LEC 29 Monday April 7 4/7/14

1

6.2 Diagonalization of matrices

Given an $n \times n$ matrix A

<u>Todays goal</u>: Find a diagonal $n \times n$ matrix D

and an invertible $n \times n$ matrix P such that

$$A = PDP^{-1}$$

<u>Ex</u> $n=2$ Fa $A = \begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}$

Recall:

eigenvalue λ	2	4
eigenvector for λ	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Define a diagonal matrix D whose diagonal entries are the eigenvalues of A (in any order)

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

Define a matrix P whose ith column is an eigenvector for A with eigenvalue the ith diagonal entry of D

So col 1 of P = eigenvector for A with eigenvalue 2 $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

col 2 of P = eigenvector for A with eigenvalue 4 $= \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$$

Note $\det(P) = 3 - 4 = -1 \neq 0$

so P is invertible

One checks

$$AP = PD$$

$$\begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 16 \\ 2 & 12 \end{bmatrix} \qquad \begin{bmatrix} 2 & 16 \\ 2 & 12 \end{bmatrix}$$

✓

Now

$$AP = PD$$

$$APP^{-1} = PDP^{-1}$$

$$A = PDP^{-1}$$

✓

the geometric meaning of P, D

Given an $n \times n$ matrix A

Given diagonal matrix D and invertible matrix P such that

$$A = PDP^{-1}$$

So

$$AP = PD$$

Write

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

For $1 \le i \le n$, compare col i of AP, PD

$$\text{col } i \text{ of } AP = A(\text{col } i \text{ of } P)$$

$$\text{col } i \text{ of } PD = P(\text{col } i \text{ of } D)$$

$$= P \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{coord } i$$

$$= \lambda_i(\text{col } i \text{ of } P)$$

compare

Col i of P is an eigenvector for A with eigenvalue λ_i

Note the cols of P are lin indep since P is invertible

<u>So</u>

- the columns of P form n lin indep eigenvectors for A
- the diagonal entries of D are the corresponding eigenvalues

How to find P, D

Given $n \times n$ matrix A

- Find n lin indep eigenvectors for A:

$$v_1, v_2, \ldots, v_n$$

- For $1 \le i \le n$ let λ_i = eigenvalue of A for v_i

- Define an $n \times n$ matrix P such that for $1 \le i \le n$,

$$\text{column } i \text{ of } P = v_i$$

Note P is invertible

- Define an $n \times n$ diag matrix D by

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

By construction

$$AP = PD$$

So

$$A = PDP^{-1}$$

Caution Given $n \times n$ matrix A

A might not have n lin indep eigenvectors.

In this case the above matrices P, D do not exist.

Thm Given $n \times n$ matrix A

the following are equivalent

(i) A has n lin indep eigenvectors

(ii) there exists a diag matrix D and invertible matrix P
such that $A = PDP^{-1}$

Ex For $A = \begin{bmatrix} 4 & -3 & 1 \\ 2 & 7 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$

Sol Recall from prev lecture

eigenvalues of A	2	1
basis for eigenspace of λ	$\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Define $P = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$ $\det P \neq 0$ so P invertible

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then $AP = PD$ so $A = PDP^{-1}$ $\square$

Ex For $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

Find a diag matrix D and an invertible matrix P such that $A = PDP^{-1}$

Sol. Find eigvalues of A

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2$$

$$\lambda = 2, 2$$

Find eigenvectors for $\lambda = 2$

$$A - 2I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Solve

$$(A - 2I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y = 0$$

$x = t$ free

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is basis for the eigenspace for $\lambda = 2$

A does not have two lin indep eigenvectors: P does not exist $\square$

Def Given $n \times n$ matrices A, B

Call A, B <u>similar</u> whenever there exists an invertible matrix P such that

$$B = P^{-1} A P$$

Def Given $n \times n$ matrix A

call A <u>diagonalizable</u> whenever A is similar to a diagonal matrix

Ex

diagonalizable	not diagonalizable
$\begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

Given $n \times n$ matrix A

Factor characteristic polynomial of A:

$$|A - \lambda I| = (-1)^n (\lambda - \lambda_1)^{e_1} (\lambda - \lambda_2)^{e_2} \cdots (\lambda - \lambda_k)^{e_k}$$

$\lambda_1, \lambda_2, \ldots, \lambda_k$ mutually distinct (the eigenvalues of A)

$e_1, e_2, \ldots, e_k$ are pos integers such that

$$e_1 + e_2 + \cdots + e_k = n$$

For $1 \le i \le k$ let V_i = eigenspace of A for λ_i. Turns out $\dim V_i \le e_i$

Let S_i = basis for V_i

Define

$$S = S_1 \cup S_2 \cup \cdots \cup S_k$$

size of $S \le e_1 + e_2 + \cdots + e_k = n$

To show

- Vectors in S are lin indep
- A is diagonalizable if and only if S has size n

these follow from next LEMMA

LEM With above notation, pick

$0 \neq w_i \in V_i$ $(1 \le i \le k)$

then $w_1, w_2, \ldots, w_k$ are lin indep

pf Given scalars $c_1, c_2, \ldots, c_k$ such that

$$c_1 w_1 + c_2 w_2 + \cdots + c_k w_k = 0 \qquad *$$

show $c_i = 0$ for $1 \le i \le k$

For $0 \le j \le k-1$ apply A^j to $*$

$$A^j (c_1 w_1 + \cdots + c_k w_k) = A^j 0 = 0$$

$$= c_1 \underbrace{A^j w_1}_{= \lambda_1^j w_1} + \cdots + c_k \underbrace{A^j w_k}_{= \lambda_k^j w_k}$$

In Matrix form:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & & & \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

nonsingular since $\lambda_1 \ldots \lambda_k$ mut dist

so $c_i = 0$ $1 \le i \le k$

$\square$

Note Given $n \times n$ matrices A, B

Assume A, B are similar

Then A, B have same char polynomial

pf There exists invertible matrix P such that

$$B = P^{-1} A P$$

$$\text{char poly of } B = |B - \lambda I|$$

$$= |P^{-1} A P - \lambda I|$$

$$= |P^{-1}(A - \lambda I) P|$$

$\det(RS) = \det R \det S$

$$= |P^{-1}|\,|A - \lambda I|\,|P|$$

$$= |P|^{-1}\,|A - \lambda I|\,|P|$$

$$= |A - \lambda I|$$

$= \text{char poly of } A$

$\square$