Lec 36 Friday April 25

7.5 Continued

Given $n \times n$ matrix A

Consider system

$$\mathbf{X}' = A\mathbf{X} \qquad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad (*)$$

$x_i = x_i(t) \qquad 1 \le i \le n$

Goal: Find the general solution

<u>thm</u> Given $n \times n$ matrix A

Let $v_1, v_2, \dots, v_k$ denote a chain of generalized eigenvectors for A

Let λ = associated eigenvalue

then each of the following is a solution to $(*)$

$$e^{\lambda t} v_1$$

$$e^{\lambda t}(t v_1 + v_2)$$

$$e^{\lambda t}\left(\frac{t^2}{2} v_1 + t v_2 + v_3\right)$$

$$e^{\lambda t}\left(\frac{t^3}{3!} v_1 + \frac{t^2}{2} v_2 + t v_3 + v_4\right)$$

$$\cdots$$

$$e^{\lambda t}\left(\frac{t^{k-1}}{(k-1)!} v_1 + \frac{t^{k-2}}{(k-2)!} v_2 + \cdots + t v_{k-1} + v_k\right)$$

the above solutions are lin indep.

pf show

$$X = e^{\lambda t}\left(\frac{t^3}{3!}v_1 + \frac{t^2}{2}v_2 + tv_3 + v_4\right)$$

is a solution to

$$X' = AX$$

By product rule

$$X' = e^{\lambda t}\left(\frac{t^2}{2}v_1 + tv_2 + v_3\right)$$

$$+ \lambda e^{\lambda t}\left(\frac{t^3}{3!}v_1 + \frac{t^2}{2}v_2 + tv_3 + v_4\right)$$

Also

$$AX =$$

$$e^{\lambda t}\left(\frac{t^3}{3!}\underbrace{Av_1}_{=\lambda v_1} + \frac{t^2}{2}\underbrace{Av_2}_{=\lambda v_2 + v_1} + t\underbrace{Av_3}_{=\lambda v_3 + v_2} + \underbrace{Av_4}_{=\lambda v_4 + v_3}\right)$$

Compare X', AX to see $X' = AX$

Other solutions are similarly verified

Set $t=0$ to see that the given solutions are lin indep

$\square$

Given $n \times n$ matrix A

Consider system

$$\underline{X}' = A\underline{X} \qquad (*)$$

To find gen solution:

- Find a basis for the underlying vector space of A consisting of a union of chains of generalized eigenvectors for A
- For each chain, the prev thm gives its contribution to the general solution for $(*)$.

<u>Ex</u> Given 7×7 matrix A

Suppose that the underlying vector space of A has a basis

$v_1, v_2, v_3, v_4, v_5, v_6, v_7$

such that

v_1, v_2 is a chain of gen. eigenvectors for A with eigenvalue λ_1

v_3, v_4, v_5 — — — λ_2

v_6, v_7 — — — λ_3

Then the system $\mathbf{X}' = A\mathbf{X}$ has gen. sol

$$c_1 e^{\lambda_1 t} v_1$$
$$+$$
$$c_2 e^{\lambda_1 t}(tv_1 + v_2)$$
$$+$$
$$c_3 e^{\lambda_2 t} v_3$$
$$+$$
$$c_4 e^{\lambda_2 t}(tv_3 + v_4)$$
$$+ \quad c_5 e^{\lambda_2 t}\left(\frac{t^2}{2}v_3 + tv_4 + v_5\right)$$
$$+ \quad c_6 e^{\lambda_3 t} v_6$$
$$+$$
$$c_7 e^{\lambda_3 t}(tv_6 + v_7)$$

$c_1, \dots, c_7$ free

□

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Ex $A = \begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

Find gen solution to

$$\mathbf{X}' = A\mathbf{X}$$

Sol Find the eigenvalues of A: (skip detail)

eigenvalues are

$$1, 1, 1$$

Find a basis for the eigenspace of A for $\lambda = 1$:

$$A - \lambda I = \begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \overset{GJ}{\sim} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Backsolve (skip detail)

Eigenspace has basis

$$\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (*)$$

A is not diagonalizable

Find chain v_1, v_2 of gen eigvectors

for A, for $\lambda = 1$

Require

$$Av_1 = \lambda v_1$$
$$(A - \lambda I) v_2 = v_1$$

Obs

$$v_1 = (A - \lambda I) v_2$$
$$\in \text{Col space } (A - \lambda I)$$
$$= \text{Span} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

Take

$$v_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \qquad (= \text{sum of vectors in} ✩)$$

Find v_2:

Write

$$v_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Require

$$\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$$a + 3b = 1$$

Take

$$a = 1, \quad b = 0, \quad c = 0$$

Now

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Define v_3 = either of ☆, say

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\mathbb{R}^3$ has basis

$$\underbrace{v_1 \quad v_2}_{\substack{\text{chain for} \\ \lambda = 1}} \qquad \underbrace{v_3}_{\substack{\text{chain for} \\ \lambda = 1}}$$

Gen sol of $\mathbf{X}' = A\mathbf{X}$ is

$$\mathbf{X} = c_1 e^t v_1 + c_2 e^t (t v_1 + v_2) + c_3 e^t v_3$$

c_1, c_2, c_3 free

□

The Jordan Normal Form

Given $n \times n$ matrix A

Given n lin indep vectors $v_1, v_2, \dots, v_n$ that form a union of chains of generalized eigenvalues for A

Let $s = \#$ chains involved

For $1 \leq i \leq s$ let

$k_i =$ length of chain i

$(\text{so } k_1 + k_2 + \dots + k_s = n)$

Let $\lambda_i =$ eigenvalue for chain i

vectors	$\underbrace{v_1 \quad v_2 \quad \dots}_{\text{chain 1}}$	$\underbrace{\dots\dots}_{\text{chain 2}}$	$\dots$	$\underbrace{\dots \quad v_n}_{\text{chain } s}$
chain length	k_1	k_2	$\dots$	k_s
eigenvalue	λ_1	λ_2	$\dots$	λ_s

Define $n \times n$ matrix Q such that for $1 \leq i \leq n$

col i of $Q = v_i$

Define an $n \times n$ matrix J by

$$J = \begin{pmatrix} \boxed{J_1} & & & 0 \\ & \boxed{J_2} & & \\ & & \ddots & \\ 0 & & & \boxed{J_s} \end{pmatrix}$$

For $1 \le i \le s$

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & 0 \\ & \lambda_i & 1 & & \\ & & \lambda_i & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda_i \end{pmatrix} \qquad k_i \times k_i$$

"Jordan block"

then

$$A Q = Q J$$

Note $\quad Q^{-1} A Q = J \quad$ so

A is similar to J

J is called the Jordan Normal form of A

<u>thm</u> With above notation,

the characteristic polynomial of A is

$$|A - \lambda I| = (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_s)^{k_s}$$

<u>pf</u> A is similar to J so

A, J have the same char poly.

J is upper triangular:

$$J = \begin{pmatrix} * & * & & & 0 \\ & * & * & & \\ & & * & * & \\ & & & \ddots & \ddots \\ & 0 & & & * \\ & & & & * \end{pmatrix}$$

char poly of J is

$$|J - \lambda I| = (J_{11} - \lambda)(J_{22} - \lambda) \cdots (J_{nn} - \lambda)$$

$$= \underbrace{(\lambda_1 - \lambda) \cdots (\lambda_1 - \lambda)}_{k_1 \text{ copies}} \underbrace{(\lambda_2 - \lambda) \cdots (\lambda_2 - \lambda)}_{k_2 \text{ copies}} \cdots \underbrace{(\lambda_s - \lambda) \cdots (\lambda_s - \lambda)}_{k_s \text{ copies}}$$

$$= (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_s)^{ks}$$

$\square$

Ex Find the Jordan Normal Form for

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix}$$

Sol We saw earlier that the vectors

$$\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

form a chain of gen eigenvectors for A, with eigenvalue $\lambda = -1$.

Define

$$Q = \begin{bmatrix} -1 & -1 & -1 \\ 5 & 5 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Then $AQ = QJ$ so $Q^{-1}AQ = J$

So J is the Jordan Normal Form for A

the Cayley-Hamilton theorem

Warm up Consider

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

then

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^4 = \mathbb{O}$$

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LEM Given $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \huge{0} \\ & & 0 & \ddots & \\ & \huge{0} & & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}$$

then

$$A^n = \mathbb{O}$$

thm (Cayley–Hamilton)

Given an $n \times n$ matrix A

consider the characteristic polynomial

$$f(\lambda) = |A - \lambda I|$$

then $f(A) = 0$

pf (Earlier we proved this assuming A is diagonalizable)

Consider Jordan Normal form for A:

$$J = \begin{pmatrix} \boxed{J_1} & & & 0 \\ & \boxed{J_2} & & \\ & & \ddots & \\ 0 & & & \boxed{J_s} \end{pmatrix}$$

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & 0 \\ & \lambda_i & 1 & & \\ & & \lambda_i & \ddots & \\ & & & \ddots & \\ & & & \lambda_i & 1 \\ 0 & & & & \lambda_i \end{pmatrix} \qquad k_i \times k_i$$

so

$$f(\lambda) = (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_s)^{k_s}$$

there exists an invertible matrix Q such that

$$A = QJQ^{-1}$$

Now

$$f(A) = f(QJQ^{-1})$$

$$= Q f(J) Q^{-1}$$

show $f(J) = 0$

For $i \le i \le s$ show $f(J_i) = 0$

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & 0 \\ & \lambda_i & 1 & & \\ & & \lambda_i & \ddots & \\ & & & \ddots & \\ & 0 & & & \lambda_i & 1 \\ & & & & & \lambda_i \end{pmatrix} \quad k_i \times k_i$$

so

$$J_i - \lambda_i I = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & \\ & 0 & & & 0 & 1 \\ & & & & & 0 \end{pmatrix} \quad k_i \times k_i$$

so

$$(J_i - \lambda_i I)^{k_i} = 0$$

obs

$(\lambda - \lambda_i)^{k_i}$ is a factor of $f(\lambda)$

so

$$f(J_i) = 0$$

$\square$

Ex For

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix}$$

We saw earlier that A has char poly

$$|A - \lambda I| = -(\lambda+1)^3$$

By Cayley-Hamilton

$$(A+I)^3 = \mathbb{O}$$

$\square$