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5.3 Homogeneous equations with constant coefficients

We consider the general nth order linear diff eq with constant coefs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad *$$

$$a_0, a_1, \ldots, a_n \in \mathbb{R} \qquad a_n \neq 0$$

Find gen sol to *

V = set of all sols to * $\qquad$ V is vector space dim n.

Find basis for V

Hunt for solutions $y \in V$ of form

$$y(x) = e^{rx}$$

Obs

$$y'(x) = r e^{rx}$$
$$y''(x) = r^2 e^{rx}$$
$$\vdots$$

* becomes

$$0 = a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_0 e^{rx}$$

$$= \underset{\substack{\neq \\ 0}}{e^{rx}} \left(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 \right)$$

$$0 = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 \qquad ☆$$

"characteristic equation for *"

☆ is polynomial eqn deg n.

Let $r_1, r_2, \dots, r_n$ denote the roots of the polynomial, ie sols to ☆

- $r_1 \dots r_n$ might not be distinct
- some of $r_1, \dots, r_n$ might be complex numbers

<u>Case</u> $r_1, r_2, \dots, r_n$ are mutually distinct and real

For $1 \le i \le n$ the function

$$y = e^{r_i x}$$

is a sol to *

So V contains

$$e^{r_1 x}, \quad e^{r_2 x}, \quad \dots, \quad e^{r_n x}$$

One checks these are lin indep and hence a basis for V

Gen solution to $*$ is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

$c_1, c_2 \ldots c_n$ free

Ex Find gen sol for

$$y'' - 4y' + 3y = 0$$

Sol char poly is

$$r^2 - 4r + 3 = 0$$

Factor $(r-3)(r-1) = 0$

$$r = 1, 3$$

$$y = c_1 e^{1x} + c_2 e^{3x}$$ c_1, c_2 free

__Next__ roots $r_1, r_2 \ldots r_n$ of ☆ not all real

Recall a complex number has form

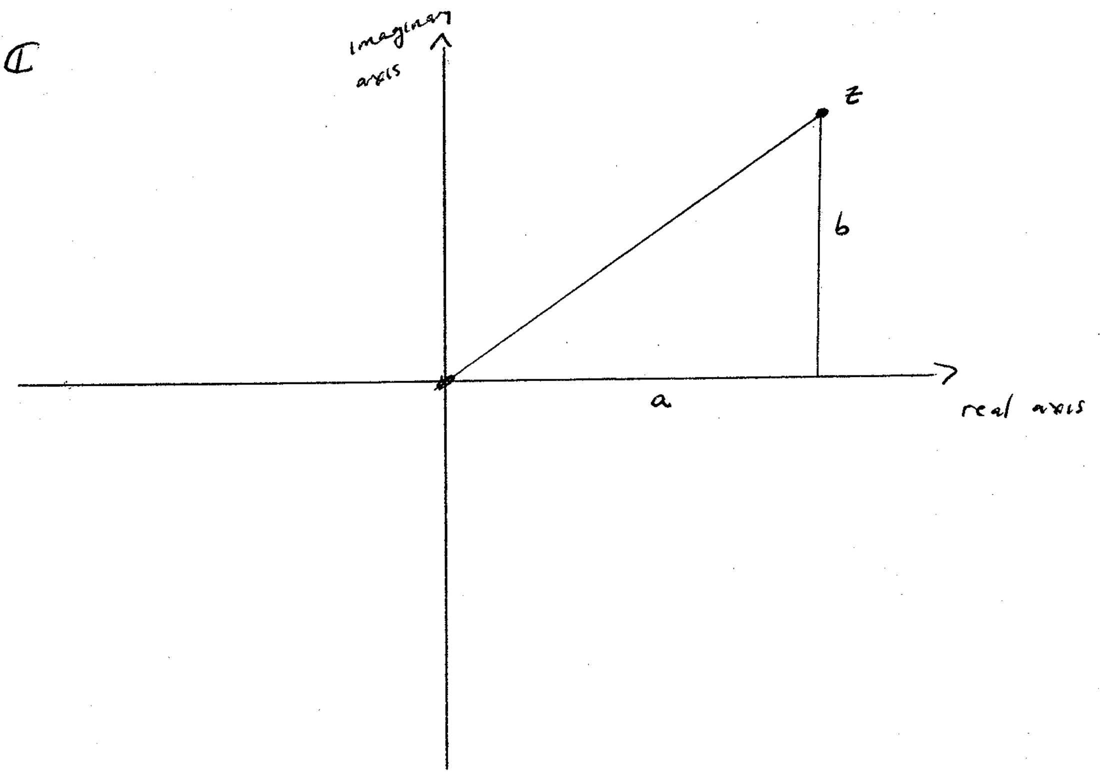
imaginary part of z

$$z = a + bi \qquad i^2 = -1 \qquad a, b \in \mathbb{R}$$

real part of z

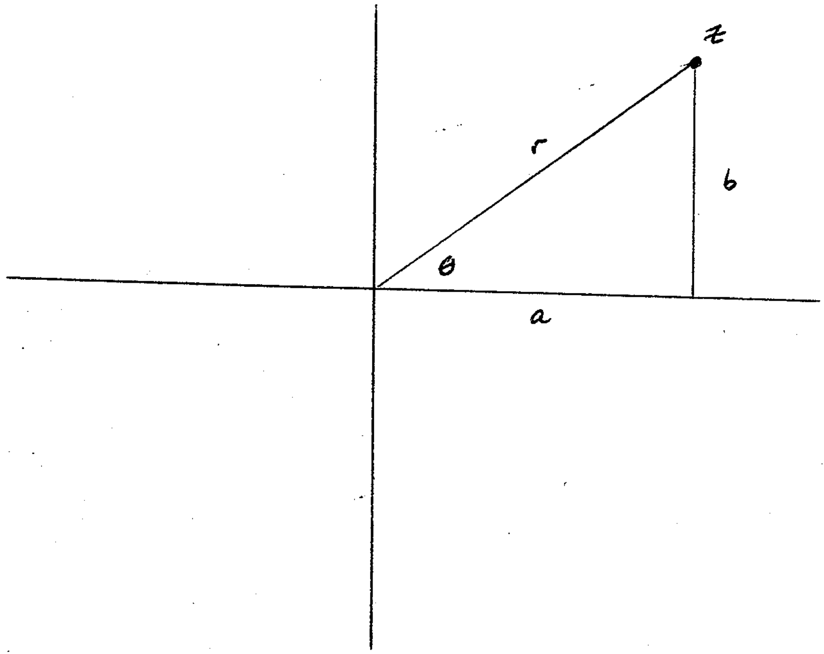
$\mathbb{C}$ = set of all complex numbers

$\mathbb{C}$



Call $\mathbb{C}$ "complex plane"

For $z = a + bi$



define

$r = \sqrt{a^2+b^2}$ "modulus" or "norm" of z

angle θ "argument" of z

$\frac{a}{r} = \cos\theta$ $\frac{b}{r} = \sin\theta$

$$z = a+bi$$
$$= r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

Recall from Calc

$$\cos\theta + i\sin\theta = e^{i\theta}$$

$$z = re^{i\theta}$$ "polar form"

Handy formulae:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

<u>Case</u> roots $r_1, r_2, \ldots, r_n$ to ✡ not all real

Conceptually, gen sol is same as before, except in vector space V of solutions, scalars are in $\mathbb{C}$ instead of $\mathbb{R}$

<u>Ex</u> Find gen sol to

$$y'' - 6y' + 13y = 0$$

<u>Sol</u> char poly is

$$r^2 - 6r + 13 = 0$$

Find roots using quadratic formula

$$r = \frac{6 \pm \sqrt{6^2 - 4 \cdot 13}}{2}$$

$$6^2 - 4 \cdot 13 = 36 - 52$$
$$= -16$$

$$r = \frac{6 \pm 4i}{2}$$

$$= 3 \pm 2i$$

Gen sol is

$$y = C_1 e^{(3+2i)x} + C_2 e^{(3-2i)x}$$

C_1, C_2 free

We now express gen sol using sin, cos:

$$e^{(3+2i)x} = e^{3x} e^{2ix}$$
$$= e^{3x}(\cos 2x + i \sin 2x)$$

$$e^{(3-2i)x} = e^{3x} e^{-2ix}$$
$$= e^{3x}(\cos 2x - i \sin 2x)$$

So sol space V has basis

$$e^{3x}\cos 2x, \qquad e^{3x}\sin 2x$$

Gen sol is

$$y = A e^{3x}\cos 2x + B e^{3x}\sin 2x$$

A, B free

<u>Next</u> roots $r_1, r_2, \ldots r_n$ to ☆ not distinct

Consider extreme case

$$r_1 = r_2 = \cdots = r_n \quad (= r)$$

Ex Given $r \in \mathbb{C}$

Given integer $n \geq 1$

Find gen sol to

$$(D - rI)^n y = 0 \qquad D = \frac{d}{dx} \qquad **$$

n	$**$
1	$y' - ry = 0$
2	$y'' - 2ry' + r^2 y = 0$
3	$y''' - 3ry'' + 3r^2 y' - r^3 y = 0$
$\vdots$	$\vdots$

Let V = vector space of all sols to $**$

$\dim V = n$

Find basis for V

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Given $y = y(x) \in V$

Write $f(x) = e^{-rx} y$

So $y = e^{rx} f(x)$

So

$$Dy = y'$$
$$= e^{rx} f'(x) + r e^{rx} f(x)$$
$$= e^{rx} f'(x) + ry$$

So
$$(D - rI) y = e^{rx} f'(x)$$
$$= e^{rx} D f(x)$$

Iterate:
$$(D - rI)^2 y = e^{rx} D^2 f(x)$$
$$(D - rI)^3 y = e^{rx} D^3 f(x)$$
$$\cdots$$
$$(D - rI)^n y = e^{rx} D^n f(x)$$

Require

$$0 = (D - rI)^n y$$

$$= \underbrace{e^{rx}}_{\neq 0} D^n f(x)$$

Require

$$0 = D^n f(x)$$ (nth derivative of $f(x)$)

Works for

$$f(x) = x^k \qquad k = 0, 1, 2, \ldots, n-1$$

So V contains

$$x^k e^{rx} \qquad k = 0, 1, 2, \ldots, n-1$$

these n functions are lin indep and hence a basis for V.

Gen sol to ** is

$$y = C_0 e^{rx} + C_1 x e^{rx} + \cdots + C_{n-1} x^{n-1} e^{rx}$$

$C_0, C_1, \ldots, C_{n-1}$ free

Ex Find gen sol to

$$y''' - 3y'' + 3y' - y = 0$$

Sol char poly is

$$r^3 - 3r^2 + 3r - 1 = 0$$

$$(r-1)^3 = 0$$

roots $r = 1, 1, 1$

Gen sol is

$$y = C_0 e^x + C_1 x e^x + C_2 x^2 e^x$$

C_0, C_1, C_2 free

Ex Find gen sol to

$$y^{(4)} - 8y'' + 16y = 0$$

Sol Char poly is

$$r^4 - 8r^2 + 16 = 0$$

Factor

$$(r^2 - 4)^2 = 0$$

$$(r-2)^2(r+2)^2 = 0$$

roots are $r = 2, 2, -2, -2.$

Gen sol is

$$y = Ae^{2x} + Bxe^{2x} + Ce^{-2x} + Dxe^{-2x}$$

A, B, C, D free