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Lec 34 Monday April 21

7.3 the eigenvalue method for linear systems

We now use eigenvalues and eigenvectors to solve 1st order linear systems of differential equations

Ex For functions

$x = x(t) \qquad y = y(t)$

find the gen solution of

$$x' = 6x - 7y \qquad (*)$$
$$y' = x - 2y$$

Sol Put $*$ in matrix form:

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{X'} = \underbrace{\begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X}$$

Find the eigenvalues of A:

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -7 \\ 1 & -2-\lambda \end{vmatrix}$$

$$= (6-\lambda)(-2-\lambda) - (-7)(1)$$

$$= -12 - 4\lambda + \lambda^2 + 7$$

$$= \lambda^2 - 4\lambda - 5$$

$$= (\lambda - 5)(\lambda + 1)$$

$\lambda = 5, -1$

A is diagonalizable

Def $D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$

Find an invertible matrix P s.t.

$$A = PDP^{-1}$$

λ	5	-1
$A - \lambda I$	$\begin{bmatrix} 1 & -7 \\ 1 & -7 \end{bmatrix}$	$\begin{bmatrix} 7 & -7 \\ 1 & -1 \end{bmatrix}$
Solve $(A-\lambda I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -7 \\ 0 & 0 \end{bmatrix}$ $x - 7y = 0$ $y = t$ $x = 7t$ $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 7 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x - y = 0$ $y = t$ $x = t$ $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
eigenvector for λ	$\begin{bmatrix} 7 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Define $P = \begin{bmatrix} 7 & 1 \\ 1 & 1 \end{bmatrix}$

Then $A = P D P^{-1}$

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Recall

$$\underline{X}' = A\underline{X}$$

$$A = PDP^{-1}$$

so

$$\underline{X}' = PDP^{-1}\underline{X}$$

so

$$P^{-1}\underline{X}' = D\,P^{-1}\underline{X} \qquad **$$

"change variables"

Define $\underline{Y} = P^{-1}\underline{X}$

so $\underline{X} = P\underline{Y}$ $\qquad \underline{Y} = \begin{bmatrix} u \\ v \end{bmatrix}$ $\qquad \begin{matrix} u = u(t) \\ v = v(t) \end{matrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$x = 7u + v$$

$$y = u + v$$

** becomes

$$\underline{Y}' = D\underline{Y}$$

So

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

So

$$u' = 5u$$

$$v' = -v$$

Gen sol fn u, v is

$$u = C_1 e^{5t}$$

$$v = C_2 e^{-t}$$

C_1, C_2 free

So gen sol fn $*$ is

$$x = 7u + v$$

$$= 7C_1 e^{5t} + C_2 e^{-t}$$

C_1, C_2 free

$$y = u + v$$

$$= C_1 e^{5t} + C_2 e^{-t}$$

gen sol in matrix form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^{5t} \begin{bmatrix} 7 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

C_1, C_2 free

Let $\underline{V}$ = set of solutions to *

$\underline{V}$ is vector space dim 2

$\underline{V}$ has basis

$$e^{5t}\begin{bmatrix}7\\1\end{bmatrix}, \qquad e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$$

Each term has form

$$e^{\lambda t}v \qquad v = \text{eigenvector of } A \text{ for eigenvalue } \lambda$$

We now consider $n \times n$ systems

Given functions

$$x_1 = x_1(t), \quad x_2 = x_2(t), \ \ldots, \quad x_n = x_n(t)$$

such that

$$\begin{aligned} x_1' &= a_{11} x_1 + \cdots + a_{1n} x_n \\ x_2' &= a_{21} x_1 + \cdots \quad a_{2n} x_n \\ &\cdots \\ x_n' &= a_{n1} x_1 + \cdots + a_{nn} x_n \end{aligned} \qquad a_{ij} \in \mathbb{R}$$

Find the gen sol

Matrix form:

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}}_{= \ \underline{X}'} = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{= \ A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{= \ \underline{X}}$$

$$\underline{X}' = A \underline{X} \qquad *$$

To keep it simple, assume A is diagonalizable

So there exists a diagonal matrix D

and an invertible matrix P such that

$$A = PDP^{-1}$$

Write

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

For $1 \le i \le n$ define

$$v_i = \text{column } i \text{ of } P$$

So

$$A v_i = \lambda_i v_i$$

P is invertible so

$$v_1, v_2, \dots, v_n \quad \text{are lin indep}$$

Let $\underline{V}$ = set of all solutions to $*$

By Last thm in prev Lec

$\underline{V}$ is vector space with dimension n

<u>thm</u> the following is a basis for $\underline{V}$:

$$e^{\lambda_1 t} v_1, \quad e^{\lambda_2 t} v_2, \quad \dots, \quad e^{\lambda_n t} v_n \qquad ☆$$

<u>pf</u> Show each term in ☆ is contained in $\underline{V}$:

Given eigenvalue λ for A with eigenvector v show

$$\underline{X} = e^{\lambda t} v$$

is a sol to $*$:

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$$\left(e^{\lambda t} v\right)' \stackrel{?}{=} A\left(e^{\lambda t} v\right)$$

$$\| \qquad\qquad\qquad \|$$

$$\lambda e^{\lambda t} v \qquad\qquad e^{\lambda t} A v$$

$$\|$$

$$\text{OK} \qquad e^{\lambda t} \lambda v$$

check the elements ☆ are lin indep

Given scalars $c_1, c_2 \ldots c_n$ such that

$$c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n = 0$$

show $c_1 = 0, \; c_2 = 0, \; \ldots, \; c_n = 0$

Set $t = 0$

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

By constr $v_1, v_2 \ldots, v_n$ are lin indep.

So $c_1 = 0, \; c_2 = 0, \; \ldots, \; c_n = 0$ ✓

$\square$

<u>Cor</u> With above notation, the general solution to

$$\underline{X}' = A\underline{X}$$

is

$$\underline{X} = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n$$

$c_1, c_2 \ldots c_n$ free

$\square$

Ex Find the general solution for

$$x' = 3x - 4y$$
$$y' = 4x + 3y$$

Sol

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underset{\overset{\shortparallel}{A}}{\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Find eigvalues of A :

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)^2 + 16$$

$$= \lambda^2 - 6\lambda + 25$$

$$\lambda = \frac{6 \mp \sqrt{36 - 100}}{2}$$

$$= \frac{6 \mp 8i}{2}$$

$$= 3 \pm 4i$$

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Find eigenvectors

λ	$3+4i$	$3-4i$
$A-\lambda I$	$\begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix}$	$\begin{bmatrix} 4i & -4 \\ 4 & 4i \end{bmatrix}$
solve $(A-\lambda I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$ $x - iy = 0$ $y = t$ $x = it$ $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$ $x + iy = 0$ $y = t$ $x = -it$ $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} -i \\ 1 \end{bmatrix}$
eigvector for λ	$\begin{bmatrix} i \\ 1 \end{bmatrix}$	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

gen sol is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{(3+4i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{(3-4i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

c_1, c_2 free

Find gen sol in trig form: 4/21/14

Recall

$$e^{(3+4i)t} = e^{3t}\left(\cos 4t + i \sin 4t\right)$$

$$e^{(3-4i)t} = e^{3t}\left(\cos 4t - i \sin 4t\right)$$

gen sol is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{3t}\left(\cos 4t + i \sin 4t\right) c_1 \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$+ \; e^{3t}\left(\cos 4t - i \sin 4t\right) c_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$= e^{3t} \cos 4t \begin{array}{c} \begin{bmatrix} c_1 i - c_2 i \\ c_1 + c_2 \end{bmatrix} \end{array} \begin{array}{l} = r \\ = s \end{array}$$

$$+ \; e^{3t} \sin 4t \begin{array}{c} \begin{bmatrix} -c_1 - c_2 \\ c_1 i - c_2 i \end{bmatrix} \end{array} \begin{array}{l} = s \\ = r \end{array}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{3t} \cos 4t \begin{bmatrix} r \\ -s \end{bmatrix} + e^{3t} \sin 4t \begin{bmatrix} s \\ r \end{bmatrix}$$

r, s free

Another way to get gen sol in trig form:

each sol has form

$$x = re^{3t}\cos 4t + se^{3t}\sin 4t$$
$$y = Re^{3t}\cos 4t + Se^{3t}\sin 4t$$

for some scalars r, s, R, S

$**$

To find R, S in terms of r, s plug $**$ into orig system. Get

$$R = -s \qquad S = r.$$

Ex Solve the system

$$x' = 3x - 4y \qquad x(0) = 2$$
$$y' = 4x + 3y \qquad y(0) = 3$$

Sol From prev example there are constants r, s such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{3t}\cos 4t \begin{bmatrix} r \\ -s \end{bmatrix} + e^{3t}\sin 4t \begin{bmatrix} s \\ r \end{bmatrix}$$

To find r, s set $t=0$:

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 1 \; 1 \begin{bmatrix} r \\ -s \end{bmatrix} + 1 \cdot 0 \cdot \begin{bmatrix} s \\ r \end{bmatrix}$$
$$= \begin{bmatrix} r \\ -s \end{bmatrix}$$

$$r = 2, \quad s = -3$$

$$x = 2e^{3t}\cos 4t - 3e^{3t}\sin 4t$$
$$y = 3e^{3t}\cos 4t + 2e^{3t}\sin 4t$$

$\square$