Lec 24 Wednesday March 26 3/26/14

5.1 Second order linear differential eqs

We now consider differential equations of the form

$$A(x)y'' + B(x)y' + C(x)y = F(x) \qquad \ast$$

$A(x), B(x), C(x), F(x)$ are functions of x with $A(x) \neq 0$

"2nd order linear"

$\ast$ called homogeneous whenever $F(x) = 0$

Consider homog diff equation

$$A(x)y'' + B(x)y' + C(x)y = 0 \qquad \star$$

Let V = set of sols to $\star$. One checks V is a vector space. This gives:

thm (Superposition principle)

If functions y_1 and y_2 are solutions to ☆,

then any linear combination

$$c_1 y_1 + c_2 y_2$$

is a solution to ☆.

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Ex Solve the initial value problem

$$y'' - 5y' = 0,$$

$$y(0) = 0, \quad y'(0) = 5$$

Sol Recall from prev lecture that

$$e^{5x}, \qquad 1$$

are both sols to $y'' - 5y' = 0$

So for all $a, b \in \mathbb{R}$

$$y = ae^{5x} + b$$

is a solution to $y'' - 5y' = 0$.

Pick a, b to satisfy init conds

Require

$$0 = y(0) = a + b$$

$$5 = y'(0) = 5a$$

So $a = 1, \; b = -1$

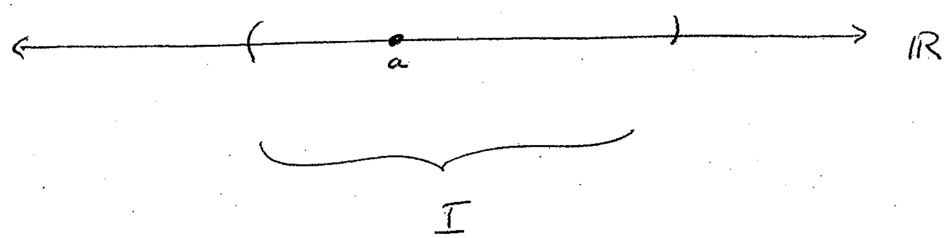
$$y = e^{5x} - 1$$

$\square$

In prev Ex we gave init values for y and y'

In gen we have:

<u>thm</u> A Given an open interval I on the real line, that contains a point a



Given functions p, q, f that are continuous on I,

Given scalars $b_0, b_1 \in \mathbb{R}$.

Then the init value problem

$$y'' + p(x)y' + q(x)y = f(x),$$

$$y(a) = b_0, \qquad y'(a) = b_1$$

has a unique solution $y = y(x)$ defined at each point in I

Thm B Given open interval I on real line

Given functions $p(x), q(x)$ that are continuous on I

Consider homog eq

$$y'' + p(x)y' + q(x)y = 0 \qquad **$$

Let V = set of all sols to $**$. (V is vector space)

then V has dimension 2

pf. Pick $a \in I$

By th A there exists $y_1 \in V$ such that

$$y_1(a) = 1, \qquad y_1'(a) = 0$$

By th A there exists $y_2 \in V$ such that

$$y_2(a) = 0, \quad y_2'(a) = 1$$

Show y_1, y_2 is basis for V

Check y_1, y_2 are lin indep: Given $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 y_1 + c_2 y_2 = 0$$

So

$$c_1 y_1' + c_2 y_2' = 0$$

Set $x = a$:

$$c_1 \underbrace{y_1(a)}_{=1} + c_2 \underbrace{y_2(a)}_{=0} = 0 \qquad c_1 = 0$$

$$c_1 \underbrace{y_1'(a)}_{=0} + c_2 \underbrace{y_2'(a)}_{=1} = 0 \qquad c_2 = 0$$

<u>Check y_1, y_2 span V:</u>

Given $y \in V$

claim

$$y = y(a)\, y_1 + y'(a)\, y_2$$

Reason: define

$$\Psi = y(a)\, y_1 + y'(a)\, y_2 - y$$

Then

$$\Psi'' + p(x)\Psi' + q(x)\Psi = 0$$

$$\Psi(a) = 0, \qquad \Psi'(a) = 0$$

Now Ψ is zero function by thm A. $\square$

Ref to Th B

Given $f, g \in V$

How to tell if f, g are lin indep :

Given $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 f + c_2 g = 0$$

take deriv

$$c_1 f' + c_2 g' = 0$$

So

$$\underbrace{\begin{pmatrix} f & g \\ f' & g' \end{pmatrix}}_{\substack{\text{if this is} \\ \text{nonsingular for some } x \in I,}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if this is nonsingular for some $x \in I$, then $c_1 = c_2 = 0$

define

$$W(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} \qquad (\det)$$

"Wronskian of f, g"

Thm C With above notation,

- If f, g are lin dependent, then $W(x) = 0$ for all $x \in I$
- If f, g are lin indep then $W(x) \neq 0$ for all $x \in I$

Ex. Find the gen sol for

$$2y'' - y' - y = 0$$

Sol. Let V = set of all sols

V = vector space dim 2

Find basis.

Hunt for $y \in V$ of form

$$y = e^{rx}$$

Find r

obs $$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

Require

$$2(r^2 e^{rx}) - 1(re^{rx}) - 1(e^{rx}) = 0$$

So

$$(2r^2 - r - 1)\underbrace{e^{rx}}_{\neq 0} = 0$$

Require

$$2r^2 - r - 1 = 0$$ "characteristic equation"

(Solve by quadratic formula or factoring)

$$(2r+1)(r-1) = 0$$

$$r = 1, \quad r = -1/2$$

so

$$y_1 = e^x, \qquad y_2 = e^{-1/2 x}$$

are solutions.

Compute Wronskian to check lin independence

$$y_1' = e^x \qquad y_2' = -1/2\, e^{-1/2 x}$$

$$W(x) = \begin{vmatrix} e^x & e^{-1/2 x} \\ e^x & -\frac{1}{2} e^{-1/2 x} \end{vmatrix}$$

$$= e^{\frac{1}{2}x}\left(-\frac{1}{2} - 1\right)$$

$$= -\frac{3}{2} e^{\frac{1}{2}x}$$

$$\neq 0$$

y_1, y_2 lin indep, hence basis for V.

Gen sol is $\quad c_1 e^x + c_2 e^{-1/2 x} \qquad c_1, c_2 \in \mathbb{R}$

Ex Find gen sol to

$$4y'' + 4y' + y = 0$$

Sol Let $V =$ v.s of all sols

$\dim V = 2$

Find basis

Hunt for $y \in V$ of form

$$y = e^{rx}$$

$$\downarrow$$

require

$$4r^2 + 4r + 1 = 0 \qquad \text{(characteristic equation)}$$

$$(2r+1)^2 = 0$$

$$r = -1/2, \ -1/2$$

$$y_1 = e^{-\frac{1}{2}x} \in V$$

need another sol

Try $y_2 = x e^{-\frac{1}{2}x}$

$$y_2' = x\left(-\tfrac{1}{2}e^{-1/2x}\right) + e^{-1/2x}$$

$$= e^{-1/2x}\left(1 - \tfrac{x}{2}\right)$$

$$y_2'' = e^{-1/2x}\left(\tfrac{-1}{2}\right) + \left(1-\tfrac{x}{2}\right)\left(\tfrac{-1}{2}\right)e^{-1/2x}$$

$$= e^{-1/2x}\left(\tfrac{x}{4} - 1\right)$$

So

$$4y_2'' + 4y' + y = e^{-\frac{1}{2}x}\left(\underbrace{x-4 + 4-2x+x}_{0}\right)$$

$$= 0 \checkmark$$

claim y_1, y_2 basis for V
Check y_1, y_2 lin indep $\checkmark$

So y_1, y_2 basis for V

Gen sol is

$$c_1 e^{-\frac{1}{2}x} + c_2 x e^{-1/2x} \qquad c_1, c_2 \in \mathbb{R}$$