Lecture 2 Friday Jan 24 1/24/14

1.2 Integrals as general and particular solutions

Consider the differential equations of the form

$$\frac{dy}{dx} = f(x) \qquad \left(\begin{array}{c}\text{slope depends} \\ \text{only on } x\end{array}\right)$$

From Calculus, the general solution is the *indefinite integral*

$$y = \int f(x)\,dx$$

Ex For the problem

$$\frac{dy}{dx} = 1 - x \qquad *$$

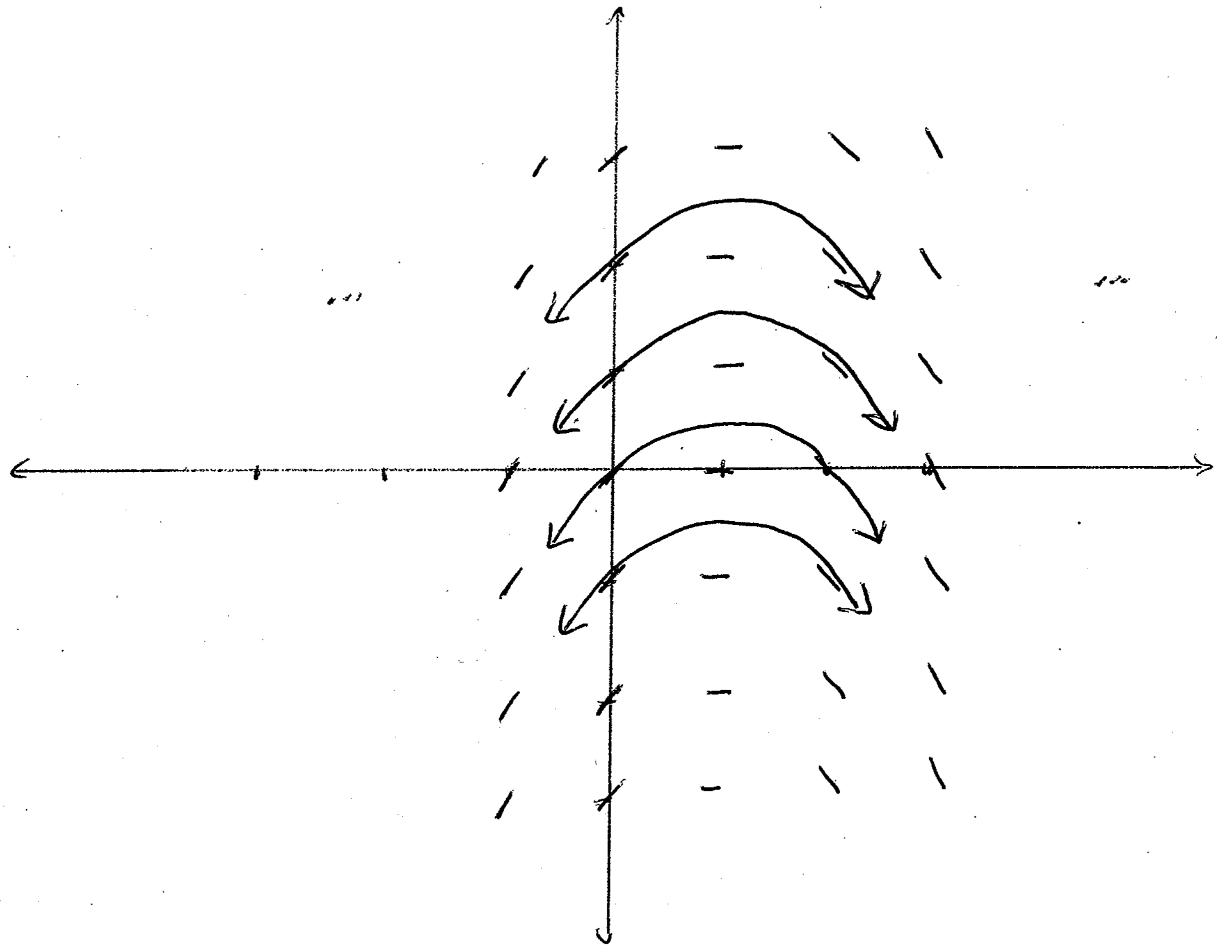
the gen sol is

$$y = \int (1-x)\,dx = x - \frac{x^2}{2} + C \qquad C \in \mathbb{R}$$

Each value of C gives particular sol,

Geometric view

For the equation (*) the slope field is



Each curve is a particular sol

Consider a <u>2nd order</u> diff equation

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$$\frac{d^2y}{dx^2} = g(x)$$

To get the general solution, integrate $g(x)$ twice.

detail: define a function $z = z(x)$ by

$$z = \frac{dy}{dx}$$

Now $$\frac{dz}{dx} = \frac{d^2y}{dx^2} = g(x)$$

So $$z = \int g(x)\,dx$$ (1st integral)

But $$\frac{dy}{dx} = z$$

So $$y = \int z(x)\,dx$$ (2nd integral)

Prob Find the general solution to

$$y'' = \sin x + 1$$

Sol Recall from calc:

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

Obs

$$\int (\sin x + 1)\, dx = -\cos x + x + C \qquad C = \text{const}$$

$$\int (-\cos x + x + C)\, dx = -\sin x + \frac{x^2}{2} + Cx + k \qquad k = \text{const}$$

Gen sol is

$$y = -\sin x + \frac{x^2}{2} + Cx + k \qquad C, k \in \mathbb{R}$$

"2-parameter" family

Each value of C and k gives a particular sol

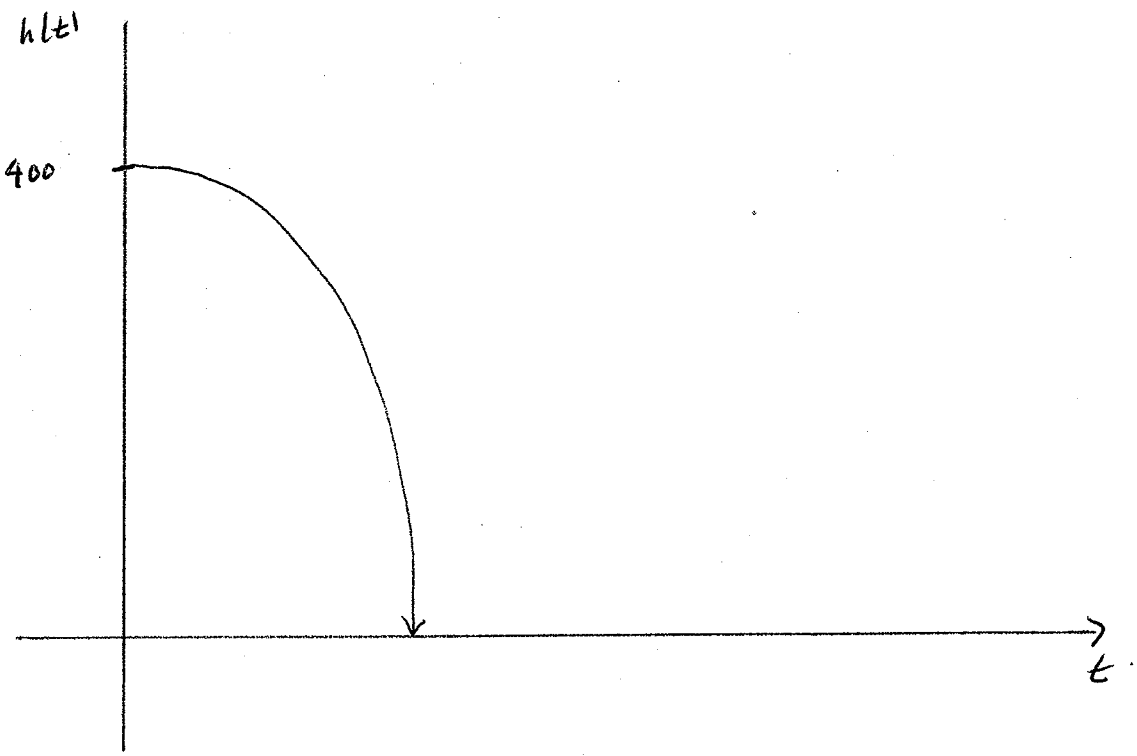
<u>Prob</u> A ball is dropped from the top of a building 400 ft high.

(i) How long does it take to reach the ground?

(ii) With what speed does the ball strike the ground?

<u>Sol</u> Define a function

$h(t)$ = height (in ft) of ball at time t (seconds) after drop



Obs.

- At each time t

$$h'(t) = \text{velocity of ball} \quad (\text{in ft/sec})$$

(call it $v(t)$)

$$\text{speed of ball} = |v(t)| \quad (\text{absolute value})$$

- Ball speeds up over time due to force of gravity

- At time t

$$\text{rate of change of } v(t) = v'(t) = \text{acceleration} \quad (\text{in ft/sec}^2)$$

(call it $a(t)$)

- From physics, near the surface of the earth

acceleration due to gravity is a constant

$$32 \text{ ft/sec}^2$$

[or 9.8 meters/sec^2]

So $a(t) = -32$

↑ since force of gravity is against $h(t)$

Our problem becomes

$$h'' = -32,$$

$$h(0) = 400; \qquad h'(0) = 0$$

We get

$$v(t) = \int -32 \, dt$$

$$= -32t + C$$

Find C:

$$0 = h'(0) = v(0) = C$$

$$v(t) = -32t$$

Now

$$h(t) = \int -32t \, dt$$

$$= -16t^2 + k$$

Find k:

At $t = 0$,

$$400 = h(0) = k$$

$$h(t) = 400 - 16t^2$$

We seek time t s.t. $h(t) = 0$:

$$0 = 400 - 16t^2$$

$$t^2 = \frac{400}{16} = 25 \qquad t \geq 0$$

so $t = 5$

• takes 5 seconds for the ball to reach the ground.

At $t = 5$

$$v(5) = -32 \times 5$$
$$= -160$$

• Ball strikes ground at 160 ft/sec □

Prob A car traveling at 60 mi/hr (88 ft/sec) skids 176 ft after its brakes are suddenly applied.

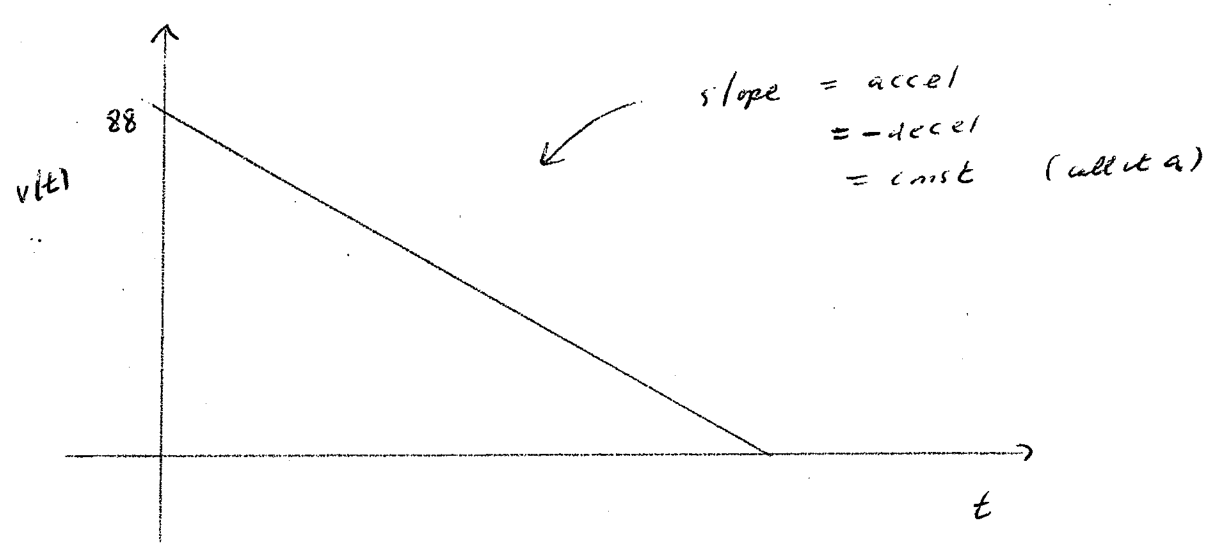
Assume that the braking system provides constant deceleration.

(i) What is the deceleration?

(ii) For how long did the skid continue?

Sol. Define

$v(t)$ = velocity of car t seconds after brakes are applied (in ft/sec)



$$v(t) = 88 + at$$

when car comes to rest at time T;

$$0 = v(T) = 88 + aT$$

so

$$T = -\frac{88}{a}$$

Find a:

Define $x(t)$ = position of car at time t

[WLOG $x(0) = 0$]

obs

$$x'(t) = v(t)$$

So

$$x(t) = \int v(t)\,dt$$

$$= \int (88 + at)\,dt$$

$$= 88t + a\frac{t^2}{2} + C$$

Set $t=0$ to find $C = 0$

$$x(t) = 88t + \frac{at^2}{2}$$

$$= t\left(88 + \frac{at}{2}\right)$$

Car skids 176 ft so

$176 = x(T)$ $\quad T = -\frac{88}{a}$

$\overset{\shortparallel}{88 \times 2}$

$= T\left(88 + \frac{aT}{2}\right)$

$= -\frac{88}{a}\left(88 - \frac{a}{2}\frac{88}{a}\right)$

$= \frac{-88}{a} 44$

$2 = -\frac{44}{a}$

$a = \frac{-44}{2}$

$= -22$

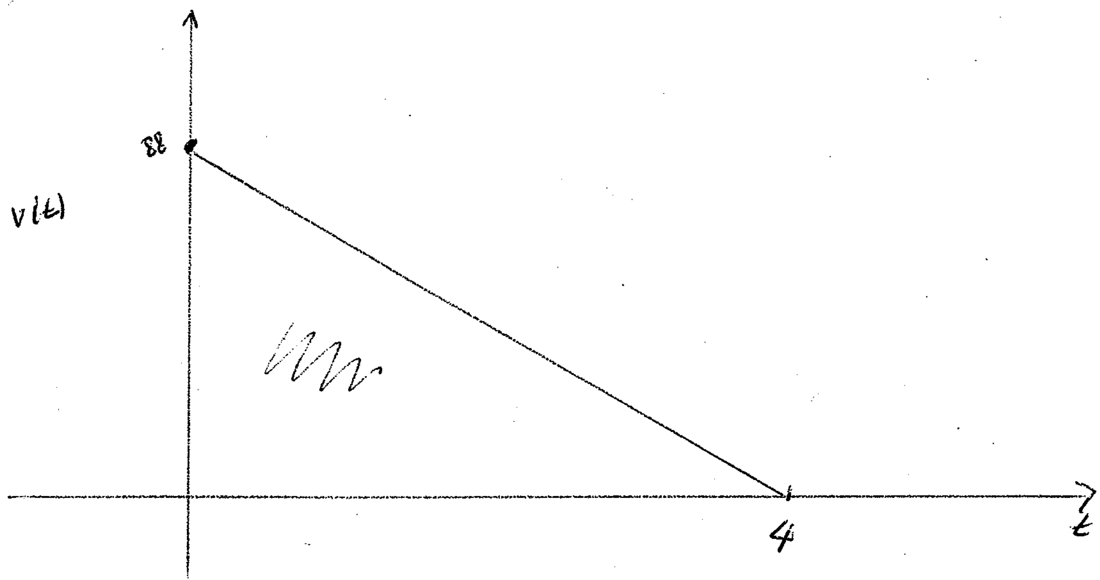
accel is $-22\ ft/sec^2$

decel is $22\ ft/sec^2$

$T = \frac{-88}{a} = \frac{-88}{-22} = 4$

skid continues for 4 seconds

Geometric interp



176 = Length of skid

= distance the car travels from $t=0$ to $t=4$

$$= \int_0^4 v(t)\,dt$$

= area under curve (shaded)

$$= \frac{1}{2} \times 4 \times 88$$

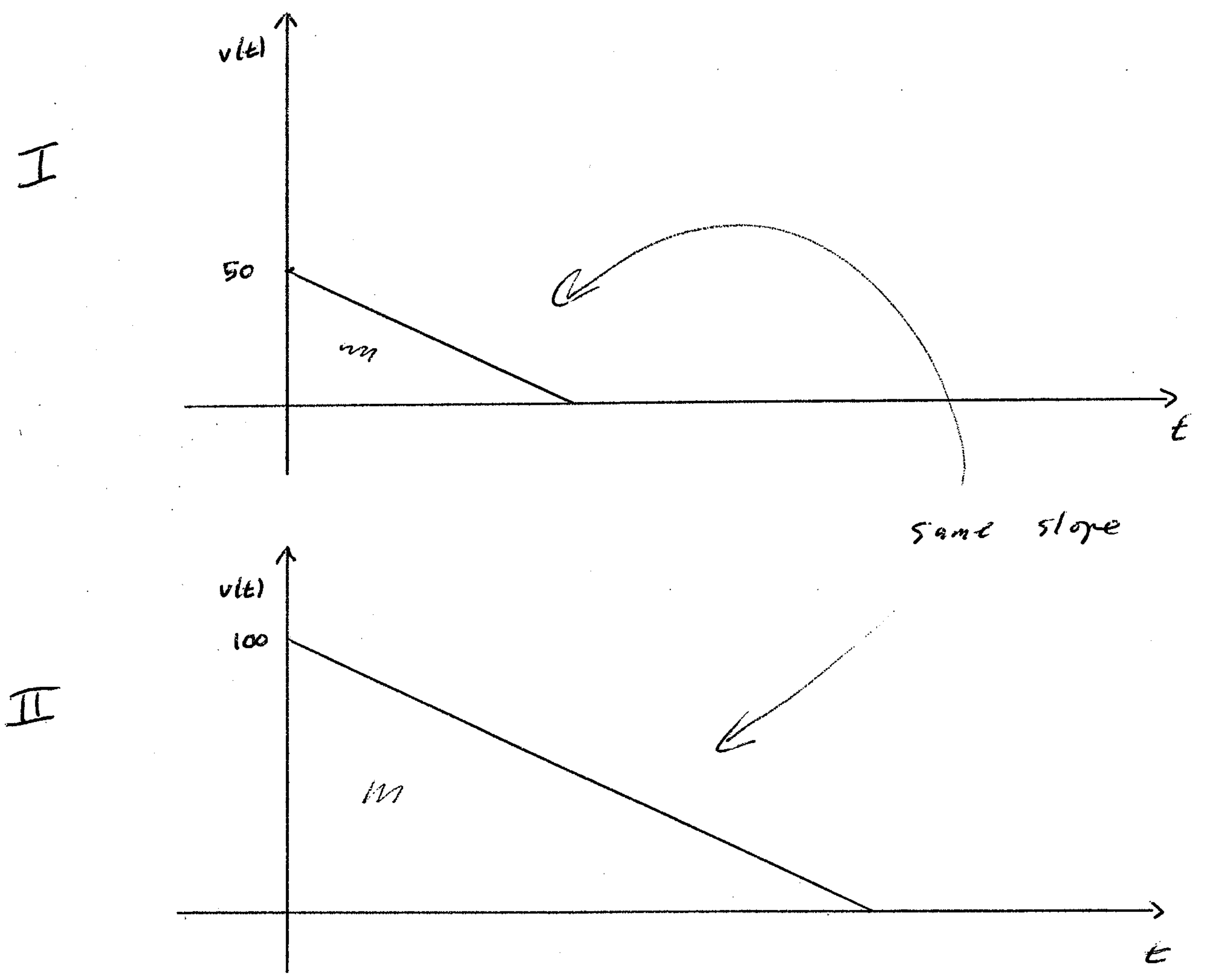
$$= 2 \times 88$$

✓

□

Prob Suppose that a car skids 15 meters (m) if it is moving at 50 km/h when brakes are applied.
Assuming that the car has the same constant decel, how far will it skid if it is moving at 100 km/h when the brakes are applied?

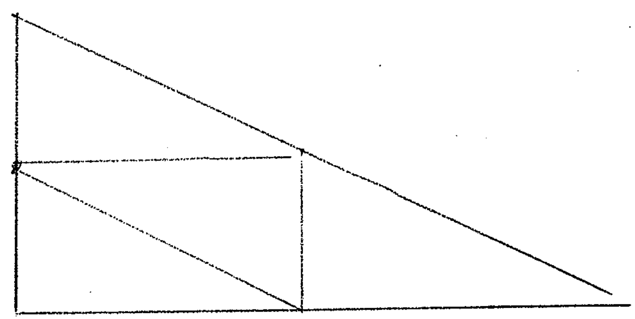
Sol Define $v(t)$ = velocity at time t after brakes applied (in km/h)



For both Cases I, II

Length of skid = distance car travels from time $t=0$ until it stops

= area under curve (shaded)



In Case I, area = 15 meters

In Case II, area is $15 \times 4 = 60$ meters

Note

In general, if we increase the speed by a factor of k then the skid length increases by a factor of k^2.