Lec 31 Friday April 11

6.3 (loose ends)

Stochastic models

A metro area has a city and several suburbs

Assume the metro area has constant total population P

Let

C_k = fraction of total pop living in city after k years $\quad k=0,1,2\ldots$

S_k = ... suburbs ...

So $\quad C_k + S_k = 1 \qquad k \geq 0$

Assume there exist pos real numbers

$a, b, c, d \qquad$ (indep of k)

such that

$$C_{k+1} = a C_k + b S_k \qquad k \geq 0 \qquad *$$

$$S_{k+1} = c C_k + d S_k$$

Using $C_k + S_k = 1$

$a + c = 1, \qquad b + d = 1$

Determine what happens to C_k, S_k as $k \to \infty$

Write * as

$$\begin{bmatrix} C_{k+1} \\ S_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C_k \\ S_k \end{bmatrix}}_{X_k} \qquad k \geq 0$$

$$X_{k+1} = A X_k \qquad k \geq 0$$

So

$$X_k = A^k X_0 \qquad k \geq 0$$

Consider

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$a+c=1 \qquad b+d=1$

Each column sum is 1

def an $n \times n$ matrix is <u>stochastic</u> provided

- each entry is nonnegative
- each column sum is 1

matrix A is stochastic

Write

$$A = \begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix} \qquad 0 < p, q < 1$$

Find eigenvalues of A:

$$|A - \lambda I| = \begin{vmatrix} p-\lambda & 1-q \\ 1-p & q-\lambda \end{vmatrix}$$

$$= (p-\lambda)(q-\lambda) - (1-p)(1-q)$$

$$= pq - p\lambda - q\lambda + \lambda^2 - 1 + p + q - pq$$

$$= \lambda^2 - (p+q)\lambda + p+q-1$$

$$= (\lambda - 1)(\lambda + 1 - p - q)$$

Eigenvalues for A are

$$1, \quad p+q-1$$

obs

$$-1 < p+q-1 < 1 \qquad \text{Since } 0 < p, q < 1$$

obs $p+q-1 \neq 1$ so A is diagonalizable

Define matrix

$$D = \begin{bmatrix} 1 & 0 \\ 0 & p+q-1 \end{bmatrix}$$

There exists an invertible matrix P such that

$$A = P D P^{-1}$$

So

$$A^k = P D^k P^{-1} \qquad k \geq 0$$

$$\underline{X}_k = P D^k P^{-1} \underline{X}_0$$

$$= P \begin{bmatrix} 1 & 0 \\ 0 & (p+q-1)^k \end{bmatrix} P^{-1} \underline{X}_0$$

Find $\lim_{k \to \infty} \underline{X}_k$

$$\lim_{k \to \infty} (p+q-1)^k = 0 \qquad \text{since} \quad -1 < p+q-1 < 1$$

So

$$\lim_{k \to \infty} \underline{X}_k = \underbrace{P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \underline{X}_0}_{\text{call this } \underline{X}}$$

Write

$$\underline{X} = \begin{bmatrix} c \\ s \end{bmatrix}$$

Find c, s:

Since

$$C_k + S_k = 1 \qquad k \geq 0$$

have

$$C + S = 1$$

Also

$$A \underline{X} = \underline{X}$$

So $\underline{X}$ is eigvector for A with eigval 1

$$(A - I)\, \underline{X} = 0$$

$$A - I = \begin{bmatrix} p-1 & 1-q \\ 1-p & q-1 \end{bmatrix}$$

$$\downarrow \text{GJ}$$

$$\begin{bmatrix} 1 & \frac{1-q}{p-1} \\ 0 & 0 \end{bmatrix}$$

Require

$$C + \frac{1-q}{p-1} S = 0 \qquad C + S = 1$$

solve for C, S:

$$C = \frac{1-q}{2-p-q} \qquad S = \frac{1-p}{2-p-q}$$

So in the long run,

	Population	
city	$\frac{1-q}{2-p-q}$	P
suburb	$\frac{1-p}{2-p-q}$	P

<u>Ex</u> With above notation

assume

$$A = \begin{bmatrix} .85 & .05 \\ .15 & .95 \end{bmatrix}$$

so

$$p = .85 \qquad 1-p = .15$$

$$q = .95 \qquad 1-q = .05$$

$$\frac{1-q}{2-p-q} = \frac{.05}{.05+.15}$$

$$= \frac{5}{5+15}$$

$$= \frac{5}{20}$$

$$= \frac{1}{4}$$

$$\frac{1-q}{2-p-q} = \frac{3}{4}$$

So in the long run

	Population
city	$\frac{1}{4}P$
suburbs	$\frac{3}{4}P$

Predator/Prey models

In a wilderness area

F_k = # foxes after k months $\qquad k = 0, 1, 2, \ldots$

R_k = # rabbits after k months

F_k, R_k are related by

More rabbits now $\rightarrow$ more foxes later

More foxes now $\rightarrow$ fewer rabbits later

Also

reproduction / disease

According to the predator/prey model

there exist scalars $a, b, c, d \in \mathbb{R}$ (indep of k)

such that

$$F_{k+1} = a F_k + b R_k \qquad k = 0, 1, 2, \ldots \qquad *$$

$$R_{k+1} = c F_k + d R_k$$

Expect

$0 < a < 1$ $\qquad$ $b > 0$

$c < 0$ $\qquad$ $d > 1$

Determine what happens to F_k, R_k in the limit $k \to \infty$

Write $\ast$ as

$$\begin{bmatrix} F_{k+1} \\ R_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{= A} \underbrace{\begin{bmatrix} F_k \\ R_k \end{bmatrix}}_{= \underline{X}_k} \qquad k \geq 0$$

$$\underline{X}_{k+1} = A\, \underline{X}_k \qquad k \geq 0$$

so

$$\underline{X}_k = A^k\, \underline{X}_0 \qquad k \geq 0$$

Assume A is diagonalizable with eigenvalues

$$\lambda_1, \lambda_2 \in \mathbb{R}$$

Write

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

there exists an invertible matrix P such that

$$A = P D P^{-1}$$

So

$$A^k = P D^k P^{-1} \qquad k \geq 0$$

So

$$X_k = P \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} P^{-1} X_0 \qquad k \geq 0$$

Multiply out RHS to find

General form of F_k, R_k is

$$F_k = r\lambda_1^k + s\lambda_2^k \qquad k \geq 0$$

$$R_k = t\lambda_1^k + u\lambda_2^k$$

where r, s, t, u are non 0 scalars in $\mathbb{R}$ that are indep of k

Note for $x \in \mathbb{R}$

$$\lim_{k \to \infty} x^k = \begin{cases} \infty & \text{if } |x| > 1 \\ 1 & \text{if } x = 1 \\ \text{no lim} & \text{if } x = -1 \\ 0 & \text{if } |x| < 1 \end{cases}$$

<u>Case</u> $\lambda_1 = 1$ and $|\lambda_2| < 1$

$$\lim_{k \to \infty} F_k = r$$

$$\lim_{k \to \infty} R_k = t$$

"stable limiting populations"

<u>Case</u> $|\lambda_1| < 1$ and $|\lambda_2| < 1$

$$\lim_{k \to \infty} F_k = 0$$

$$\lim_{k \to \infty} R_k = 0$$

"mutual extinction"

<u>Case</u> $|\lambda_1| > 1$ or $|\lambda_2| > 1$

$$\lim_{k \to \infty} F_k = \infty$$

$$\lim_{k \to \infty} R_k = \infty$$

"mutual population explosion"