Lec 38 Wednesday April 30

8.1 Continued

Recall For the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 11 & -15 \\ 6 & -8 \end{bmatrix}}_{=A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x(0) = 1, \quad y(0) = 1$$

the sol is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{At} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In general we have

<u>thm</u> Given an $n \times n$ matrix A

Given scalars $b_1, b_2, \dots, b_n$

Consider the initial value problem

$$\underline{X}' = A\underline{X} \qquad \underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad *$$

$$x_i(0) = b_i \qquad 1 \le i \le n$$

then the unique solution is

$$\underline{X} = e^{At} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

<u>pf</u> the general solution to $*$ is

$$\underline{X} = e^{At} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad c_i \text{ free}$$

To find c_i set $t = 0$:

$$\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \underline{X}(0) = \underbrace{e^0}_{= I} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

so $\qquad c_i = b_i \qquad 1 \le i \le n$

$\square$

COR Given $n \times n$ matrix A

Consider system

$$\underline{X}' = A\underline{X} \qquad \underline{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad (*)$$

Let $\underline{X}$ a solution to $(*)$. Then

$$\underline{X}(t) = e^{At}\,\underline{X}(0)$$

pf Define

$$b_i = x_i(0) \qquad 1 \le i \le n$$

By prev thm,

$$\underline{X}(t) = e^{At} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$= e^{At}\,\underline{X}(0)$$

$\square$

Fundamental matrices

Given $n \times n$ matrix A

Consider system

$$\underline{X}' = A\underline{X} \qquad \underline{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad (*)$$

Recall the set of solutions to $*$ is a vector space of dim n.

A <u>fundamental matrix</u> for $(*)$ is any $n \times n$ matrix valued function

$$\underline{\Phi} = \underline{\Phi}(t)$$

such that

the columns of $\underline{\Phi}$ form a **basis** for the sol space of $(*)$

For example,

e^{At} is a fundamental matrix for $*$.

<u>thm</u> With above notation

let $\Phi = \Phi(t)$ denote a fundamental

matrix for $(*)$ then

(i) $\Phi(t) = e^{At}\,\Phi(0)$

(ii) $\Phi(0)$ is invertible

(iii) $e^{At} = \Phi(t)\,\Phi(0)^{-1}$

pf (i) For $1 \le i \le n$ show

column i of $\Phi(t)$ = column i of $e^{At}\,\Phi(0)$

$\overset{\text{matrix mult}}{=}$

$\underset{\text{by prev Cor}}{=}$ e^{At} (column i of $\Phi(0)$)

(ii) Suppose $\Phi(0)$ is not invertible.

Then the columns of $\Phi(0)$ are lin dependent.

So there exist scalars $c_1, c_2, \ldots, c_n$ (not all 0)

such that

$$\Phi(0)\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Now

$$\begin{aligned} \Phi(t)\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} &= e^{At}\,\Phi(0)\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= e^{At}\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned}$$

Now the columns of $\Phi(t)$ are lin dependent, a contradiction.

(iii) By (i), (ii) $\square$

<u>Ex</u> Consider system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix}}_{= A} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{matrix} x = x(t) \\ y = y(t) \end{matrix} \qquad (*)$$

Find a fundamental matrix $\Phi(t)$ and use it to find e^{At}

<u>Sol</u> Find eigvalues of A (skip detail)

eigenvalues are

$$\lambda = 3, 2$$

Find eigenvectors for A (skip detail)

λ	3	2
eigenvector	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

By section 7.5

sol space for $*$ has basis

$$e^{3t}\begin{bmatrix}3\\2\end{bmatrix}, \qquad e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$$

So a fundamental matrix for $(*)$ is

$$\Phi(t) = \begin{bmatrix} 3e^{3t} & e^{2t} \\ 2e^{3t} & e^{2t} \end{bmatrix}$$

Note

$$\Phi(0) = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

So

$$\Phi(0)^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

So

$$e^{At} = \Phi(t)\,\Phi(0)^{-1}$$

$$= \begin{bmatrix} 3e^{3t} & e^{2t} \\ 2e^{3t} & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= e^{3t} \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} + e^{2t} \begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix}$$

$\square$

Next goal:

Given $n \times n$ matrix A

How to find e^A (without assuming A is diagonalizable)

Let J = Jordan Normal Form for A.

Recall

$$J = \begin{pmatrix} \boxed{J_1} & & & O \\ & \boxed{J_2} & & \\ & & \ddots & \\ O & & & \boxed{J_s} \end{pmatrix}$$

For $1 \le i \le s$,

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & O \\ & \lambda_i & 1 & & \\ & & \lambda_i & \ddots & \\ & O & & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix} \quad k_i \times k_i$$

There exists an invertible matrix Φ such that

$$A = \Phi J \Phi^{-1}$$

Now

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots$$

$$= \varphi\left(I + J + \frac{J^2}{2} + \frac{J^3}{3!} + \cdots \right)\varphi^{-1}$$

$$= \varphi e^J \varphi^{-1}$$

Find e^J

$$e^J = \begin{pmatrix} \boxed{e^{J_1}} & & & 0 \\ & \boxed{e^{J_2}} & & \\ & 0 & \ddots & \\ & & & \boxed{e^{J_s}} \end{pmatrix}$$

For $1 \le i \le s$ find

$$e^{J_i}$$

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \mathbb{O} \\ & \lambda_i & 1 & \\ & & \ddots & \ddots \\ 0 & & & 1 \\ & & & \lambda_i \end{pmatrix} \qquad k_i \times k_i$$

$$= \lambda_i \underbrace{\begin{pmatrix} 1 & & \mathbb{O} \\ & 1 & \\ 0 & \ddots & \\ & & 1 \end{pmatrix}}_{= I} + \underbrace{\begin{pmatrix} 0 & 1 & & \mathbb{O} \\ & 0 & 1 & \\ & & \ddots & \ddots \\ \mathbb{O} & & & 1 \\ & & & 0 \end{pmatrix}}_{= N_i}$$

N_i is nilpotent

$$N_i^{k_i} = \mathbb{O}$$

$$e^{J_i} = e^{\lambda_i I + N_i}$$

$$= e^{\lambda_i I} e^{N_i}$$

$$= e^{\lambda_i} \left(I + N_i + \frac{N_i^2}{2} + \cdots + \frac{N_i^{k_i - 1}}{(k_i - 1)!} \right)$$

(finite sum)

Ex Solve the system

$$x' = 3x - y \qquad x = x(t)$$

$$y' = x + y \qquad y = y(t)$$

$$x(0) = 1 \qquad y(0) = 0$$

Sol Write in matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X}$$

Find eigenvalues of A (skip detail)

eigvals are $\lambda = 2, 2$

Find Jordan Normal Form for A (skip detail)

$$J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A = Q J Q^{-1} \qquad Q^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Find e^{At}

$$e^{At} = \varphi e^{Jt} \varphi^{-1}$$

Find e^{Jt}

$$J = 2I + N \qquad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$N^2 = 0$$

$$e^{Jt} = e^{(2I+N)t}$$

$$= e^{2t} e^{Nt}$$

$$= e^{2t}(I + Nt)$$

$$= e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$e^{At} = e^{2t} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix}$$

the sol is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{At} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = e^{At} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{2t} \begin{bmatrix} 1+t \\ t \end{bmatrix}$$

$\square$