Lec 39 Friday May 2

8.2 Nonhomogeneous Linear Systems

Consider the system

$x' = 3x + 4y$

$y' = 3x + 2y + t^2$

extra terms make the system nonhomog

$x = x(t) \quad y = y(t)$

Find the general sol.

Sol Put in matrix form

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{= \mathbb{X}'} = \underbrace{\begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}}_{= A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{= \mathbb{X} = \mathbb{X}(t)} + \underbrace{\begin{bmatrix} 0 \\ t^2 \end{bmatrix}}_{= F = F(t)} \quad (*)$$

To solve (*), first find the gen sol for the homog system

$$\mathbb{X}' = A\,\mathbb{X} \qquad (**)$$

Find eigvals of A

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda) - 12$$

$$= \lambda^2 - 5\lambda - 6$$

$$= (\lambda - 6)(\lambda + 1)$$

$$\lambda = 6, -1$$

Find the eigenvectors for A

λ	6	-1
$A - \lambda I$	$\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}$
EV	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Sol space for (**) has basis

$$e^{6t}\begin{bmatrix}4\\3\end{bmatrix}, \qquad e^{-t}\begin{bmatrix}-1\\1\end{bmatrix}$$

Gen sol to (**) is

$$c_1 e^{6t}\begin{bmatrix}4\\3\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} \qquad c_1, c_2 \text{ free}$$

the gen sol to (*) is

$$c_1 e^{6t}\begin{bmatrix}4\\3\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} + \mathbf{X}_p(t)$$

c_1, c_2 free

where $\mathbf{X}_p(t)$ is any particular sol to (*)

<u>Find $\mathbf{X}_p(t)$</u>

Proceed by analogy with Sec 7.5

- List terms in $F(t)$ along with their derivatives:

$$t^2, t, 1$$

- In this List, check for duplications with terms in gen sol to (*)

  (No dup this time)

Guess $\bar{X}_p(t)$ has form

$$\bar{X}_p(t) = at^2 + bt + c$$

$\bar{X}_p$ where $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

with entries undep of t

Find a, b, c

<u>obs</u>

$$\bar{X}_p' = 2at + b$$

<u>Require</u>

$$2at + b = A\left(at^2 + bt + c\right) + t^2 f$$

$f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Compare coefs of t^2, t, 1

coefs of	Requirement
t^2	$Aa = -f$
t	$Ab = 2a$
1	$Ac = b$

Find a

$$Aa = -f$$

$$a = -A^{-1}f$$

$$A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix} \qquad A^{-1} = \frac{-1}{6}\begin{bmatrix} 2 & -4 \\ -3 & 3 \end{bmatrix}$$

$$A^{-1}f = A^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$a = -A^{-1}f = \frac{1}{6}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

<u>Find b</u>

$$Ab = 2a$$

$$b = 2A^{-1}a$$

$$= \frac{1}{18}\begin{bmatrix} 20 \\ -21 \end{bmatrix}$$

<u>Find c</u>

$$Ac = b$$

$$c = A^{-1}b$$

$$= \frac{1}{108}\begin{bmatrix} -124 \\ 123 \end{bmatrix}$$

So our particular sol to $(*)$ is

$$\bar{X}_p(t) = \frac{1}{6}\begin{bmatrix} -4 \\ 3 \end{bmatrix} t^2 + \frac{1}{18}\begin{bmatrix} 20 \\ -21 \end{bmatrix} t + \frac{1}{108}\begin{bmatrix} -124 \\ 123 \end{bmatrix} \quad (\star)$$

Ex Ref to prev example, find the particular solution that satisfies the init conds

$$x(0) = -\frac{123}{108} \qquad y(0) = \frac{123}{108}$$

Sol the general solution to (*) is

$$\mathbf{X}(t) = c_1 e^{6t}\begin{bmatrix} 4 \\ 3 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \underset{\uparrow \text{ from } (\star)}{\mathbf{X}_p(t)}$$

To find c_1, c_2 set $t=0$:

$$\frac{123}{108}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 4 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{108}\begin{bmatrix} -124 \\ 123 \end{bmatrix}$$

$$\frac{1}{108}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 4 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\frac{1}{108}$$

$$= \frac{1}{7}\begin{bmatrix} 1 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\frac{1}{108}$$

$$= \frac{1}{756}\begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \square$$

__Ex__ Find a particular sol to

$$x' = 4x + y + e^t$$
$$y' = 6x - y - e^t$$

__Sol__ Matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X = X(t)} + \underbrace{e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{F = F(t)} \qquad (*)$$

__Find eigvalues of A__ (skip detail)

eigvals are

$$\lambda = 5, -2$$

Gen sol to $X' = AX$ involves terms

$$e^{5t}, \; e^{-2t}$$

__List terms in $F(t)$ along with their derivatives:__

$$e^t$$

(no dup with e^{5t}, e^{-2t})

<u>Guess</u> there is a partic sol to $(*)$ of form

$$\mathbf{X}_p(t) = a e^{t}$$

$\mathbf{X}_p$

where $\quad a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

Find a

$$\mathbf{X}_p' = a e^{t}$$

<u>Require</u>

$$a e^{t} = A(a e^{t}) + e^{t} f \qquad f = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

So

$$a = Aa + f$$

$$(I - A) a = f$$

$$a = (I-A)^{-1} f$$

$$I - A = \begin{bmatrix} -3 & -1 \\ -6 & 2 \end{bmatrix} \qquad (I-A)^{-1} = \frac{-1}{12}\begin{bmatrix} 2 & 1 \\ 6 & -3 \end{bmatrix}$$

$$a = (I - A^{-1}) f = \frac{-1}{12}\begin{bmatrix} 1 \\ 9 \end{bmatrix} \qquad \text{so } \mathbf{X}_p(t) = \frac{-1}{12}\begin{bmatrix} 1 \\ 9 \end{bmatrix} e^{t}$$

$\square$

Ex Find a particular solution to

$$x' = x - 5y + \cos(2t)$$
$$y' = x - y$$

Sol Matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underset{\substack{\shortparallel \\ A}}{\begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}} \underset{\substack{\shortparallel \\ X = X(t)}}{\begin{bmatrix} x \\ y \end{bmatrix}} + \underbrace{\cos(2t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\substack{\shortparallel \\ F = F(t)}} \qquad (*)$$

Find eigenvalues of A (skip detail)

eigenvalues are

$$\lambda = 2i, \ -2i \qquad i^2 = -1$$

Gen sol to $X' = AX$ involves terms

$$e^{2it} = \cos 2t + i \sin 2t$$
$$e^{-2it} = \cos 2t - i \sin 2t$$

List terms in $F(t)$ along with their derivatives

$\cos(2t)$, $\sin(2t)$ (duplication)

<u>Guess</u> there is a particular sol to (*) of form

$$\underset{\underline{X}_p}{\overset{}{\underline{X}_p(t)}} = a\cos(2t) + b\sin(2t) + ct\cos(2t) + dt\sin(2t)$$

where $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$.

Find a, b, c, d

<u>Obs</u>

$$\underline{X}_p' = -2a\sin(2t) + 2b\cos(2t)$$
$$+ c\left(\cos(2t) - 2t\sin(2t)\right)$$
$$+ d\left(\sin(2t) + 2t\cos(2t)\right)$$

$$\overset{\text{require}}{=} A\left(a\cos(2t) + b\sin(2t) + ct\cos(2t) + dt\sin(2t)\right)$$
$$+ f\cos(2t)$$

$f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Compare coefs for

$\cos(2t)$, $\sin(2t)$, $t\cos(2t)$ $t\sin(2t)$

coefs of	requirement
$\cos(2t)$	$2b + c = Aa + f$
$\sin(2t)$	$-2a + d = Ab$
$t\cos(2t)$	$2d = Ac$
$t\sin(2t)$	$-2c = Ad$

$$d = \frac{1}{2} A c$$

$-2a + d = Ab$ becomes

$$-2a + \frac{1}{2} A c = Ab \qquad (1)$$

Using $2b + c = Aa + f$

$$2Ab + Ac = \underset{\underset{-4a}{\parallel}}{A^2 a} + Af \qquad A^2 + 4I = 0 \quad \text{(Cayley Ham)} \qquad (2)$$

Combine (1), (2) to get

$$Ac = \frac{1}{2} Af$$

$$c = \frac{1}{2} f = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$$d = \frac{1}{2} A c = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

b free take $b = 0$

$$a = \frac{1}{2} d = \frac{1}{8} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\square$