LEC 30 Wednesday April 9 4/9/14

6.3 Applications involving powers of matrices

Given diagonalizable $n \times n$ matrix A

Eigenvalues

$\lambda_1, \lambda_2, \ldots, \lambda_n$ (not nec distinct)

write

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Recall there exists an invertible matrix P such that

$$A = P D P^{-1}$$

(col i of P is eigenvector for A and eigenvalue λ_i)

LEM

$$A^k = P D^k P^{-1} \qquad k = 0, 1, 2, \ldots$$

pf

$$A^k = \underbrace{AA\cdots A}_{k}$$

$$= (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})$$

$$= P\,D\,\underbrace{P^{-1}P}_{I}\,D\,\underbrace{P^{-1}P}_{I}\,D \cdots\cdots DP^{-1}$$

$$= P\,D^k\,P^{-1}$$

$\square$

Ex For $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$

Find entries of A^{100}

Sol Find eigenvalues of A

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(-1-\lambda) - (-3)(2)$$

$$= -4 - 3\lambda + \lambda^2 + 6$$

$$= \lambda^2 - 3\lambda + 2$$

$$= (\lambda - 2)(\lambda - 1)$$

Eigenvalues are

$2, 1$

Define

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Find P:

λ	2	1
$A-\lambda I$	$\begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix}$	$\begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}$
solve $(A-\lambda I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by GJ	$\begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}$ $x - 3/2\, y = 0$ $y = t$ free $x = \frac{3}{2}t$ $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x - y = 0$ $y = t$ free $x = t$ $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
eigenvector for λ	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

By construction

$$AP = PD$$

$$A = PDP^{-1}$$

<u>Find P^{-1}</u>

$$P^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{3-2} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

For $k \geq 0$

$$A^k = P D^k P^{-1}$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot 2^k - 2 & 3 - 3 \cdot 2^k \\ 2^{k+1} - 2 & 3 - 2^{k+1} \end{bmatrix}$$

Set $k = 100$ to get ans $\square$

Ex For $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

find the entries of A^k for all $k \geq 0$

Sol First try small k

n	A^k
0	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
1	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$
2	$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
3	$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$
4	$\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$
5	$\begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}$
6	$\begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix}$
$\vdots$	$\vdots$

For $k \geq 0$ define

$$s_n = (1,1)\text{-entry of } A^k$$

k	0	1	2	3	4	5	6	...
s_k	1	1	2	3	5	8	13	...

obs

$$s_{k+1} = s_k + s_{k-1} \qquad k \geq 1$$

" Fibonacci sequence ".

$$A^k = \begin{bmatrix} s_k & s_{k-1} \\ s_{k-1} & s_{k-2} \end{bmatrix} \qquad k \geq 1$$

$(s_{-1} = 0)$

<u>Find s_k in closed form</u>

Find eigenvalues of A:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 1$$

$$= \lambda^2 - \lambda - 1$$

Solve $\lambda^2 - \lambda - 1 = 0$

$$\lambda = \frac{1 \pm \sqrt{5}}{2} \qquad \text{(quadratic formula)}$$

Eigenvalues of A are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

$\lambda_1 \neq \lambda_2$ so A is diagonalizable

Define

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

There exists an invertible matrix P such that

$$A = P D P^{-1}$$

Write

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

For $k \geq 0$

$$A^k = P D^k P^{-1}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{ad-bc}$$

$$s_k = (1,1)\text{-entry of } A^k$$

$$= r\lambda_1^k + s\lambda_2^k$$

where $r = \dfrac{ad}{ad-bc}$ $\quad s = \dfrac{-bc}{ad-bc}$

Solve for r, s using init conditions

$s_0 = 1, \quad s_1 = 1$

Recall

$$r\lambda_1^k + s\lambda_2^k = s_k \qquad *$$

$k=0$: $\quad r + s = 1$

$k=1$: $\quad \lambda_1 r + \lambda_2 s = 1$

Solve this linear system for r, s to get

$$r = \frac{\lambda_2 - 1}{\lambda_2 - \lambda_1} \qquad s = \frac{\lambda_1 - 1}{\lambda_1 - \lambda_2}$$

Evaluate $*$ using this and

$\lambda_1 + \lambda_2 = 1, \quad \lambda_1\lambda_2 = -1$

to get

$$s_k = \frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\sqrt{5}}$$

so

$$s_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] \qquad k = 0, 1, 2, \ldots$$

Next goal — Cayley-Hamilton thm

Given diagonalizable $n \times n$ matrix A

eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not nec dist)

Write

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

there exists an invert matrix P s.t.

$$A = P D P^{-1}$$

LEM With above notation the following are equivalent for all poly $f(\lambda)$:

(i) $f(A) = 0$

(ii) $f(D) = 0$

(iii) $f(\lambda_i) = 0$ for $1 \le i \le n$

pf $(i) \leftrightarrow (ii)$

Recall $A^k = P D^k P^{-1}$ for $k \geq 0$

$(ii) \leftrightarrow (iii)$

Obs

$$D^k = \begin{pmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{pmatrix}$$ for $k \geq 0$

$\square$

<u>thm</u> (Cayley - Hamilton)

Given $n \times n$ matrix A

Eigenvalues

$\lambda_1, \lambda_2, \dots, \lambda_n$

char poly of A:

$$|A - \lambda I| = (-1)^n \underbrace{(\lambda-\lambda_1)(\lambda-\lambda_2) \cdots (\lambda-\lambda_n)}_{f(\lambda)}$$

then $f(A) = 0$

pf (For simplicity assume A is diagonalizable)

By def of char poly

$f(\lambda_i) = 0 \qquad 1 \le i \le n$

Now $f(A) = 0$ by prev LEM. $\square$

Ex let A = 3×3 matrix with eigenvalues

$1, 2, 3$

Find A^{-1} as a polynomial in A

Sol define poly

$$f(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3)$$

$$= \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

obs

$$0 = f(A)$$

$$= A^3 - 6A^2 + 11A - 6I$$

$$A^3 - 6A^2 + 11A = 6I$$

$$A \underbrace{\frac{A^2 - 6A + 11I}{6}}_{\text{must be } A^{-1}} = I$$

$$A^{-1} = \frac{A^2 - 6A + 11I}{6}$$

$\square$