Lec 27 Wednesday April 2

5.5 Nonhomogeneous Equations and undetermined coefficients

We now consider differential equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \qquad *$$

$a_0, a_1, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$ $\qquad$ $f(x)$ = a function of x

Recall associated homog eqn

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad **$$

Recall gen sol to * is

$$y_p + c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \qquad c_1, c_2, \ldots, c_n \text{ free}$$

where y_p is particular sol to * and

$y_1, \ldots, y_n$ is basis for sol space of **.

Next goal: How to find y_p using

method of undetermined coefficients

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Ex Find a particular solution to

$$y'' - y' - 6y = \underbrace{2\sin 3x}_{f(x)} \qquad *$$

Sol (1) List all the terms that show up in

$f(x)$, $f'(x)$, $f''(x)$, ...

List: $\sin 3x$, $\cos 3x$

[List must be finite for method to work]

(2) Determine if functions in List are sols to associated homog eqn

$$y'' - y' - 6y = 0 \qquad **$$

Check char poly

$$r^2 - r - 6 = 0$$

$$(r-3)(r+2) = 0$$

$$r = 3, -2$$

e^{3x}, e^{-2x} is basis for sol space of **

$\sin 3x$, $\cos 3x$ are not sols to **

(3) Search for particular sol $y = y_p$ to *

of form

$$y = A\sin 3x + B\cos 3x$$

Scalars A, B to be determined

Obs

$$y = A\sin 3x + B\cos 3x$$

$$y' = 3A\cos 3x - 3B\sin 3x$$

$$y'' = -9A\sin 3x - 9B\cos 3x$$

Eval * using this:

$$0 = y'' - y' - 6y - 2\sin 3x$$

$$= -9A\sin 3x - 9B\cos 3x$$

$$- \left(3A\cos 3x - 3B\sin 3x\right)$$

$$- 6\left(A\sin 3x + B\cos 3x\right)$$

$$- 2\sin 3x$$

$$= \sin 3x \left(\underbrace{-9A + 3B - 6A - 2}_{\| \; -15A + 3B - 2} \right)$$

$$+ \cos 3x \left(\underbrace{-9B - 3A - 6B}_{\| \; -3A - 15B} \right)$$

The functions

$$\sin 3x, \quad \cos 3x$$

are lin indep, so

$$-15A + 3B = 2$$
$$-3A - 15B = 0$$

Solve this system for A, B to get

$$A = \frac{-5}{39}, \qquad B = \frac{1}{39}$$

So

$$y_p = \frac{-5\sin 3x + \cos 3x}{39}$$

is a particular sol to $*$.

□

Ex Find a particular sol to

$$4y'' + 4y' + y = 3xe^x \qquad *$$

Sol (1)

$$\frac{d}{dx}\left(xe^x\right) = xe^x + e^x$$

$$\frac{d}{dx}\left(e^x\right) = e^x$$

List: xe^x, e^x

(2) For assoc hom equation

$$4y'' + 4y' + y = 0 \qquad **$$

char eq is

$$4r^2 + 4r + 1 = 0$$

$$(2r+1)^2 = 0$$

$$r = -1/2, \; -1/2$$

$$e^{-\frac{x}{2}}, \quad xe^{-\frac{x}{2}}$$

is basis for the sol space of **.

xe^x, e^x are not sols to **

(3) Search for particular sol $y = y_p$ to * of form

$$y = Axe^x + Be^x$$

scalars A, B to be determined

$$y = Axe^x + Be^x$$

$$y' = A(xe^x + e^x) + Be^x$$

$$= Axe^x + (A+B)e^x$$

$$y'' = Axe^x + (2A+B)e^x$$

Evaluate * using this

table of coefficients

	xe^x	e^x	coef
y	A	B	1
y'	A	$A+B$	4
y''	A	$2A+B$	4
RHS	3	0	

xe^x, e^x are lin indep, so

$A \cdot 1 + A \cdot 4 + A \cdot 4 = 3$

$B \cdot 1 + (A+B) 4 + (2A+B) \cdot 4 = 0$

So

$$9A = 3$$

$$12A + 9B = 0$$

So

$$A = \frac{1}{3}, \qquad B = -\frac{4}{9}$$

So

$$y_p = \frac{3xe^x - 4e^x}{9}$$

is a particular sol'n to *

□

Ex Find a particular sol to

$$y''' + y' = -\sin x \qquad *$$

Sol (1) List: $\sin x, \cos x$

(2) For the assoc homog eq

$$y''' + y' = 0 \qquad **$$

charply is

$$r^3 + r = 0$$
$$\| $$
$$r(r^2+1)$$
$$\| $$
$$r(r-i^0)(r+i^0)$$

$$r = 0, i^0, -i^0$$

$$1 = e^{0x}, \quad e^{i^0 x}, \quad e^{-i^0 x}$$

is basis for solspace of **

Recall $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

so $\sin x, \cos x$ are sols to **

(3) Search for a partic sol $y = y_p$ to * of form

$$y = Ax\sin x + Bx\cos x$$

Reason Write * as

$$(D^3 + D)y = -\sin x$$

$$D = \frac{d}{dx}$$

$$D(D^2+1)y = -\sin x$$

$$D(D+i)(D-i)y = -\sin x$$

But $(D^2+1)(-\sin x) = 0$

$(D+i)(D-i)(-\sin x) = 0$

So

$$D(D+i)^2(D-i)^2 y = 0 \qquad ***$$

char eq is

$$r(r+i)^2(r-i)^2 = 0$$

$r = 0, i, i, -i, -i$

sol space of *** has basis

$1, \; e^{ix}, \; xe^{ix}, \; e^{-ix}, \; xe^{-ix}$

Another basis is

$1, \; \sin x, \; \cos x, \; x\sin x \; x\cos x$

these are sols to **

so ignore

$y = Ax\sin x + Bx\cos x$

Find y', y'', y'''

$\frac{d}{dx}$:

	$\sin x$	$\cos x$	$x\sin x$	$x\cos x$
$\sin x$	0	-1	1	0
$\cos x$	1	0	0	1
$x\sin x$	0	0	0	-1
$x\cos x$	0	0	1	0

$= M$

$\frac{d^2}{dx^2}$:

	sinx	cosx	xsinx	xcosx
sinx	−1	0	0	−2
cosx	0	−1	2	0
xsinx	0	0	−1	0
xcosx	0	0	0	−1

$= M^2$

$\frac{d^3}{dx^3}$:

	sinx	cosx	xsinx	xcosx
sinx	0	1	−3	0
cosx	−1	0	0	−3
xsinx	0	0	0	1
xcosx	0	0	−1	0

$= M^3$

* becomes

	$\sin x$	$\cos x$	$x\sin x$	$x\cos x$	coef
y'	A	B	$-B$	A	1
y'''	$-3A$	$-3B$	B	$-A$	1
RHS	-1	0	0	0	

$-2A = -1$ $\qquad -2B = 0$

$A = \frac{1}{2}$ $\qquad B = 0$

$y_p = \frac{1}{2} x\sin x$

is a part sol to *