Lec 33 Friday April 18

7.2 Matrices and Linear systems

We now consider systems of differential equations in matrix form

Ex Given functions

$x = x(t), \quad y = y(t)$

such that

$$x' = 2x + 4y + 3e^t \qquad *$$
$$y' = 5x - y - t^2$$

In matrix form $*$ becomes

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\substack{\| \\ \mathbb{X}' \\ \| \\ \mathbb{X}'(t)}} = \underbrace{\begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix}}_{\substack{\| \\ A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\substack{\| \\ \mathbb{X} \\ \| \\ \mathbb{X}(t)}} + \underbrace{\begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}}_{\substack{\| \\ F \\ \| \\ F(t)}}$$

$$\mathbb{X}' = A\mathbb{X} + F$$

$\underline{X}(t)$ and $F(t)$ are examples of <u>matrix valued functions</u>

For each input t they output a 2×1 matrix

<u>Derivatives and matrix valued functions</u>

<u>Ex</u> Given matrix valued function

$$P(t) = \begin{bmatrix} t & t^2 \\ 1 & t \end{bmatrix}$$

Find

$$\frac{d}{dt} P(t)$$

<u>Sol</u> The entries of $\frac{d}{dt} P(t) = P'(t)$
are the derivatives of the entries of $P(t)$

$$\frac{d}{dt} P(t) = \begin{bmatrix} \frac{d}{dt} t & \frac{d}{dt} t^2 \\ \frac{d}{dt} 1 & \frac{d}{dt} t \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix}$$

Ex For matrix valued functions

$A = A(t)$, $\quad \mathbb{X} = \mathbb{X}(t)$

show $\quad (A\mathbb{X})' = A'\mathbb{X} + A\mathbb{X}' \qquad *$

(ie prod rule of derivative works for matrix valued functions)

Sol: list $A, \mathbb{X}$ are 2×2

write

$$A = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$$

$$\mathbb{X} = \begin{bmatrix} x(t) & y(t) \\ z(t) & w(t) \end{bmatrix}$$

Compute (i,j)-entry on each side of $*$. Try $i=1$, $j=1$

LHS: $(1,1)$ entry of $(A\mathbb{X})'$

$$= (ax + bz)'$$

$$= (ax)' + (bz)'$$

$$= a'x + ax' + b'z + bz'$$

RHS:

(1,1)-entry of $A'X + AX'$

$$= a'x + b'z \quad + \quad ax' + bz'$$

(same) $\square$

Ex Consider the system

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad *$$

$$\underline{X}' \qquad\qquad A \qquad\qquad \underline{X}$$

Show that each of

$$\underline{X}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\underline{X}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

is a sol to $*$

Sol check

$$\underline{X}_1' = A\underline{X}_1$$

$$\mathbb{X}_1' \stackrel{?}{=} A\,\mathbb{X}_1$$

$$\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} 3e^{2t} - e^{2t} \\ 5e^{2t} - 3e^{2t} \end{bmatrix}$$

Similarly check

$$\mathbb{X}_2' = A\,\mathbb{X}_2$$

$\square$

Ex Ref to prev Example show

$\mathbb{X}_1, \mathbb{X}_2$ are lin indep.

Sol Given scalars $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \mathbb{X}_1 + c_2 \mathbb{X}_2 = 0 \qquad \star$$

Show $c_1 = 0, \; c_2 = 0$

Define 2×2 matrix such that

col1 is $\mathbb{X}_1$

col2 is $\mathbb{X}_2$.

$$\begin{bmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{bmatrix}$$

$\star$ becomes

$$\begin{bmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If there exists $t \in \mathbb{R}$ such that is nonsingular then only sol is $c_1 = 0, c_2 = 0$

Consider determinant

$$W(t) = \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix}$$

"the Wronskian determinant of the sols Φ_1, Φ_2"

$$W(t) = e^{2t} \begin{vmatrix} 1 & e^{-2t} \\ 1 & 5e^{-2t} \end{vmatrix}$$

$$= \underbrace{e^{2t} e^{-2t}}_{1} \underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}}_{\substack{\shortparallel \\ 5-1 \\ \shortparallel \\ 4}}$$

$$= 4$$

$$\neq 0$$

Above coeff matrix is nonsingular for all t so

only sol is $c_1 = 0, \quad c_2 = 0$

Φ_1, Φ_2 are lin indep

$\square$

Ex Solve the system

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad *$$

$\mathbb{X}' =$ $\qquad$ $= A$ $\qquad$ $= \mathbb{X}$

subject to initial condition

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Sol $\quad *$ is same as above

We have 2 lin indep sols

$$\mathbb{X}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbb{X}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Hunt for sol of form

$$\mathbb{X} = c_1\mathbb{X}_1 + c_2\mathbb{X}_2 \qquad c_1, c_2 \in \mathbb{R}$$

so

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + c_2\begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

To find c_1, c_2 set $t=0$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

So

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$= \frac{1}{5-1} \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 28 \\ -8 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$c_1 = 7, \quad c_2 = -2$

Sol is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 7e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$\square$

Some theory

For $n \geq 1$ consider general 1st order linear system

$$x_1' = p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + f_1(t)$$
$$x_2' = p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + f_2(t)$$
$$\cdots$$
$$x_n' = p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + f_n(t)$$

(each function p_{ij}, f_i is continuous on open interval I)

Matrix form:

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}'}_{\underline{X}'} = \underbrace{\begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix}}_{P = P(t)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\underline{X} = \underline{X}(t)} + \underbrace{\begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}}_{F = F(t)}$$

$$\underline{X}' = PX + F$$

Call the system homogeneous whenever

$$f_i(t) = 0 \quad 1 \leq i \leq n$$

ie

$$F = 0$$

so

$$\underline{X}' = P\underline{X}$$

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thm Ref to the above homog system.
The set of sols is a vector space of dim n.

pf let V = set of sols to $\mathbf{X}' = P\mathbf{X}$

Show V is vector space

Given $\mathbf{X}_1, \mathbf{X}_2$ in V show $\mathbf{X}_1 + \mathbf{X}_2 \in V$

$$\mathbf{X}_1' = P\mathbf{X}_1 \qquad \mathbf{X}_2' = P\mathbf{X}_2$$

$$(\mathbf{X}_1 + \mathbf{X}_2)' \stackrel{?}{=} P(\mathbf{X}_1 + \mathbf{X}_2)$$

$$(\mathbf{X}_1 + \mathbf{X}_2)' = \mathbf{X}_1' + \mathbf{X}_2' = P\mathbf{X}_1 + P\mathbf{X}_2 \qquad P(\mathbf{X}_1 + \mathbf{X}_2) = P\mathbf{X}_1 + P\mathbf{X}_2 \quad \checkmark$$

Given $\mathbf{X} \in V$ Given $a \in \mathbb{R}$ show $a\mathbf{X} \in V$

$$(a\mathbf{X})' \stackrel{?}{=} P(a\mathbf{X})$$

$$(a\mathbf{X})' = a\mathbf{X}' = a\,P\mathbf{X} \qquad P(a\mathbf{X}) = a\,P\mathbf{X} \quad \checkmark$$

Show $\dim V = n$

For $1 \le i \le n$ define

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i \text{ coord}$$

Pick $a \in I$

By Last thm in prev lec

For $1 \le i \le n$ there exists a unique sol $X_i = X_i(t)$

s.t

$$X_i(a) = e_i$$

show $X_1, X_2, \ldots X_n$ form bases for V

One checks (using Wronskian) that

$$X_1, X_2, \ldots, X_n \quad \text{are lin indep}$$

Show they span V

Given any $X \in V$

Write

$$X(a) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

then

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

because each side is in V and satisfies the same init conds.

So $X_1, \ldots, X_n$ form bases for V

$\square$