Lec 25 Friday March 28 3/28/14

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5.2 General solutions of linear equations

In Section 5.1 we considered diff equations that are 2nd order linear.

We now consider nth order linear for all n:

nth derivative

$$P_0(x)\, y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_{n-1}(x)\, y' + P_n(x)\, y = F(x) \qquad *$$

$P_0(x), \ldots, P_n(x)$, $F(x)$ are functions of x with $P_0(x) \neq 0$

"nth order linear"

Notes

- $*$ called homogeneous if $F(x) = 0$

- In the homog case set of sols V to $*$ is a vector space.

  this gives:

thm (superposition principle)

Ref. to * assume $F(x) = 0$

Let $y_1, y_2, \ldots, y_n$ denote solutions. Then any linear combination

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

is a solution.

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Ex Solve the initial value problem

$$y''' - 6y'' + 11y' - 6y = 0 \quad \star$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 3 \quad \star\star$$

Sol Hunt for sols of form

$$y = e^{rx}$$

Find r

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

$$y''' = r^3 e^{rx}$$

Require

$$0 = r^3 e^{rx} - 6r^2 e^{rx} + 11 r e^{rx} - 6 e^{rx}$$

$$= \underset{\neq 0}{e^{rx}} \left(r^3 - 6r^2 + 11r - 6 \right)$$

So

$$0 = r^3 - 6r^2 + 11r - 6$$

"characteristic eqn"

Factor if poss

$$0 = (r-1)(r-2)(r-3)$$

$$r = 1, 2, 3$$

Found ☆ has sols

$$y_1(x) = e^x, \quad y_2(x) = e^{2x}, \quad y_3(x) = e^{3x}$$

So for all $c_1, c_2, c_3 \in \mathbb{R}$

$$y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

is a sol to ☆

Find c_1, c_2, c_3 such that ☆☆ holds

$$y'(x) = c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x}$$

$$y''(x) = c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x}$$

Require

$$c_1 + c_2 + c_3 = 0 \qquad (y(0) = 0)$$

$$c_1 + 2c_2 + 3c_3 = 0 \qquad (y'(0) = 0)$$

$$c_1 + 4c_2 + 9c_3 = 3 \qquad (y''(0) = 3)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 4 & 9 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 8 & 3 \end{array}\right] \quad \begin{array}{l} r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 3 \end{array}\right] \quad r_3' = r_3 - 3r_2$$

Back solve

$$c_3 = 3/2$$

$$c_2 = -2c_3 = -3$$

$$c_1 = -c_2 - c_3 = 3/2$$

$$y = \frac{3}{2} e^{x} - 3e^{2x} + 3/2\, e^{3x}$$

$\square$

Note on initial value problems for hom Linear

In prev Ex order is $n=3$

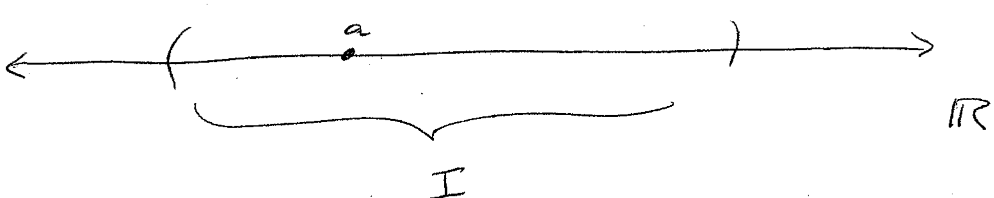
we gave init conds on y, y', y'' (3 constraints)

For general n, give init conds on

$$y, y', y'', \dots, y^{(n-1)}$$

(n constraints)

thm A Given an open interval I on the real line, that contains a point a



Given functions $p_1(x), \dots, p_n(x), f(x)$ that are contin on I

Given scalars in $\mathbb{R}$:

$$b_0, b_1, \dots, b_{n-1}$$

then the init value problem

$$y^{(n)} + p_1(x)\, y^{(n-1)} + \dots + p_n(x)\, y = f(x)$$

$$y(a) = b_0, \quad y'(a) = b_1, \quad \dots, \quad y^{(n-1)}(a) = b_{n-1}$$

has a unique sol $y = y(x)$ defined at each point $x \in I$

Thm B Given open interval I on the real line

Given functions

$$p_1(x),\ p_2(x),\ \ldots,\ p_n(x)$$

that are continuous on I

Consider homog diff eq

$$y^{(n)} + p_1(x)\, y^{(n-1)} + \cdots + p_n(x)\, y = 0 \qquad **$$

Let V = set of sols to ** (V = vector space)

then V has dimension n

[pf sim to pf of thm B in Section 5.1]

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Ref to th B

Wish to find basis for V

Given $f_1, f_2, \ldots, f_n \in V$

How to tell if $f_1, f_2, \ldots, f_n$ are lin indep:

Given scalars $c_1, c_2, \dots, c_n$ such that

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$$

Take rth derivative for $0 \le r \le n-1$:

$$\begin{aligned} c_1 f_1 + \dots \quad & c_n f_n = 0 \\ c_1 f_1' + \dots \quad & c_n f_n' = 0 \\ c_1 f_1'' + \dots \quad & c_n f_n'' = 0 \\ & \dots \\ c_1 f_1^{(n-1)} + \dots + \quad & c_n f_n^{(n-1)} = 0 \end{aligned}$$

So

$$\begin{pmatrix} f_1 & \dots & f_n \\ f_1' & \dots & f_n' \\ f_1'' & \dots & f_n'' \\ & \dots & \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\uparrow$

If this is nonsing for some $x \in I$ then $c_1 = c_2 = \dots = c_n = 0$

Def

$$W(x) = \begin{vmatrix} f_1 & \dots & f_n \\ f_1' & \dots & f_n' \\ f_1'' & \dots & f_n'' \\ & \dots & \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix} \qquad (\det)$$

" Wronskian of $f_1, \dots, f_n$ "

Thm C With above notation,

- If $f_1, f_2, \ldots, f_n$ are lin dependent, then

$$W(x) = 0 \quad \text{for all } x \in I$$

- If $f_1, f_2, \ldots, f_n$ are lin independent, then

$$W(x) \neq 0 \quad \text{for all } x \in I.$$

Ex Consider these functions on $\mathbb{R}$:

$$e^x, \quad \cos x, \quad \sin x$$

Are they lin indep?

Sol Compute Wronskian

$$W(x) = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix}$$

$$= e^x \begin{vmatrix} 1 & \cos x & \sin x \\ 1 & -\sin x & \cos x \\ 1 & -\cos x & -\sin x \end{vmatrix}$$

↑ expand down col 1

[$S = \sin x$ $\quad C = \cos x$]

$| \; | =$

$$\begin{vmatrix} -s & c \\ -c & -s \end{vmatrix} - \begin{vmatrix} c & s \\ -c & -s \end{vmatrix} + \begin{vmatrix} c & s \\ -s & c \end{vmatrix}$$

$$\begin{matrix} \shortparallel & & \shortparallel & & \shortparallel \\ s^2+c^2 & & 0 & & c^2+s^2 \\ \shortparallel & & & & \shortparallel \\ 1 & & & & 1 \end{matrix}$$

$= 2$

$W(x) = 2e^x$

$\neq 0$

So

e^x, $\cos x$, $\sin x$

are lin indep. $\square$

Given open interval I on real line

Given functions

$$p_1(x),\ p_2(x),\ \dots,\ p_n(x),\quad f(x)$$

that are contin on I

Consider non hom diff eq

$$y^{(n)} + p_1(x)\,y^{(n-1)} + \dots + p_n(x)\,y = f(x) \qquad (*)$$

Associated homog diff eq is

$$y^{(n)} + p_1(x)\,y^{(n-1)} + \dots + p_n(x)\,y = 0 \qquad (**)$$

Let y_p = particular sol to $*$

obs

- Let y = any sol to $(*)$. Then $y - y_p$ is a sol to $(**)$
- Let y = any sol to $(**)$ Then $y + y_p$ is a sol to $(*)$

(Routine to check)

Thm With above notation.

let V = set of sols for (**) (V is vector sp dim n)

let

$y_1, y_2, \ldots, y_n$

denote a basis for V.

Then the gen solution to * is

$$y_p + c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \qquad c_1, c_2, \ldots, c_n \in \mathbb{R}$$

pf By obs above thm. $\square$