Lec 40 Monday May 5

8.2 Continued

We continue to discuss nonhomogeneous systems of diff. equations

We now give a solution method called

<u>variation of parameters</u>

Consider an $n \times n$ system

$$X' = P(t)X + F(t) \qquad *$$

where

$P(t) = n \times n$ matrixed valued function of t

$F(t) = n \times 1$ ...

$X = X(t) = n \times 1$ ...

Associated homog system is

$$X' = P(t)X \qquad **$$

Recall the set of solutions for ** is a vector space of dim n. Fix a basis

$$\underline{X}_1(t),\ \underline{X}_2(t), \dots, \underline{X}_n(t)$$

So the gen solution to (**) is

$$\underline{X}_c(t) = c_1 \underline{X}_1(t) + c_2 \underline{X}_2(t) + \dots + c_n \underline{X}_n(t) \qquad c_1, \dots, c_n \text{ free}$$

Define an $n \times n$ matrix $\underline{\Phi} = \underline{\Phi}(t)$ s.t. for $1 \le i \le n$

column i of $\underline{\Phi}(t)$ is $\underline{X}_i(t)$

So $\underline{\Phi}$ is a fundamental matrix for (**). Obs

$$\underline{X}_c(t) = \underline{\Phi}(t) \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{= C}$$

We now seek a particular solution to (*) of form

$$\underline{X}_p(t) = \underline{\Phi}(t)\, \underline{U}(t) \qquad U(t) = n \times 1 \text{ matrix valued function of } t$$

<u>Find $U(t)$</u>

By product rule

$$\Psi_p'(t) = \Phi'(t)\, u(t) + \Phi(t)\, u'(t)$$

Require

$$\underbrace{\Phi'(t)}_{\substack{\| \\ P(t)\,\Phi(t)}}\, u(t) + \Phi(t)\, u'(t) = P(t)\,\Phi(t)\,u(t) + F(t)$$

Require

$$\Phi(t)\, u'(t) = F(t)$$

So

$$u'(t) = \Phi(t)^{-1}\, F(t)$$

So

$$u(t) = \int \Phi(t)^{-1}\, F(t)\, dt$$

In summary

$$\Psi_p(t) = \Phi(t) \int \Phi(t)^{-1}\, F(t)\, dt \qquad \star$$

For the special case

$$P(t) = \text{indep of } t$$
$$\overset{\shortparallel}{A}$$

we may take

$$\Phi(t) = e^{At}$$

In this case ☆ becomes

$$X_p(t) = e^{At} \int e^{-At} F(t)\,dt$$

Ex With above notation solve the init value problem

$$X' = AX + F(t) \qquad (*)$$

subject to

$$x_i(0) = b_i \qquad 1 \le i \le n$$

Sol Write

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

so $X(0) = B$

Gen sol to * is

$$\mathbb{X}(t) = \mathbb{X}_c(t) + \mathbb{X}_p(t)$$

$$\mathbb{X}_c(t) = \Phi(t)\, C = e^{At} C$$

$$\mathbb{X}_p(t) = e^{At}\int e^{-At} F(t)\, dt \overset{\text{wloG}}{=} e^{At}\int_0^t e^{-At} F(t)\, dt$$

To Find C set $t=0$

$$\mathbb{X}(0) = C + 0$$

$$\mathbb{X}(0) = B$$

$$C = B$$

So

$$\mathbb{X}(t) = e^{At} B + e^{At}\int_0^t e^{-At} F(t)\, dt \qquad \square$$

Ex Solve the init value problem.

$$x' = 3x - y + 7 \qquad x = x(t),$$
$$y' = 9x - 3y + 5 \qquad y = y(t)$$

$x(0) = 3 \qquad y(0) = 5$

Sol Matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X = X(t)} + \underbrace{\begin{bmatrix} 7 \\ 5 \end{bmatrix}}_{F = F(t)}$$

$X(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = B$

Find e^{At}

Find eigenvalues of A

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) + 9$$
$$= \lambda^2$$

eigenvals of A are

0, 0

A ~~is~~ nilpotent $\qquad A^2 = 0$ by Cayley-Hamilton

$$e^{At} = I + At = \begin{pmatrix} 1+3t & -t \\ 9t & 1-3t \end{pmatrix}$$

Also

$$e^{-At} = I - At$$

$$= \begin{pmatrix} 1-3t & t \\ -9t & 1+3t \end{pmatrix}$$

So

$$e^{-At} F(t) = \begin{pmatrix} 1-3t & t \\ -9t & 1+3t \end{pmatrix} \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 7-16t \\ 5-48t \end{pmatrix}$$

$$\int_0^t e^{-At} F(t)\, dt = \begin{pmatrix} 7t-8t^2 \\ 5t-24t^2 \end{pmatrix}$$

Sol is

$$X(t) = e^{At} X(0) + e^{At} \int_0^t e^{-At} F(t)\, dt$$

$$= e^{At} \left(X(0) + \int_0^t e^{-At} F(t)\, dt \right)$$

$$= \begin{pmatrix} 1+3t & -t \\ 9t & 1-3t \end{pmatrix} \begin{pmatrix} 3+7t-8t^2 \\ 5+5t-24t^2 \end{pmatrix} = \begin{pmatrix} 3+11t+8t^2 \\ 5+17t+24t^2 \end{pmatrix}$$

$\square$

Ex Solve the initial value problem

$$x' = x + 2y + 75e^{2t}$$

$$y' = 2x - 2y$$

where $x(0) = 0, \quad y(0) = 0$

Sol Matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X = X(t)} + \underbrace{\begin{bmatrix} 75 \\ 0 \end{bmatrix} e^{2t}}_{F(t)}$$

$$X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find e^{At}

Find eigenvalues of A (skip detail)

$$\lambda = -3, 2$$

Find eigenvectors for A (skip detail)

λ	-3	2
eigvector	$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \qquad A = P D P^{-1}$$

$$e^{At} = P e^{Dt} P^{-1}$$

$$= \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \frac{1}{5}$$

$$= \frac{1}{5} e^{-3t} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} + \frac{1}{5} e^{2t} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

to get e^{-At} replace $t \to -t$:

$$e^{-At} = \frac{1}{5} e^{3t} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} + \frac{1}{5} e^{-2t} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

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$$e^{-At} F(t) =$$

$$\frac{1}{5} e^{5t} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 75 \\ 0 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 75 \\ 0 \end{bmatrix}$$

$$= e^{5t} \begin{bmatrix} 15 \\ -30 \end{bmatrix} + \begin{bmatrix} 60 \\ 30 \end{bmatrix}$$

$$\int_0^t e^{-At} F(t)\, dt = \frac{(e^{5t}-1)}{5} \begin{bmatrix} 15 \\ -30 \end{bmatrix} + t \begin{bmatrix} 60 \\ 30 \end{bmatrix}$$

$$= e^{5t} \begin{bmatrix} 3 \\ -6 \end{bmatrix} + \begin{bmatrix} -3 \\ 6 \end{bmatrix} + t \begin{bmatrix} 60 \\ 30 \end{bmatrix}$$

Sol is

$$\underline{X}(t) = e^{At} \underbrace{\underline{X}(0)}_{\parallel \atop 0} + e^{At} \int_0^t e^{-At} F(t)\, dt$$

$$= \left(\frac{1}{5} e^{-3t} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} + \frac{1}{5} e^{2t} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \right) \left(e^{5t} \begin{bmatrix} 3 \\ -6 \end{bmatrix} + \begin{bmatrix} -3 \\ 6 \end{bmatrix} + t \begin{bmatrix} 60 \\ 30 \end{bmatrix} \right)$$

$$= e^{2t} \begin{bmatrix} 3 \\ -6 \end{bmatrix} + t e^{2t} \begin{bmatrix} 60 \\ 30 \end{bmatrix} + e^{-3t} \begin{bmatrix} -3 \\ 6 \end{bmatrix}$$

$\square$

Ex Solve init value prob

$$x' = 3x - y +$$
$$y' = 9x - 3y + t^{-2}$$

$$x(1) = 3$$
$$y(1) = 7$$

Sol Matrix Form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 \\ t^{-2} \end{bmatrix}}_{F(t)}$$

$$X(1) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

As before

$$e^{At} = I + At$$
$$= \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix}$$

$$e^{-At} F(t) = \begin{bmatrix} 1-3t & t \\ -9t & 1+3t \end{bmatrix} \begin{bmatrix} 0 \\ t^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} t^{-1} \\ t^{-2} + 3t^{-1} \end{bmatrix}$$

$$\int_1^t e^{-At} F(t)\,dt = \begin{bmatrix} \ln t \\ -t^{-1} + 3\ln t \end{bmatrix}$$

Sol is

$$\underset{\sim}{X}(t) = \underset{\substack{\shortparallel \\ e^{At} e^{-A} X(1)}}{X_c(t)} + \underset{\substack{\shortparallel \\ e^{At} \int_1^t e^{-At} F(t)\,dt}}{X_p(t)}$$

$$= e^{At} \Big(\underset{\substack{\shortparallel \\ \begin{bmatrix} -2 & 1 \\ -9 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix}}}{e^{-A} X(1)} + \int_1^t e^{-At} F(t)\,dt \Big)$$

$$= \begin{bmatrix} 2 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \ln t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\square$