# Lec 37 Monday April 28

## 8.1 Matrix Exponentials and Linear systems

Motivation Given a scalar $a$

the differential eqn

$$x' = ax \qquad x = x(t)$$

has the gen solution

$$x = e^{at} c \qquad c = \text{free scalar} \qquad (*)$$

Now let $A = n \times n$ matrix.

Consider the system

$$\mathbf{X}' = A\mathbf{X} \qquad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$x_i = x_i(t) \quad 1 \le i \le n.$$

By analogy with $(*)$, the general sol should be

$$\mathbf{X} = e^{At} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad c_1, c_2 \ldots c_n \text{ free} \qquad (**)$$

$(**)$ is correct, once we understand the meaning of $e^{At}$

Recall from calculus: for a variable $z$

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$$

$$= \sum_{i=0}^{\infty} \frac{z^i}{i!}$$

<u>Def</u> For an $n \times n$ matrix $A$ let

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots$$

$$= \sum_{i=0}^{\infty} \frac{A^i}{i!}$$

(For time being ignore issue of convergence)

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We now consider some special cases

**Def** An $n \times n$ matrix $A$ is called <u>nilpotent</u> whenever there exists a positive integer $r$ such that

$$A^r = 0$$

In this case

$$e^A = I + A + \frac{A^2}{2} + \cdots + \frac{A^{r-1}}{(r-1)!}$$

Ex  Given a diagonal $n \times n$ matrix $A$.
Find $e^A$

Sol  Write

$$A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & 0 & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

For $i \geq 0$

$$A^i = \begin{bmatrix} \lambda_1^i & & & 0 \\ & \lambda_2^i & & \\ & 0 & \ddots & \\ & & & \lambda_n^i \end{bmatrix}$$

So

$$e^A = \begin{bmatrix} e^{\lambda_1} & & & 0 \\ & e^{\lambda_2} & & \\ & 0 & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix}$$

**Ex** Given a diagonalizable $n \times n$ matrix $A$, find $e^A$

**Sol** there exists a diagonal matrix $D$ and an invertible matrix $P$ such that

$$A = PDP^{-1}$$

For $i \geq 0$,

$$A^i = \underbrace{AA \cdots A}_{i}$$

$$= \underbrace{PDP^{-1}}_{1} \underbrace{PDP^{-1}}_{2} \cdots \underbrace{PDP^{-1}}_{i}$$

$$= P \underbrace{D \cdots D}_{i} P^{-1}$$

$$= PD^iP^{-1}$$

So
$$e^A = I + A + \frac{A^2}{2} + \cdots$$

$$= PP^{-1} + PDP^{-1} + \frac{PD^2P^{-1}}{2} + \cdots$$

$$= P\left(I + D + \frac{D^2}{2} + \cdots\right)P^{-1}$$

$$= Pe^DP^{-1}$$

with $e^D$ found by prev ex. $\square$

**LEM** Given $n \times n$ matrix $A$,

define a matrix valued function

$$\mathbb{X}(t) = e^{At}$$ (Caution: $X(t)$ is $n \times n$)

then

$$\mathbb{X}' = A\mathbb{X}$$

**pf**

$$X' = \frac{d}{dt} e^{At}$$

$$= \frac{d}{dt}\left( I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \cdots \right)$$

$$= \emptyset + A1 + \frac{A^2}{2} 2t + \frac{A^3}{3!} 3t^2 + \cdots$$

$$= A\left( I + At + \frac{A^2t^2}{2} + \cdots \right)$$

$$= A e^{At}$$

$$= A\mathbb{X} \qquad \square$$

**thm** Given $n \times n$ matrix $A$

Consider system

$$\underline{X}' = A\underline{X} \tag{$*$}$$

(Here $\underline{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is column vector)

then

(i) Each column of $e^{At}$ is a sol to $*$.

(ii) the columns of $e^{At}$ are lin indep.

(iii) the columns of $e^{At}$ form a basis for the solution space of $(*)$

(iv) the general sol to $(*)$ is

$$\underline{X} = e^{At}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad c_1, c_2, \ldots, c_n \text{ free scalars}$$

pf (i) By prev lem

(ii) $e^{At}$ is invertible indeed inverse is $e^{-At}$

(iii) Sol space of $*$ is vectr space of dim $n$.
Result follows by (i), (ii)

(iv) By (iii) $\square$

Ex Solve the system

$x' = 11x - 15y$ $\quad\quad x = x(t),$

$y' = 6x - 8y$ $\quad\quad y = y(t)$

$x(0) = 1, \quad y(0) = 1$

Sol Write in matrix form

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{X'} = \underbrace{\begin{bmatrix} 11 & -15 \\ 6 & -8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X}$$

Find eigenvalues of A (skip detail)

eigvalues are

$\lambda = 2, 1$

Find eigenvectors for A (skip detail)

| $\lambda$ | 2 | 1 |
|---|---|---|
| eigenvectr | $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ | $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ |

Define

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

Note

$$P^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

$$A = P D P^{-1}$$

$$e^{At} = P e^{Dt} P^{-1}$$

$$= \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} 10 & -15 \\ 6 & -9 \end{bmatrix} + e^{t} \begin{bmatrix} -9 & 15 \\ -6 & 10 \end{bmatrix}$$

Gen sol is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{At} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

To find $c_1, c_2$ set $t=0$:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underbrace{e^{0}}_{= I} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$c_1 = 1, \qquad c_2 = 1$

Sol is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{At} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} -5 \\ -3 \end{bmatrix} + e^{t} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$\square$