4/23/14
1

Lec 35 Wednesday April 23

7.5 Multiple eigenvalue Solutions

Given $n \times n$ matrix A

Consider system

$$\underline{X}' = A \underline{X} \qquad \underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$x_i = x_i(t) \qquad 1 \le i \le n$$

Find gen solution.

Cases: A is diagonalizable — We did 7.3

A arb — today + Fri

Ex For functions

$$x = x(t), \qquad y = y(t)$$

find the gen solution of

$$x' = -2x + y,$$
$$y' = -x - 4y$$

Sol Matrix form

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{X'} = \underbrace{\begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} \qquad (*)$$

Find the eigenvalues of A:

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 1 \\ -1 & -4-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(-4-\lambda) + 1$$

$$= \lambda^2 + 6\lambda + 9$$

$$= (\lambda + 3)^2$$

Eigenvalues are

$-3, -3$

Find a basis for the eigenspace of A for $\lambda = -3$

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \overset{GJ}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is basis for the eigenspace

So

$$Av_1 = -3v_1$$

So

$$X = e^{-3t} v_1$$

is a solution to $X' = AX$ (but not gen sol)

Need another sol that is lin indep of first one.

Guess every sol to $*$ has form

$$x = ae^{-3t} + bte^{-3t}$$

$$y = ce^{-3t} + dte^{-3t}$$

with scalars a, b, c, d to be determined

Plug form into orig system

$$x' = \frac{d}{dt} \quad e^{-3t}(a+bt)$$

$$= e^{-3t} b \quad -3e^{-3t}(a+bt)$$

$$= e^{-3t}\left(b - 3a \quad - 3bt\right)$$

$$y' = \frac{d}{dt} \quad e^{-3t}(c + dt)$$

$$= e^{-3t}\left(d - 3c \quad - 3dt\right)$$

Orig system becomes

$$b - 3a - 3bt = -2(a+bt) + 1(c+dt)$$

$$d - 3c - 3dt = -(a+bt) - 4(c+dt)$$

Compare coefs of t to get

$$c = b - a$$
$$d = -b$$

a, b free

$*$ has gen sol

$$x = a e^{-3t} + b t e^{-3t}$$
$$y = (b-a) e^{-3t} - b t e^{-3t}$$

a, b free

In matrix form the gen sol is

$$\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix} = a e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + b e^{-3t} \begin{bmatrix} t \\ 1-t \end{bmatrix}$$

$$= a e^{-3t} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{= v_1} + b e^{-3t} \left(t \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{= v_1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

call this v_2

$$= a e^{-3t} v_1 + b e^{-3t} (t v_1 + v_2)$$

a, b free

Find meaning of v_2

Recall $\qquad Av_1 = \lambda v_1 \qquad \lambda = -3$

So $\qquad (A - \lambda I) v_1 = 0$

Find $\qquad (A - \lambda I) v_2$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = v_1$$

$$(A - \lambda I) v_2 = v_1$$

Gen sol to * is

$$a e^{\lambda t} v_1 + b e^{\lambda t} (t v_1 + v_2) \qquad a, b \text{ free}$$

$$\lambda = -3$$

where

$$(A - \lambda I) v_1 = 0, \qquad (A - \lambda I) v_2 = v_1, \qquad v_1 \neq 0$$

the sequence v_1, v_2 is called a

chain of generalized eigenvectors for $\lambda = -3$

Def Given any $n \times n$ matrix A

let $\lambda =$ an eigenvalue of A

A sequence of vectors

$$v_1, v_2, \ldots, v_k$$

is called a chain of generalized eigenvectors for λ

whenever $\qquad v_1 \neq 0$

$$(A - \lambda I) v_1 = 0$$

$$(A - \lambda I) v_2 = v_1$$

$$(A - \lambda I) v_3 = v_2$$

$$\vdots$$

$$(A - \lambda I) v_k = v_{k-1}$$

[In this case $v_1, v_2, \ldots, v_k$ are lin indep]

__Ex__ For

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix}$$

Find a chain of generalized eigenvectors

__Sol__ Find the eigenvalues of A

$$|A-\lambda I| = \begin{vmatrix} 0-\lambda & 0 & 1 \\ -5 & -1-\lambda & -5 \\ 4 & 1 & -2-\lambda \end{vmatrix}$$

$$= -\lambda \underbrace{\begin{vmatrix} -1-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix}}_{\lambda^2+3\lambda+7} - 0 + 1 \underbrace{\begin{vmatrix} -5 & -1-\lambda \\ 4 & 1 \end{vmatrix}}_{4\lambda-1}$$

$$= -(\lambda+1)^3$$

$\lambda = -1, -1, -1$

Find basis for eigenspace of A for $\lambda = -1$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{bmatrix}$$

Solve

$$(A - \lambda I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A - \lambda I$$

$$\downarrow \quad GJ$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$z = t$ free
$y = 5t$
$x = -t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$$

Eigenspace has basis

$$v_1 = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$$

Hunt for a vector v_2 such that

$$(A - \lambda I)v_2 = v_1 \qquad \lambda = -1$$

Write

$$v_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Require

$$\begin{bmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$$

$$a + c = -1$$

$$4a + b - c = 1$$

take

$$a = -1 \qquad b = 5 \qquad c = 0$$

(many sols – just pick one)

$$v_2 = \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}$$

Hunt for vector v_3 such that

$$(A - \lambda I) v_3 = v_2 \qquad \lambda = -1$$

Write $$v_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Require

$$\begin{bmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}$$

$$a + c = -1$$

$$4a + b - c = 0$$

Take $a = -1 \qquad b = 4 \qquad c = 0$

$$v_3 = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

By construction for $\lambda = -1$

$$(A-\lambda I)v_1 = 0, \qquad v_1 \neq 0$$

$$(A-\lambda I)v_2 = v_1$$

$$(A-\lambda I)v_3 = v_2$$

So v_1, v_2, v_3 is a chain of generalized eigenvectors for A,

<u>Thm</u> For any $n \times n$ matrix A,
there exists a basis for the underlying vector space
that is a union of chains of generalized eigenvectors

<u>Note</u> Above A is diagonalizable if and only if
each chain of gen eigenvectors has length 1

Ex Ref to prev example,

find the general sol to

$$X' = A X \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad *$$

Sol let v_1, v_2, v_3 be the chain of gen eigenvectors

from prev ex.

We show the gen sol to $*$ is

$$X = C_1 e^{\lambda t} v_1 + C_2 e^{\lambda t}(tv_1 + v_2) + C_3 e^{\lambda t}\left(\tfrac{t^2}{2} v_1 + tv_2 + v_3\right)$$

$\lambda = -1$

scalars c_1, c_2, c_3 free

* let V = set of all sols to $*$

V is vector space dim 3

One checks V contains

$$e^{\lambda t} v_1, \quad e^{\lambda t}(tv_1 + v_2), \quad e^{\lambda t}\left(\tfrac{t^2}{2} v_1 + tv_2 + v_3\right)$$

the elements are lin indep (set $t=0$ to see this)

so they form a basis for V $\square$