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Lec 28 Friday April 4

6.1 Introduction to Eigenvalues

Motivation Consider the function

$$y(x) = e^{3x}$$

So $y' = 3y$

So $Dy = 3y$ $\qquad D = \frac{d}{dx}$

the operator D sends y to a scalar multiple of y.

Def Given an $n \times n$ matrix A

Given a nonzero column n-vector v

v is called an eigenvector for A whenever there exists a scalar λ such that

$$Av = \lambda v$$

We call λ the eigenvalue of A associated with v.

Ex For $A = \begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$

- For $v = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ one checks

$$Av = 3v$$

So $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is an eigenvector for A with eigenvalue 3

- For $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ one checks

$$Av = 2v$$

So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigvector for A with eigenvalue **2**

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<u>Ex</u> Fn $A = \begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}$

find all the eigenvalues of A

<u>Sol</u> let λ = an eigenvalue of A

there exists an eigenvector v for A and λ:

$$Av = \lambda v$$

So

$$\begin{aligned} 0 &= Av - \lambda v \\ &= Av - \lambda I v \\ &= (A - \lambda I)v \end{aligned}$$

So

v is in the solution space of $A - \lambda I$
$\neq 0$

$A - \lambda I$ is singular

So

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= |A - \lambda I| \end{aligned}$$

$$|A-\lambda I| = \begin{vmatrix} 10-\lambda & -8 \\ 6 & -4-\lambda \end{vmatrix}$$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

$$= \underbrace{(10-\lambda)(-4-\lambda)}_{\substack{\shortparallel \\ -40-6\lambda+\lambda^2}} \; - \underbrace{(-8)(6)}_{\substack{\shortparallel \\ 48}}$$

$$= \lambda^2 - 6\lambda + 8$$

$$= (\lambda-2)(\lambda-4)$$

Require

$$(\lambda-2)(\lambda-4) = 0$$

"characteristic equation"

characteristic polynomial

$$\lambda = 2, 4.$$

the eigenvalues of A are 2, 4 $\square$

Ex For $A = \begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}$

For each eigenvalue 2, 4 of A find an associated eigenvector.

Sol

λ	2	4
$A - \lambda I$	$\begin{bmatrix} 8 & -8 \\ 6 & -6 \end{bmatrix}$	$\begin{bmatrix} 6 & -8 \\ 6 & -8 \end{bmatrix}$
Solve $(A-\lambda I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by applying GJ to $A - \lambda I$	$\begin{bmatrix} 1 & -1 \\ 6 & -6 \end{bmatrix}$ $r_1' = \frac{1}{8} r_1$ $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $r_2' = r_2 - r_1$ $x - y = 0$ $y = t$ free $x = t$ $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -4/3 \\ 6 & -8 \end{bmatrix}$ $r_1' = \frac{1}{6} r_1$ $\begin{bmatrix} 1 & -4/3 \\ 0 & 0 \end{bmatrix}$ $r_2' = r_2 - 6r_1$ $x - 4/3\, y = 0$ $y = t$ free $x = \frac{4}{3} t$ $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$ (take $t = 3$)
eigenvector	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$
check:	$\begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \stackrel{?}{=} 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓	$\begin{bmatrix} 10 & -8 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} \stackrel{?}{=} 4\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ ✓

Ex For $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

find all the eigenvalues of A.
For each eigenvalue find an associated eigenvector

Sol let λ = an eigenvalue of A

Require

$$0 = |A - \lambda I|$$

$$= \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix}$$

$$= (-\lambda)^2 - (-1)(1)$$

$$= \lambda^2 + 1$$

$$= (\lambda - i)(\lambda + i) \qquad i^2 = -1$$

$$\lambda = i, -i$$

λ	i	$-i$
$A-\lambda I$	$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$	$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$
Solve $(A-\lambda I)\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by applying GJ to $A-\lambda I$	$\begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix}$ $r_1' = ir_1$ $\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$ $r_2' = r_2 + r_1$ $x + iy = 0$ $y = t$ free $x = -it$ $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} -i \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -i \\ -1 & i \end{bmatrix}$ $r_1' = -ir_1$ $\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$ $r_2' = r_2 + r_1$ $x - iy = 0$ $y = t$ free $x = it$ $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$
eigenvector	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
check	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -i \\ 1 \end{bmatrix} \overset{?}{=} i\begin{bmatrix} -i \\ 1 \end{bmatrix}$ ✓	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} \overset{?}{=} -i\begin{bmatrix} i \\ 1 \end{bmatrix}$ ✓

We record some ideas

<u>thm</u>

Given an $n \times n$ matrix A

Given a scalar λ (possibly in $\mathbb{C}$)

the following are equivalent:

(i) λ is an eigenvalue of A

(ii) $A - \lambda I$ is singular I = $n \times n$ identity matrix

(iii) the determinant

$$|A - \lambda I| = 0$$

"characteristic equation"

this is a polynomial in the variable λ that has degree n, called the characteristic polynomial

Thm Given an $n \times n$ matrix A

Given an eigenvalue λ for A

then for each nonzero column n-vector v,
the following are equivalent:

(i) v is an eigenvector for A and λ

(ii) v is contained in the null space
(solution space) of $A - \lambda I$
"eigenspace for A and λ"

Ex For $A = \begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Find the eigenvalues of A.

For each eigenvalue find a basis for the corresp eigenspace

Sol Let λ = eigenvalue of A

Require

$$0 = |A - \lambda I|$$

$$= \begin{vmatrix} 4-\lambda & -3 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

(expand along row 3)

$$= (2-\lambda) \underbrace{\begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix}}$$

$$\parallel$$

$$(4-\lambda)(-1-\lambda) - (-3)(2)$$

$$\parallel$$

$$-4 - 3\lambda + \lambda^2 + 6$$

$$\parallel$$

$$\lambda^2 - 3\lambda + 2$$

$$\parallel$$

$$(\lambda-2)(\lambda-1)$$

$$= -(\lambda-2)^2(\lambda-1)$$

Eigenvalues of A are $2, 2, 1$

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λ	2	1
$A-\lambda I$	$\begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
Solve $(A-\lambda I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ using GJ	$\begin{bmatrix} 1 & -3/2 & \frac{1}{2} \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $r_1' = \frac{1}{2} r_1$ $\begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $r_2' = r_2 - 2r_1$ $y = s$ free $z = t$ free $x = \frac{3}{2}s - \frac{t}{2}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & \frac{1}{3} \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $r_1' = \frac{1}{3} r_1$ $\begin{bmatrix} 1 & -1 & 1/3 \\ 0 & 0 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$ $r_2' = r_2 - 2r_1$ $\begin{bmatrix} 1 & -1 & \frac{1}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $z = 0$ $y = t$ free $x = y - z/3$ $= t$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
basis for eigenspace	$\begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

checks

$$\begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix} \stackrel{?}{=} 2\begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix} \quad \checkmark$$

$$\begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \stackrel{?}{=} 2\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \quad \checkmark$$

$$\begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \stackrel{?}{=} 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \checkmark$$

Notes

Given $n \times n$ matrix A

Given eigenvector v for A, with eigenvalue λ

So $Av = \lambda v$

- For an integer $k \geq 0$

  v is an eigenvector for A^k with eigenvalue λ^k

ex $k=2$

$$A^2 v = A\underbrace{Av}_{=\lambda v}$$
$$= \lambda Av$$
$$= \lambda \lambda v$$
$$= \lambda^2 v$$

- Assume A is nonsingular. Then $\lambda \neq 0$. Moreover

  v is eigenvector for A with eigenvalue λ^{-1}

$$Av = \lambda v$$
$$A^{-1} Av = A^{-1} \lambda v$$
$$Iv = \lambda A^{-1} v$$
$$v = \lambda A^{-1} v$$

$\lambda \neq 0$ since $v \neq 0$

$$\lambda^{-1} v = A^{-1} v$$