Lec 23 Monday March 24 3/24/14

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4.7 General vector spaces

In Section 4.2 we saw the axioms for a general vector space. Since then we mainly considered the vector space $\mathbb{R}^n$. We now consider other vector spaces

Ex Pick scalars a, b, c and consider the diff eq

$$ay'' + by' + cy = 0 \qquad *$$

let V = set of solutions to $*$

then V is a vector space

check: Set of all functions $y = y(x)$ is a vector space

Show V is subspace

Given y_1, y_2 in V show $y_1 + y_2 \in V$:

$$ay_1'' + by_1' + cy_1 = 0$$

$$ay_2'' + by_2' + cy_2 = 0$$

So

$$0 = a\underbrace{(y_1'' + y_2'')}_{\substack{\shortparallel \\ (y_1+y_2)''}} + b\underbrace{(y_1' + y_2')}_{\substack{\shortparallel \\ (y_1+y_2)'}} + c(y_1+y_2)$$

So $\quad y_1 + y_2 \in V$

Given $y \in V$ and $\alpha \in \mathbb{R}$ show $\alpha y \in V$:

$$ay'' + by' + cy = 0$$

$$0 = \alpha(ay'' + by' + cy)$$

$$= a\underbrace{\alpha y''}_{\substack{\shortparallel \\ (\alpha y)''}} + b\underbrace{\alpha y'}_{\substack{\shortparallel \\ (\alpha y)'}} + c\alpha y$$

So $\quad \alpha y \in V$

$\square$

Ex For the diff equation

$$y'' - 5y' = 0 \qquad **$$

let V = set of solutions to $**$

Find a basis for vector space V

Sol. Find the gen sol to $**$

View $y = y(x)$

Change vars

$$u = y'$$

So $**$ becomes

$$u' - 5u = 0$$

$$u' = 5u$$

$$u = Ce^{5x} \qquad C \text{ free}$$

$$y' = Ce^{5x}$$

$$y = \int Ce^{5x}\,dx$$

$$= C_1 e^{5x} + C_2 \qquad C_1, C_2 \text{ free}$$

$$V = \left\{ C_1 e^{5x} + C_2 \;\middle|\; C_1, C_2 \in \mathbb{R} \right\}$$

So the vector space

$$V = \mathrm{Span}\{ e^{5x}, 1 \}$$

Obs the functions

$$e^{5x}, \quad 1$$

are lin indep (since e^{5x} is not constant)

So

$e^{5x}, 1$ is a basis for V $\square$

Consider a polynomial in the variable x

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + \underset{\neq 0}{a_n} x^n \qquad a_0, a_1, \ldots, a_n \in \mathbb{R}$$

$n =$ <u>degree</u> of f

Ex For $n \geq 0$ define

$\mathcal{P}_n =$ set of polynomials with degree at most n.

Show $\mathcal{P}_n$ is a vector space

Sol Show $\mathcal{P}_n$ is a subspace of the vector space of all functions of x

Given $f_1(x)$, $f_2(x) \in \mathcal{P}_n$ show $f_1(x) + f_2(x) \in \mathcal{P}_n$:

Write

$$f_1(x) = a_0 + a_1 x + \cdots + a_n x^n$$
$$f_2(x) = b_0 + b_1 x + \cdots + b_n x^n$$

a_n might be 0 since degree of f is <u>at most</u> n

$$f_1(x) + f_2(x) = a_0 + b_0 + (a_1 + b_1) x + \cdots + (a_n + b_n) x^n$$
$$= c_0 + c_1 x + \cdots + c_n x^n \qquad c_i = a_i + b_i \quad 0 \leq i \leq n$$

$=$ poly of degree $\leq n$ ✓

$\in \mathcal{P}_n$.

Given $f(x) \in \mathcal{P}_n$ Given $\alpha \in \mathbb{R}$ show $\alpha f(x) \in \mathcal{P}_n$:

Write

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$
$$\alpha f(x) = \alpha a_0 + (\alpha a_1) x + \cdots + (\alpha a_n) x^n \quad \in \mathcal{P}_n \ ✓$$

Ex Find a basis for $\mathcal{P}_n$.

Sol

$$\mathcal{P}_n = \left\{ a_0 + a_1 x + \cdots + a_n x^n \;/\; a_0, a_1, \ldots, a_n \in \mathbb{R} \right\}$$

So

$$\mathcal{P}_n = \operatorname{Span}\left\{ 1, x, x^2, \ldots, x^n \right\}$$

To show

$$1, x, x^2, \ldots, x^n \qquad *$$

is a basis for $\mathcal{P}_n$, need to show $*$ is lin indep.

Given scalars $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that

$$a_0 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0 \qquad ☆$$

show

$$0 = a_0 = a_1 = \cdots = a_n$$

Pick $n+1$ distinct scalars

$$x_0, x_1, x_2, \ldots, x_n$$

For $0 \le i \le n$, ☆ gives

$$a_0 1 + a_1 x_i + a_2 x_i^2 + \cdots + a_n x_i^n = 0$$

This yields

$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}}_{\text{this Vandermonde matrix is nonsingular.}} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Above equation has only triv solution

$a_i = 0 \qquad 0 \le i \le n$

So

$1, x, x^2, \ldots, x^n$

is a basis for P_n $\square$

Ex. Let V denote the set of all 2×2 matrices of form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \qquad a, b \in \mathbb{R}$$

Show V is a vector space under usual matrix addition and scalar mult

Sol. Recall set of all 2×2 matrices is a vector space.

Show V is a subspace

Given $u, v \in V$ show $u+v \in V$:

$$u = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \qquad v = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

$a, b, A, B \in \mathbb{R}$

$$u+v = \begin{pmatrix} a+A & -b-B \\ b+B & a+A \end{pmatrix}$$

11 entry, 22 entry same
12 entry, 21 entry opposite

$u+v \in V$ ✓

Given $u \in V$ and $\alpha \in \mathbb{R}$ show $\alpha u \in V$

Write

$$u = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$\alpha u = \begin{pmatrix} \alpha a & -\alpha b \\ \alpha b & \alpha a \end{pmatrix}$$

11 entry, 22 entry same ✓
12 entry, 21 entry opp

$\alpha u \in V$ ✓

Ex For above V find basis

Sol

$$V = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| \ a, b \in \mathbb{R} \right\}$$

So

$$V = \operatorname{Span}\left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{= I}, \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{= i} \right\}$$

One checks I, i are lin indep and hence form a basis for V

Ex Fn above matrix $\overset{\circ}{i}$ find $\overset{\circ}{i}^2$

Sol
$$\overset{\circ}{i}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= -I$$

<u>Note</u> The above vector space V is a model for the set of complex numbers

$$\begin{matrix} a+bi \\ (i^2=-1) \end{matrix} \iff aI + b\overset{\circ}{i}$$

The complex numbers form a 2 dim'l vector space over $\mathbb{R}$.

Ex let V = set of all sequences

$a_0, a_1, a_2, \ldots,$ $\qquad a_n \in \mathbb{R}$

Such that

$$a_{n-1} - 2a_n + a_{n+1} = 0 \qquad n = 1, 2, 3, \ldots$$

Show V is a vector space.

<u>Clarify what is in V</u>

take $a_0 = 1$ $\quad a_1 = 1$ $\quad$ find a_2

$$0 = a_0 - 2a_1 + a_2$$
$$= 1 - 2 + a_2$$

$$a_2 = 1$$

Find a_3

$$0 = a_1 - 2a_2 + a_3$$
$$= 1 - 2 + a_3$$

$$a_3 = 1$$

In general

$$a_n = 1 \qquad \text{all } n$$

so V contains the sequence

$$1, 1, 1, \ldots\ldots$$

Call this sequence I

Suppose

$$a_0 = 0, \qquad a_1 = 1$$

Find a_2

$$0 = a_0 - 2a_1 + a_2$$
$$= 0 - 2 + a_2$$

$$a_2 = 2$$

Find a_3

$$0 = a_1 - 2a_2 + a_3$$
$$= 1 - 4 + a_3$$

$$a_3 = 3$$

In general

$$a_n = n \qquad n = 0, 1, 2, \ldots$$

So V contains the sequence

$$0, 1, 2, 3, \ldots \qquad \text{(call this sequence } X\text{)}$$

Given any $a_0, a_1 \in \mathbb{R}$ we can recursively find $a_2, a_3, \ldots$ to get a sequence

$$a_0, a_1, a_2, \ldots$$

in V.

To show V is vs, show V is subspace of the vs of all ∞ sequences

Given $u, v \in V$ show $u+v \in V$:

Write

$$u: \quad a_0, a_1, a_2, \ldots$$

Since $u \in V$

$$a_{n-1} - 2a_n + a_{n+1} = 0 \qquad n = 1, 2, \ldots$$

Write

$$v: \quad b_0, b_1, b_2, \ldots$$

Since $v \in V$,

$$b_{n-1} - 2b_n + b_{n+1} = 0 \qquad n = 1, 2, \ldots$$

$$u+v: \quad \underset{c_0}{\underset{\shortparallel}{a_0+b_0}}, \quad \underset{c_1}{\underset{\shortparallel}{a_1+b_1}}, \quad \underset{c_2}{\underset{\shortparallel}{a_2+b_2}}, \ldots$$

Require

$$0 \overset{?}{=} \underset{a_{n-1}+b_{n-1}}{\underset{\shortparallel}{c_{n-1}}} - 2\underset{a_n+b_n}{\underset{\shortparallel}{c_n}} + \underset{a_{n+1}+b_{n+1}}{\underset{\shortparallel}{c_{n+1}}} \qquad n = 1, 2, \ldots$$

$$= \underbrace{a_{n-1} - 2a_n + a_{n+1}}_{= 0} + \underbrace{b_{n-1} - 2b_n + b_{n+1}}_{= 0}$$

$$= 0 \; \checkmark$$

Given $u \in V$ and $\alpha \in \mathbb{R}$ show $\alpha u \in V$:

Write $u = a_0, a_1, a_2 \ldots$

$$\alpha u = \alpha a_0, \alpha a_1, \alpha a_2, \ldots$$

Require
$$0 \stackrel{?}{=} \underbrace{\alpha a_{n-1} - 2(\alpha a_n) + \alpha a_{n+1}}_{\parallel} \qquad n = 1, 2 \ldots$$

$$\alpha(\underbrace{a_{n-1} - 2a_n + a_{n+1}}_{\overset{\parallel}{0}})$$

ok

$\square$

Ex For the above vs V show

I, X is basis for V

Sol. Show I, X span V

element

$a_0, a_1, a_2 \ldots$

of V is equal to

$$a_0 I + (a_1 - a_0) X$$

One checks I, X lin indep, hence form basis for V $\square$