Lec 32 Wed April 16

7.1 First-order systems and applications

We introduce First Order systems with an example

Ex Given two functions

$$x(t), \quad y(t)$$

Assume

$$\begin{aligned} x' &= y \\ y' &= 6x - y \end{aligned} \qquad *$$

Find the general solution.

$*$ is a

First order linear system of diff equations

First order: $x'', x''', \ldots$ $y'', y''', \ldots$ do not appear

linear: each eq is linear in x, x', y, y'

system: more than one eq and unknown x, y

Sol Eliminate y:

$$x'' = y'$$
$$= 6x - y$$
$$= 6x - x'$$

So

$$x'' + x' - 6x = 0$$

Char eq

$$r^2 + r - 6 = 0$$
$$(r-2)(r+3) = 0$$
$$r = 2, -3$$

$$x = Ae^{2t} + Be^{-3t} \qquad A, B \in \mathbb{R}$$

Also

$$y = x'$$
$$= 2Ae^{2t} - 3Be^{-3t}$$

the gen sol to * is

$$x = Ae^{2t} + Be^{-3t} \qquad A, B \in \mathbb{R}$$
$$y = 2Ae^{2t} - 3Be^{-3t}$$

$\square$

Ex An ant runs around on a table top (the x-y plane)

After t seconds its location is

$(x(t), y(t))$

Assume

$$x' = -2y$$
$$y' = 2x$$

Describe the ant's motion.

Sol Elim y

$$x'' = -2y'$$
$$= -4x$$

$$x'' + 4x = 0$$

$$r^2 + 4 = 0$$
$$r = 2i, -2i$$

$$x = A\cos 2t + B\sin 2t \qquad A, B \in \mathbb{R}$$
$$y = \frac{-1}{2}x' = A\sin 2t - B\cos 2t$$

the ant moves along some curve in $\mathbb{R}^2$. To find the curve, elim t

$$A\cos 2t + B\sin 2t = x$$

$$-B\cos 2t + A\sin 2t = y$$

View $\cos 2t$, $\sin 2t$ as unknowns

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{A^2+B^2}\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\cos 2t = \frac{Ax - By}{A^2+B^2}$$

$$\sin 2t = \frac{Ay + Bx}{A^2+B^2}$$

For all t

$$1 = \cos^2(2t) + \sin^2(2t)$$

$$= \frac{(Ax - By)^2 + (Ay + Bx)^2}{(A^2+B^2)^2}$$

$$= \frac{x^2+y^2}{A^2+B^2}$$

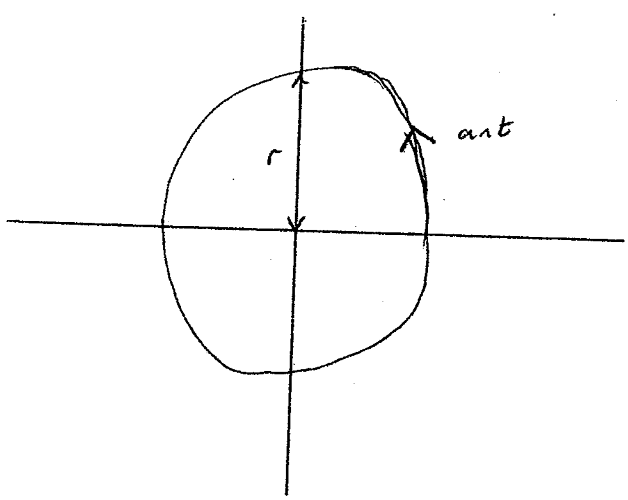
Write

$$r^2 = A^2 + B^2 \qquad 0 \le r \in \mathbb{R}$$

so

$$x^2+y^2 = r^2$$

Ant moves around a circle of radius r



At each time t

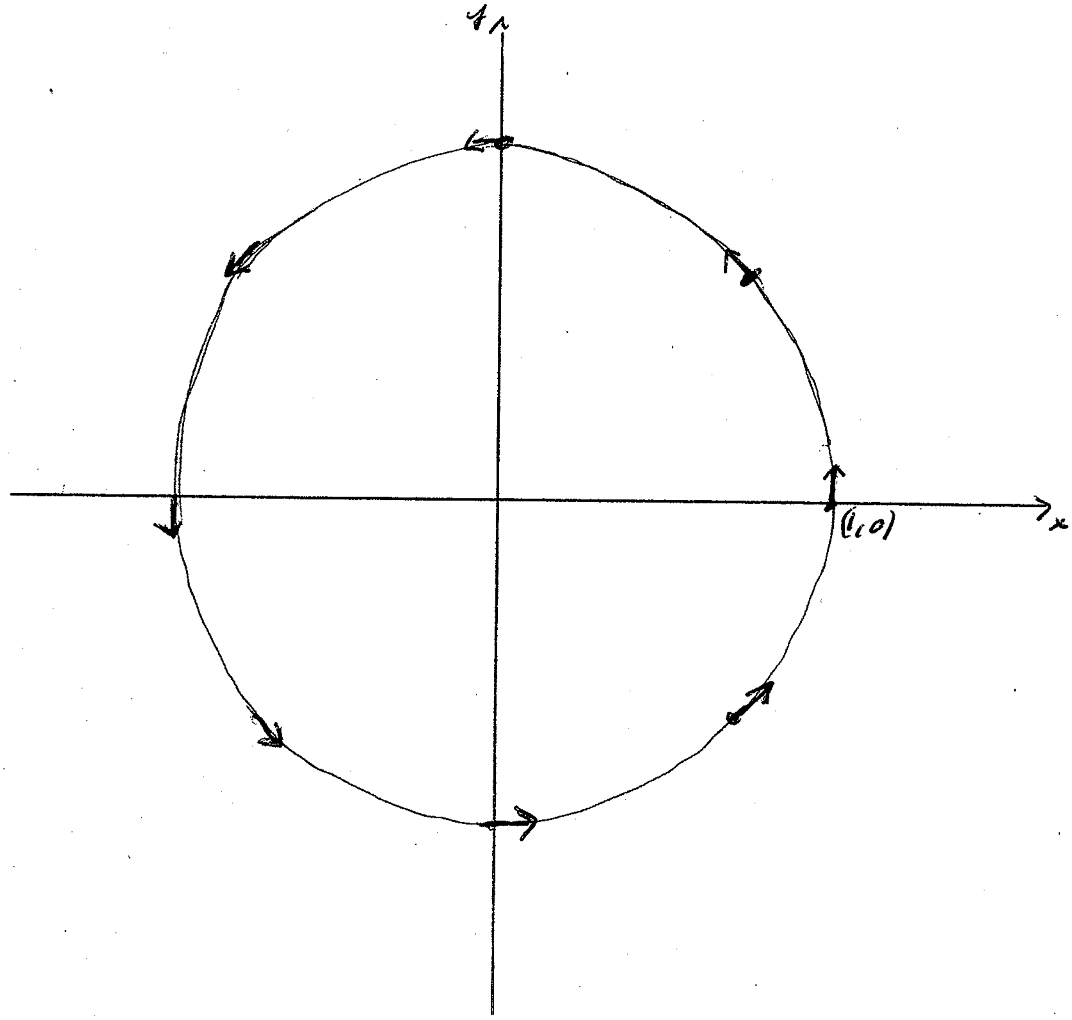
ant's position vector is $(x,y) = (x(t), y(t))$

ant's velocity vector is $(x',y') = (x'(t), y'(t))$

At each point (x,y) imagine an arrow that describes the velocity vector

"direction field"

position		velocity	
x	y	x'	y'
1	0	0	2
0	1	-2	0
-1	0	0	-2
0	-1	2	0
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\sqrt{2}$	$\sqrt{2}$
$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\sqrt{2}$	$-\sqrt{2}$
$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\sqrt{2}$	$-\sqrt{2}$
$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\sqrt{2}$	$\sqrt{2}$



ant travels c.clockwise with constant speed

Reduction to 1st order systems

Ex Consider the differential equation

$$x''' - 3x'' + 3x' - x = \sin(t)$$

Express this as a 1st order system.

Sol Introduce new variables

$$y = x'$$
$$z = x''$$

then

$$x' = y$$
$$y' = z$$
$$z' = x'''$$
$$= \sin(t) + 3\underset{z}{\underset{\shortparallel}{x''}} - 3\underset{y}{\underset{\shortparallel}{x'}} + x$$
$$= x - 3y + 3z + \sin(t)$$

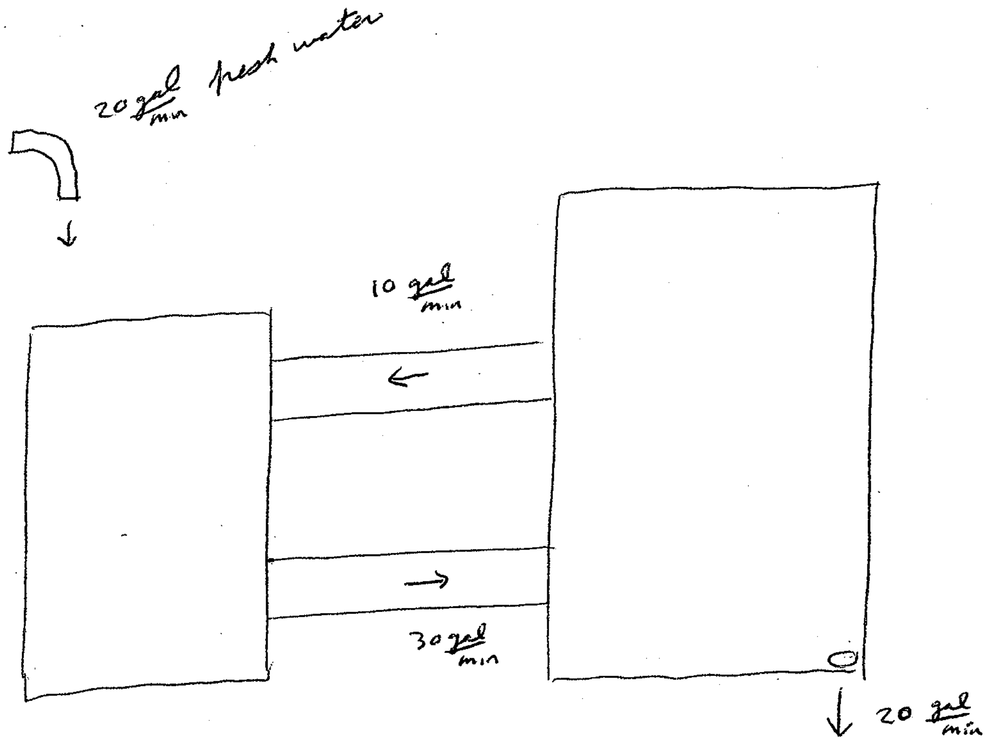
1st order system is

$$x' = y$$
$$y' = z$$
$$z' = x - 3y + 3z + \sin(t)$$

□

An application in which a 1st order system comes up naturally

Ex Consider two tanks of brine as shown



gal brine:	100	200
# Lb salt after t min:	$x(t)$	$y(t)$

Find the differential equations satisfied by x, y

Sol Describe $x'(t)$

After t minutes and a short time interval Δt

$$x(t+\Delta t) = x(t) + \Delta t \; 10 \; \frac{y(t)}{200}$$

Lb salt Lb salt min $\frac{\text{gal}}{\text{min}}$ $\frac{\text{Lb salt}}{\text{gal}}$

$$- \Delta t \; 30 \; \frac{x(t)}{100}$$

min $\frac{\text{gal}}{\text{min}}$ $\frac{\text{Lb salt}}{\text{gal}}$

So

$$x'(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

$$= \frac{y(t)}{20} - \frac{3\,x(t)}{10}$$

So

$$x' = \frac{-3}{10} x + \frac{1}{20} y$$

Describe $y'(t)$

After t minutes and a short time interval Δt,

$$y(t+\Delta t) = y(t) + \Delta t \quad 30 \quad \frac{x(t)}{100}$$

Lb salt Lb salt min $\frac{\text{gal}}{\text{min}}$ $\frac{\text{Lb salt}}{\text{gal}}$

$$- \Delta t \quad 10 \quad \frac{y(t)}{200}$$

min $\frac{\text{gal}}{\text{min}}$ $\frac{\text{Lb salt}}{\text{gal}}$

$$- \Delta t \quad 20 \quad \frac{y(t)}{200}$$

min $\frac{\text{gal}}{\text{min}}$ $\frac{\text{Lb salt}}{\text{gal}}$

$$y'(t) = \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

$$= \frac{3}{10} x(t) - \frac{3}{20} y(t)$$

So

$$y' = \frac{3}{10} x - \frac{3}{20} y$$

So $x = x(t)$, $y = y(t)$ satisfy the 1st order system

$$x' = -\frac{3}{10} x + \frac{1}{20} y$$

$$y' = \frac{3}{10} x - \frac{3}{20} y$$

$\square$

For $n \geq 1$

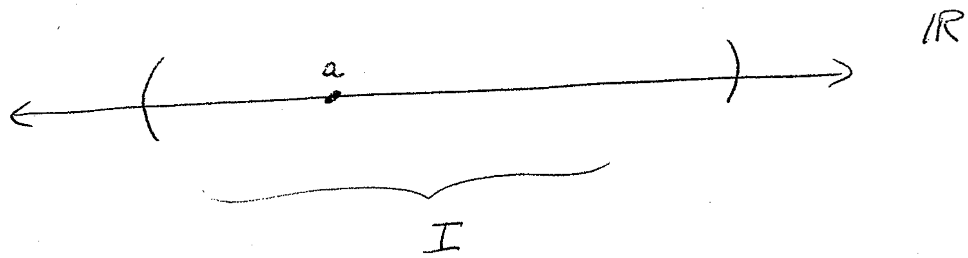
consider the general 1st order linear system involving functions

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad \ldots, \quad x_n = x_n(t)$$

Has form

$$\left.\begin{aligned}
x_1' &= p_{11}(t)\,x_1 + p_{12}(t)\,x_2 + \cdots + p_{1n}(t)\,x_n + f_1(t) \\
x_2' &= p_{21}(t)\,x_1 + p_{22}(t)\,x_2 + \cdots \quad p_{2n}(t)\,x_n + f_2(t) \\
&\ldots \\
x_n' &= p_{n1}(t)\,x_1 + \cdots \quad p_{nn}(t)\,x_n + f_n(t)
\end{aligned}\right\} \; *$$

<u>thm</u> Given an open interval I on the real line that contains a pt a



Given functions

$p_{ij}(t)$ $\quad 1 \le i, j \le n$

$f_i(t)$ $\quad 1 \le i \le n$

that are continuous at each point in I

Then for any scalars $b_1, b_2, \ldots, b_n$ in $\mathbb{R}$

the system $*$ has a unique solution

$x_1, x_2, \ldots, x_n$

defined on I such that

$x_1(a) = b_1, \quad x_2(a) = b_2, \quad \ldots, \quad x_n(a) = b_n$

$\square$

Ex Solve the system

$$x' = -y \qquad x(0) = 2$$
$$y' = 10x - 7y \qquad y(0) = -7$$

Sol First find gen sol

$$x'' = -y' = -10x + 7y$$
$$= -10x - 7x'$$

$$x'' + 7x' + 10x = 0$$

char eq
$$r^2 + 7r + 10 = 0$$
$$(r+2)(r+5) = 0$$
$$r = -2, -5$$

$$x = Ae^{-2t} + Be^{-5t}$$
$$y = -x' = 2Ae^{-2t} + 5Be^{-5t}$$

To find A, B set $t = 0$:

$$2 = A + B$$
$$-7 = 2A + 5B$$

solve for A, B to get

$$A = \frac{17}{3} \qquad B = \frac{-11}{3}$$

$\square$