{=} \mathbb{O} \qquad P_2^2 \stackrel{?}{=} P_2$$

$$A \stackrel{?}{=} 5\, P_1 - 2\, P_2$$

$$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \stackrel{?}{=} \frac{5}{7}\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} - \frac{2}{7}\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

✓

$$e^{At} = e^{\lambda_1 t} P_1 + e^{\lambda_2 t} P_2$$

$$= e^{5t}\, \frac{1}{7}\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} + e^{-2t}\, \frac{1}{7}\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

□

Ex Apply thm to

$$A = \begin{bmatrix} 3 & -4 & 16 \\ 0 & 6 & -15 \\ 0 & 2 & -5 \end{bmatrix}$$

Sol Find eigenvalues of A (skip detail)

$\lambda_1 = 0$ $\lambda_2 = 1$ $\lambda_3 = 3$

Find P_1, P_2, P_3

$$P_1 = \frac{(A - 1I)(A - 3I)}{(0-1)(0-3)}$$

$$= \begin{bmatrix} 0 & 4 & -12 \\ 0 & -5 & 15 \\ 0 & -2 & 6 \end{bmatrix}$$

$$P_2 = \frac{(A - 0I)(A - 3I)}{(1-0)(1-3)}$$

$$= \begin{bmatrix} 0 & -4 & 10 \\ 0 & 6 & -15 \\ 0 & 2 & -5 \end{bmatrix}$$

$$P_3 = \frac{(A - 0I)(A - 1I)}{(3-0)(3-1)}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

One checks

$$P_1 + P_2 + P_3 = I$$

$$P_i P_j = \delta_{ij} P_i$$

$$0 P_1 + 1 P_2 + 3 P_3 = A$$

$$e^{At} = e^{0t} P_1 + e^{t} P_2 + e^{3t} P_3$$