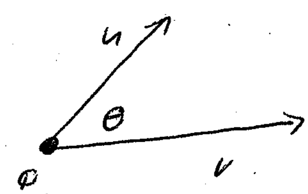
Lec 22 Friday March 14

4.6 Orthogonal vectors in $\mathbb{R}^n$

Motivation Given nonzero vectors in $\mathbb{R}^3$, View as arrows



The vectors u, v create angle θ.

u, v are orthogonal whenever $\theta = 90°$

We now generalize these ideas to $\mathbb{R}^n$

Def Given vectors u, v in $\mathbb{R}^n$:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

define

$$u \cdot v = u^t v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

"dot product of u, v"

Ex $\quad n=3 \quad$ For

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad v = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$u \cdot v = 1 \cdot 4 + 2 \cdot 1 + 3 \cdot 1 = 9$$

$\square$

Properties of dot product in $\mathbb{R}^n$

$$u \cdot v = v \cdot u$$

$$u \cdot (v+w) = u \cdot v + u \cdot w \qquad u, v, w \in \mathbb{R}^n$$

$$(cu) \cdot v = c(u \cdot v) \qquad c \in \mathbb{R}$$

$$u \cdot u \geq 0$$

$u = 0$ if and only if $u = 0$

Generalize this to arbitrary vector spaces:

Def Given a vector space V

By an <u>inner product</u> on V we mean a function $\langle \, , \rangle$ that sends each pair of vectors u, v to a scalar $\langle u, v \rangle$ such that the following axioms hold

- $\langle u, v \rangle = \langle v, u \rangle$
- $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$ $\quad u, v, w \in V$, $c \in \mathbb{R}$
- $\langle cu, v \rangle = c\langle u, v \rangle$
- $\langle u, u \rangle \geq 0$
- $\langle u, u \rangle = 0$ if and only if $u = 0$

Ex The dot product is an example of an inner product on $\mathbb{R}^n$

In what follows we work with dotproduct in $\mathbb{R}^n$

<u>Next general goal</u>: use dot product to define angle between two vectors.

Given a vector in $\mathbb{R}^n$

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

Define

$$|u| = \sqrt{u \cdot u}$$ "length of u"

↑
pos square root

So

$$|u| \geq 0$$

$|u| = 0$ if and only if $u = 0$

<u>thm</u> (Cauchy-Schwarz inequality)

Given vectors u, v in $\mathbb{R}^n$

$$|u \cdot v| \leq |u|\,|v| \qquad \star$$

ordinary absolute value   length

<u>pf</u> <u>Case</u> $u = 0$:

Then both sides of $\star$ are 0 ✓

<u>Case</u> $u \neq 0$:

For all $x \in \mathbb{R}$

$$0 \leq (xu + v) \cdot (xu + v)$$

$$= \underbrace{u \cdot u}_{= a}\, x^2 + \underbrace{2\; u \cdot v}_{= b}\, x + \underbrace{v \cdot v}_{= c}$$

$$= ax^2 + bx + c$$

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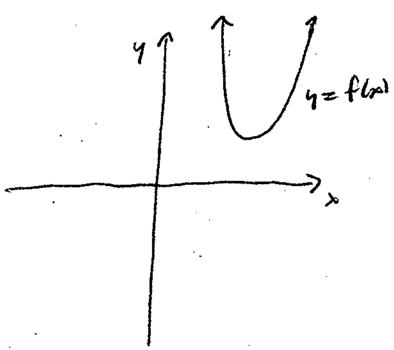
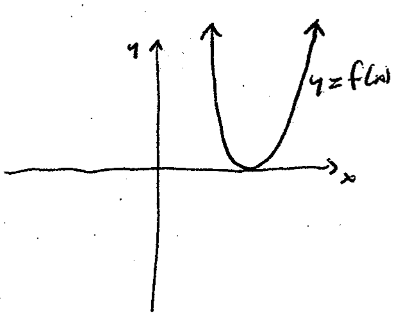
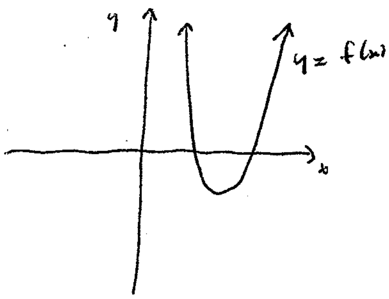
Consider the function

$$f(x) = ax^2 + bx + c$$

Graph of $f(x)$ does not fall below the x-axis

Cases	Allowed ?
	yes
	yes
	NO

So $f(x)$ does not have 2 distinct zeros.

the zeros of $f(x)$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(quadratic formula)

($a \neq 0$ since $u \neq 0$)

Require

$$b^2 - 4ac \leq 0$$

So

$$b^2 \leq 4ac$$

$$(2\, u \cdot v)^2 \leq 4\,(u \cdot u)(v \cdot v)$$

So

$$(u \cdot v)^2 \leq (u \cdot u)(v \cdot v)$$

take sq roots:

$$|u \cdot v| \leq |u|\,|v| \qquad \square$$

Given non0 vectors u, v in $\mathbb{R}^n$

We now define angle between u, v.

Obs

$$|u \cdot v| \leq |u| |v|$$

so

$$\frac{|u \cdot v|}{|u| \, |v|} \leq 1$$

so

$$-1 \leq \frac{u \cdot v}{|u| \, |v|} \leq 1$$

there exists a unique angle θ

$$0 \leq \theta \leq \pi$$

such that

$$\cos \theta = \frac{u \cdot v}{|u| \, |v|}$$

Call θ the <u>angle between u, v</u>

By constr

$$u \cdot v = |u|\,|v| \cos\theta$$

Special Case:

$$\theta = \frac{\pi}{2} \iff \cos\theta = 0 \iff u \cdot v = 0$$

Def vectors u, v in $\mathbb{R}^n$ are called <u>orthogonal</u>

whenever $u \cdot v = 0$

thm Given non0 vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$

Assume any 2 are orthogonal

then $v_1, v_2, \ldots, v_k$ are linearly independent.

pf Given scalars $c_1, c_2, \ldots c_k$ such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \mathbb{0}$$

show $c_i = 0 \quad (1 \le i \le k)$

Obs

$$0 = v_i \circ \mathbb{0}$$

$$= v_i \circ \left(c_1 v_1 + \cdots + c_k v_k \right)$$

$$= c_1 v_i \circ v_1 + c_2 v_i \circ v_2 + \cdots + c_k v_i \circ v_k$$

$v_i \circ v_j = 0$ if $i \neq j$

$$= c_i \, v_i \circ v_i$$

$v_i \circ v_i \neq 0$ since $v_i \neq 0$

So

$$c_i = 0$$

$\square$

Def Given a subspace V of $\mathbb{R}^n$

Define

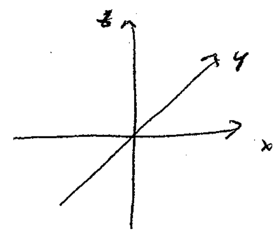
$V^\perp$ = set of vectors in $\mathbb{R}^n$ that are orthogonal to each vector in V

$$= \left\{ u \in \mathbb{R}^n \;\middle|\; u \cdot v = 0 \text{ for all } v \in V \right\}$$

" orthogonal complement of V in $\mathbb{R}^n$ "

One checks $V^\perp$ is a subspace of $\mathbb{R}^n$

Ex $\mathbb{R}^3$



take $V = \text{span of } \begin{pmatrix}1\\0\\0\end{pmatrix} = x\text{-axis}$

$V^\perp = \text{span of } \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

= yz plane

obs

$$\underset{1}{\underset{\parallel}{\dim V}} + \underset{2}{\underset{\parallel}{\dim(V^\perp)}} = \underset{3}{\underset{\parallel}{\dim(\mathbb{R}^3)}}$$

Next goal: show that for any subspace V of $\mathbb{R}^n$,

$$\dim V + \dim(V^\perp) = n$$

— o —

Ex Given any matrix A, say

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Define $V = Row(A)$

Find $V^\perp$

Sol. Given vector in $V^\perp$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Require this vector is orthog to each row of A

$$1x + 2y + 3z = 0$$
$$4x + 5y + 6z = 0$$

$V^\perp$ = solution space of A
= $Null(A)$

thm Given $m \times n$ matrix A, define

$$V = \text{Row}(A)$$

then

$$V^{\perp} = \text{Null}(A)$$

thm Given a subspace V of $\mathbb{R}^n$,

$$\underbrace{\dim V}_{= m} + \dim(V^{\perp}) = n$$

pf Pick a basis for V:

$$v_1, v_2, \ldots, v_m$$

Define an $m \times n$ matrix A such that for $1 \le i \le m$,

$$i\text{th row vector of } A = v_i$$

So $\text{Row}(A) = \text{Span}(v_1, v_2, \ldots, v_m) = V$

So $V^{\perp} = \text{Null}(A)$

Apply Gauss-Jordan

$$A \xrightarrow{GJ} E \quad (\text{ech form})$$

$$\dim V = \dim \text{Row}(A) = \# \text{ pivot cols of } E$$

$$\dim(V^{\perp}) = \dim \text{Null}(A) = \# \text{ non pivot cols of } E$$

$$= n - \# \text{ pivot cols in } E$$

$$= n - \dim(V)$$

$\square$

thm (Triangle inequality)

Given $u, v \in \mathbb{R}^n$

$$|u+v| \le |u| + |v| \qquad *$$

pf

$$|u+v|^2 = (u+v)\cdot(u+v)$$

$$= \underset{\|}{u\cdot u} + 2\underset{\wedge|}{\underbrace{u\cdot v}} + \underset{\|}{v\cdot v}$$

$$\quad |u|^2 \qquad |u||v| \qquad |v|^2$$

$$\le |u|^2 + 2|u||v| + |v|^2$$

$$= (|u| + |v|)^2$$

Take square root of each side to get * $\square$