Lec 3 Monday Jan 27

1.3 Slope fields and solution curves

We now consider differential equations of the form

$$\frac{dy}{dx} = f(x,y)$$

There is no uniform method that gives the solution in closed form

But we can use the slope field to estimate the sols.

Ex Describe the sols to

$$\frac{dy}{dx} = x^2+y^2-1 \qquad \ast$$

Sol: sketch slope field

- at origin $(x,y) = (0,0)$ slope is -1
- $\underbrace{\text{at all pts } (x,y) \text{ with } x^2+y^2=1}_{\text{circle abt origin with radius 1}}$, slope is 0
- For any radous r,

  consider circle abt origin with radius r

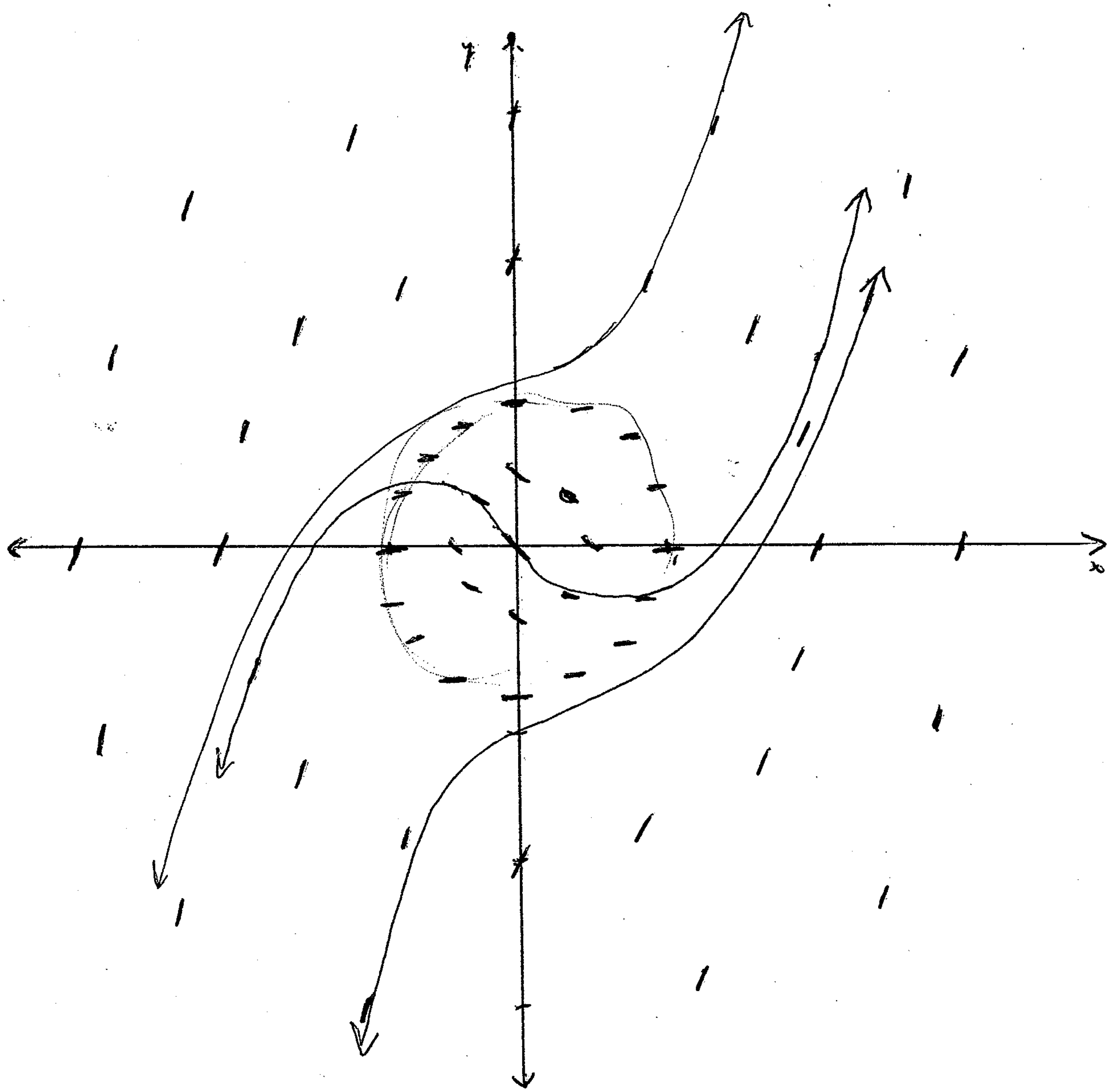
  For each pt (x,y) on this circle

  $$x^2+y^2 = r^2$$

  So slope at (x,y) is r^2-1

r	r^2-1
0	-1
$\frac{1}{2}$	$-3/4$
1	0
2	3
3	8

slope field



Each curve gives a sol to $*$.

Ex For the initial value problem

$$\frac{dy}{dx} = x^2+y^2-1, \qquad y(0)=0$$

Estimate $y(2)$

Sol the curve thru origin goes close to $(2,1)$. So $y(2) \doteq 1$. $\square$

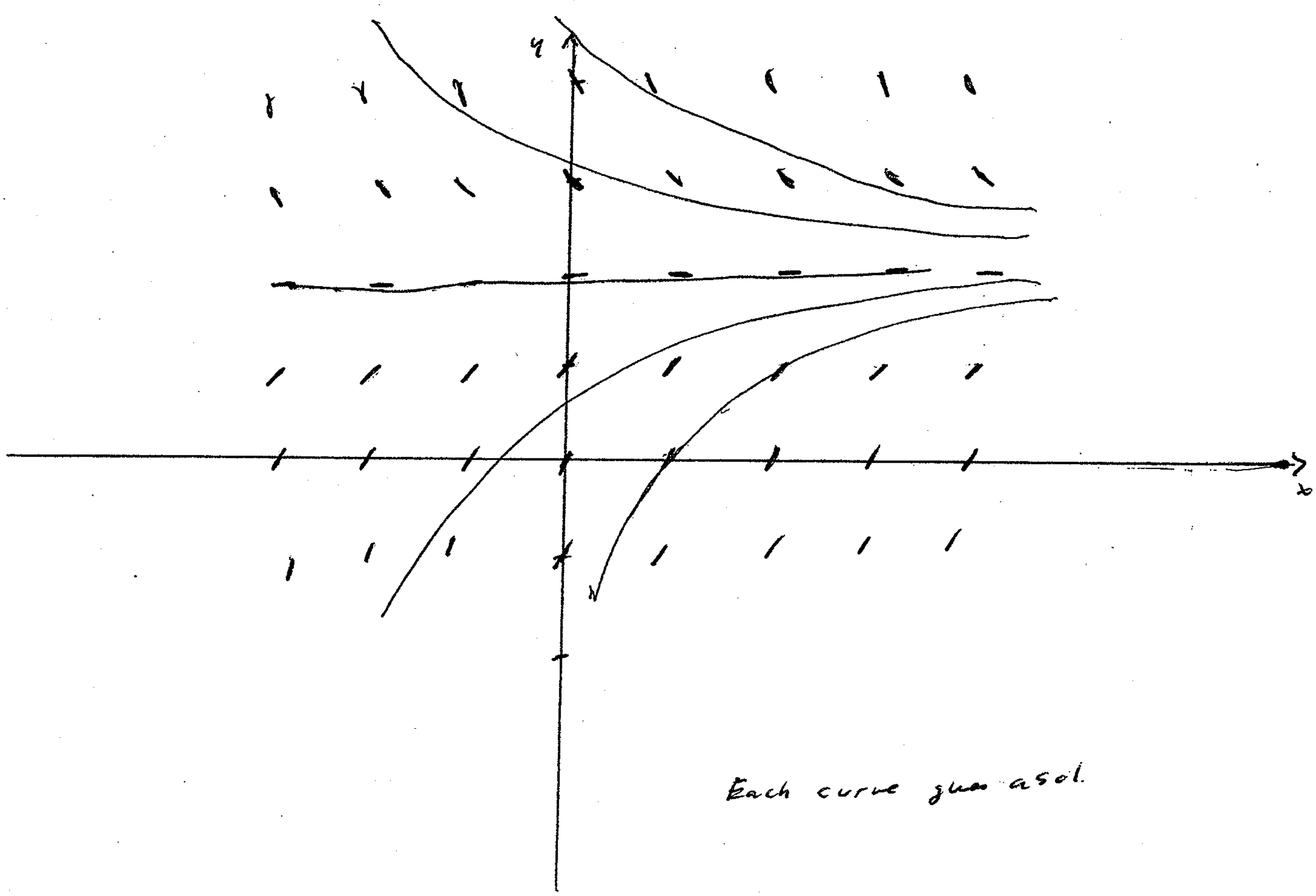
Ex Describe the sols to

$$\frac{dy}{dx} = -y + 2$$

Sol sketch slope field

- On line $y = 0$ slope is 2
- on line $y = 1$ slope is 1
- on line $y = 2$ slope is 0
- on line $y = 3$ slope is -1

...



Each curve gives a sol.

- the function

$$y = 2$$

is a particular solution.

- For any solution curve, as x grows, y approaches 2

Ex For the init value problem

$$\frac{dy}{dx} = -y + 2 \qquad y(0) = 10$$

Estimate $y(1000)$

Sol $y(1000) \approx 2$ $\square$

For the previous two examples,

For any pt (a,b) in $\mathbb{R}^2$, there is a unique solution curve thru that point

"start at (a,b) and follow the arrows in the slope field".

Caution For a given diff. equation

$$\frac{dy}{dx} = f(x,y)$$

and a given pt (a,b) in $\mathbb{R}^2$

- A solution curve thru (a,b) may not exist.
- The solution curve thru (a,b) might not be unique.

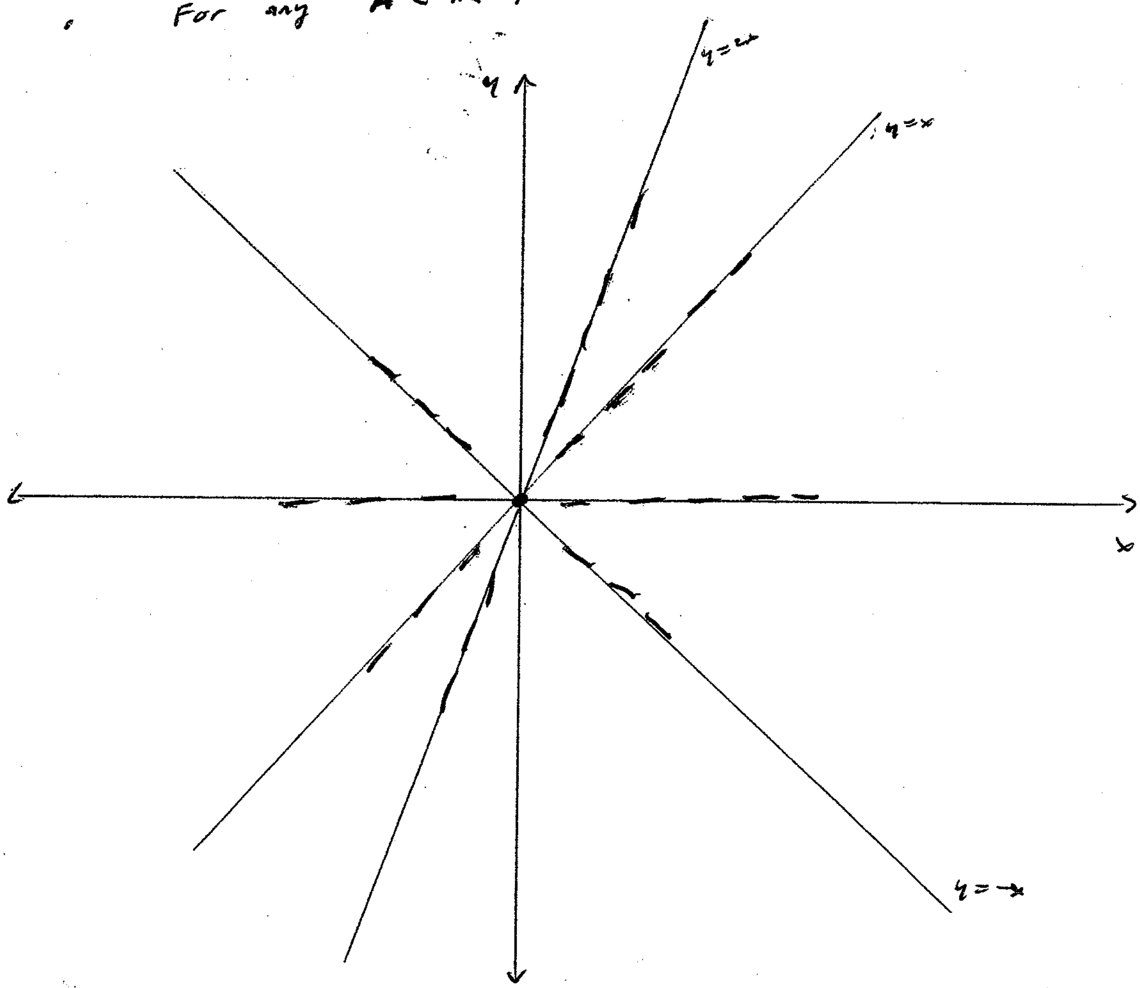
The following examples illustrate what can go wrong.

Ex Find all sols to

$$\frac{dy}{dx} = \frac{y}{x}$$

Sol First sketch slope field

- On line $y=0$ slope is 0
- on line $y=x$ slope is 1
- on line $y=-x$ slope is -1
- on line $y=2x$ slope is 2
- For any $A \in \mathbb{R}$, on the line $y=Ax$ the slope is A



It appears that each (non vertical) line thru origin gives a solution curve

Gen sol should be

$$y = Cx \qquad C = \text{const}$$

check

$$\frac{dy}{dx} = C$$
$$= \frac{y}{x} \quad \checkmark$$

<u>Observe</u>

- For any pt (a,b) with $a = 0$ and $b \neq 0$ there is no solution curve thru (a,b).
- At the origin $(a,b) = (0,0)$ there are ∞ many solution curves thru (a,b)
- For (a,b) with $b \neq 0$ there is unique sol curve thru (a,b)

Essential problem:

$$\frac{dy}{dx} = \frac{y}{x} \leftarrow \text{RHS is not defined for } x = 0$$

Ex Describe the solutions to

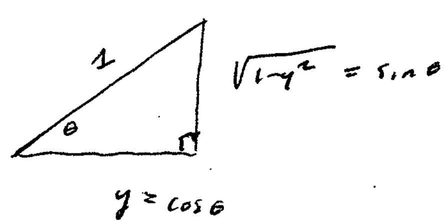
$$\frac{dy}{dx} = -\sqrt{1-y^2}$$

Sol. obs: At any pt (a,b) in $\mathbb{R}^2$,
no solution curve thru (a,b) unless $-1 \le b \le 1$

sketch slope field

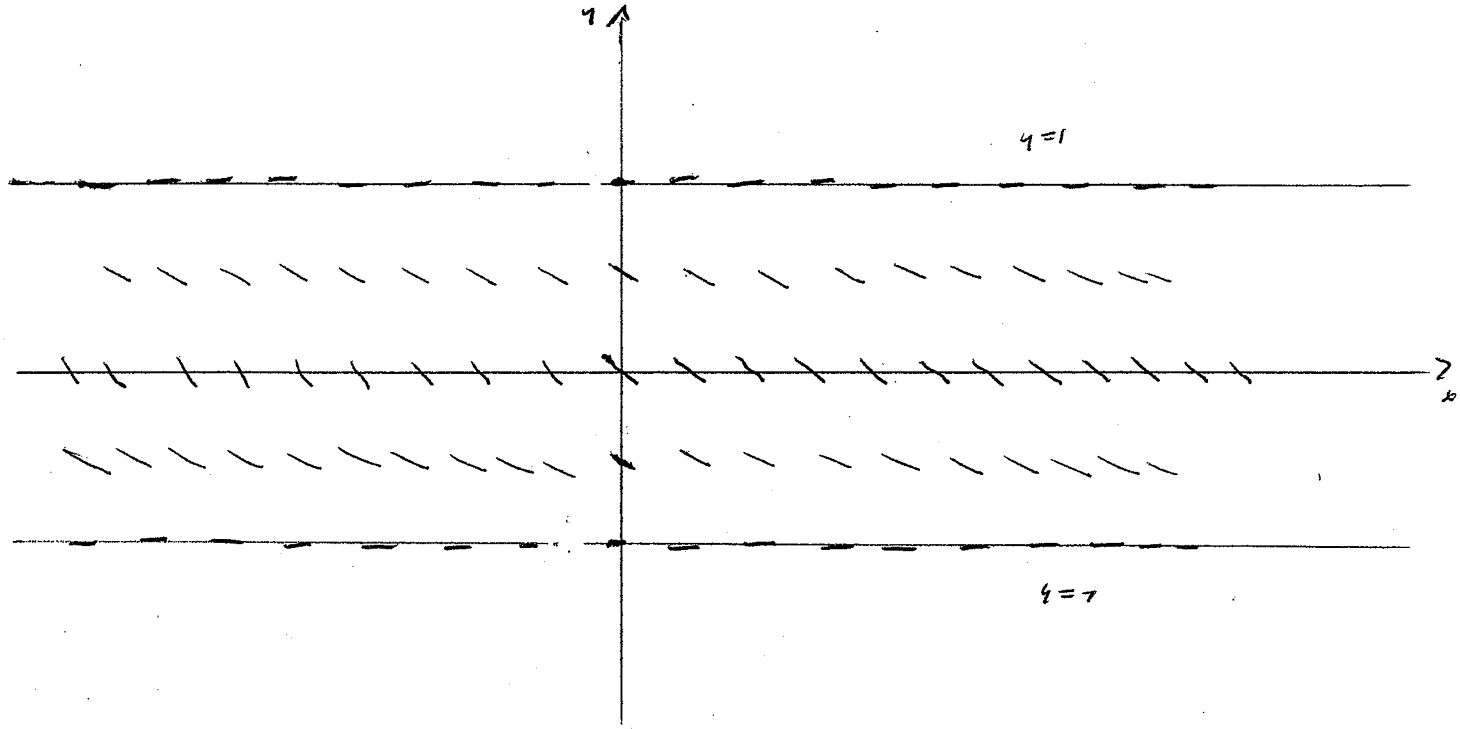
- on the line $y = 0$ slope is -1
  - ... $y = 1$ ... 0
  - ... $y = -1$ ... 0
  - .. $y = \frac{1}{2}$ .. $-\frac{\sqrt{3}}{2}$
  - .. $y = -1/2$ .. $-\frac{\sqrt{3}}{2}$

trig view:

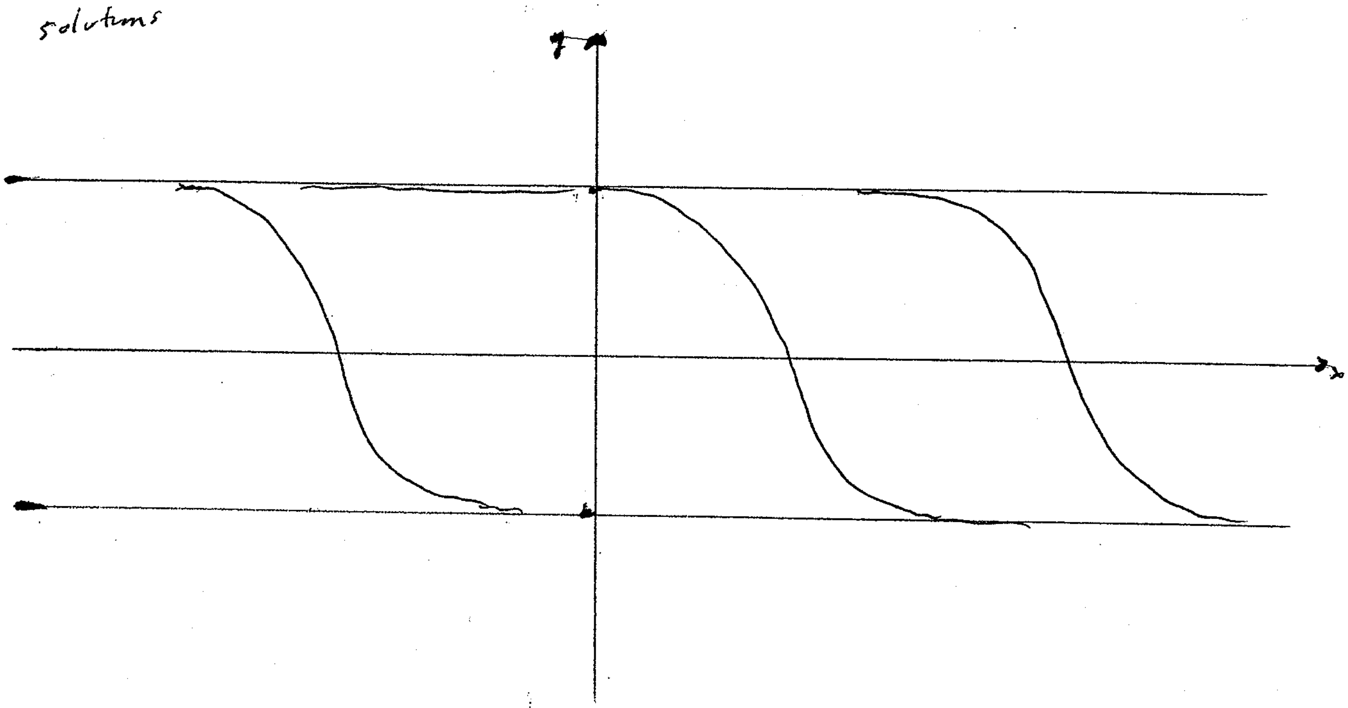


Fn $-1 \le b \le 1$ view $b = \cos\theta$ $(0 \le \theta < \pi)$
on line $y = b$ slope is $-\sin\theta$

slope field:



solutions

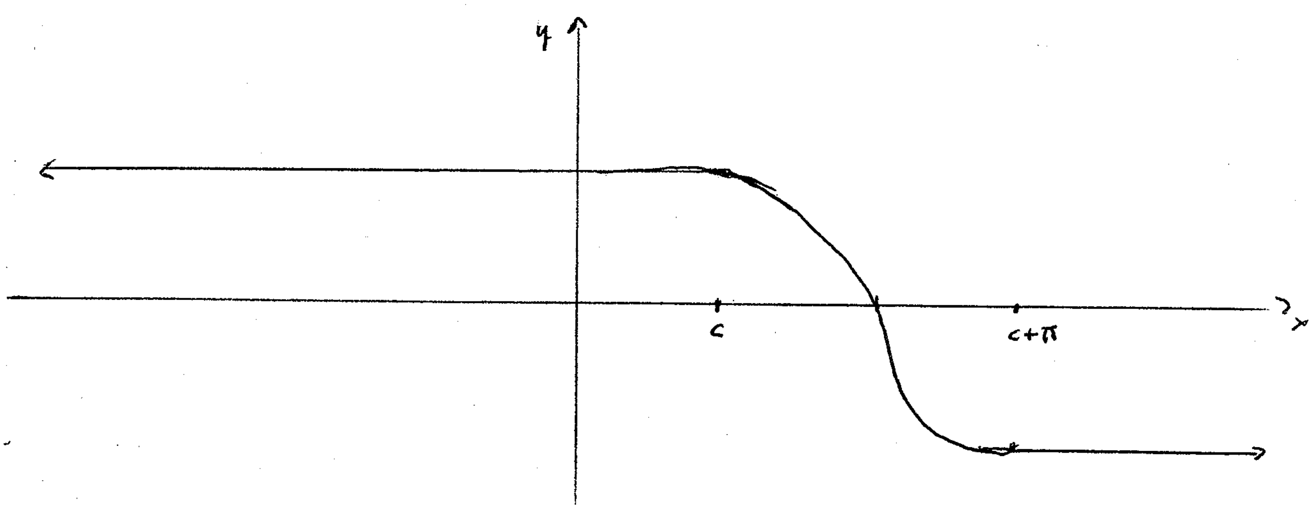


Precise desc of sols:

Pick any $c \in \mathbb{R}$

Define a function y by:

$$y = \begin{cases} \cos(x-c) & \text{if} \quad c \le x \le c+\pi \\ 1 & \text{if} \quad x < c \\ -1 & \text{if} \quad x > c+\pi \end{cases}$$



check that the function y is a particular sol,

Find $\frac{dy}{dx}$

For $c \le x \le c+\pi$,

$$y = \cos(x-c)$$

$$\frac{dy}{dx} = -\sin(x-c)$$

$$= -\sqrt{1-\cos(x-c)^2}$$

$$= -\sqrt{1-y^2}$$ ✓

For $x < c$,

$$y = 1$$

$$\frac{dy}{dx} = 0$$

$$= -\sqrt{1-1^2}$$

$$= -\sqrt{1-y^2}$$ ✓

For $x > c+\pi$,

$$y = -1$$

$$\frac{dy}{dx} = 0$$

$$= -\sqrt{1-(-1)^2}$$

$$= -\sqrt{1-y^2}$$ ✓

We have given some particular sols

the lines $y = 1$, $y = -1$ are also particular sols.

Turns out there are no other sols.

So for any pt (a,b) in $\mathbb{R}^2$

- If $b > 1$ or $b < -1$ then No sol passes thru (a,b)
- If $b = 1$ or $b = -1$ then ∞ sols ""
- If $-1 < b < 1$ then unique sol ""

thm Given a pt (a,b) in $\mathbb{R}^2$

and consider the init value problem

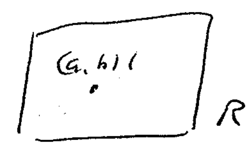
$$\frac{dy}{dx} = f(x,y) \qquad y(a)=b \qquad *$$

Assume that both

$$f(x,y), \qquad \frac{\partial f(x,y)}{\partial y}$$

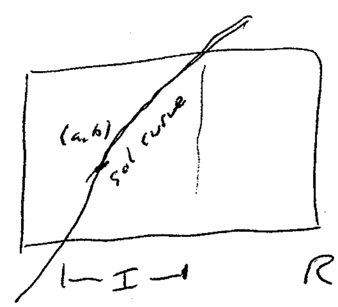
are defined and continuous on a rectangle R

that contains (a,b) in its interior



then for some open interval I containing a

$\exists$ unique sol to $*$ defined for all $x \in I$



Back to

$$\frac{dy}{dx} = -\sqrt{1-y^2}$$

Here $$f(x,y) = -\sqrt{1-y^2}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{2}\frac{-2y}{\sqrt{1-y^2}} = \frac{y}{\sqrt{1-y^2}}$$

Not def for $y = \mp 1$.