Lec 21 Wednesday March 12 3/12/14

4.5 Row and Column spaces

Given $m \times n$ matrix A

For $1 \le i \le m$ let

$r_i =$ ith row of A "ith row vector of A"

view $r_i \in \mathbb{R}^n$ (by taking transpose)

Ex $m = 2$ $n = 3$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$r_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad r_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad r_1, r_2 \in \mathbb{R}^3$$

Def For an $m \times n$ matrix A, define

Row(A) = subspace of $\mathbb{R}^n$ spanned by the row vectors $r_1, r_2, \ldots r_m$ of A. "Row space of A"

the <u>row rank</u> of A is

the dimension of Row(A)

<u>Ex</u> Find a basis for the row space of

$$A = \begin{bmatrix} 2 & 5 & 2 \\ 1 & 5 & -9 \\ 1 & 2 & 3 \end{bmatrix}$$

<u>Sol</u> Apply GJ to put A in row echelon form.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & -9 \\ 2 & 5 & 2 \end{bmatrix} \quad r_1 \leftrightarrow r_2$$

this elem row op does not change the rowspace

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & -12 \\ 0 & 1 & -4 \end{bmatrix} \quad \begin{matrix} r_2' = r_2 - r_1 \\ r_3' = r_3 - 2r_1 \end{matrix}$$

these elem row ops do not change the rowspace

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 1 & -4 \end{bmatrix} \quad r_2' = \frac{1}{3} r_2$$

this elem row op does not change the row space

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \quad r_3' = r_3 - r_2$$

"Ech Form"

By constr

- Row (A) = Row (E)
- Nonzero row vectors for E are lin indep and hence form a basis for Row (E)

Therefore

Nonzero row vectors for E form a basis for Row(A):

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}$$

the row rank of A is 2 $\square$

Given $m \times n$ matrix A

- elementary row ops do not change the row space
- Any matrix B that is row equivalent to A has
  $\mathrm{Row}(A) = \mathrm{Row}(B)$
- $A \xrightarrow{GJ} E$ (ech form)

  $\mathrm{Row}(A) = \mathrm{Row}(E)$
- Non zero row vectors for E form a basis for $\mathrm{Row}(E) = \mathrm{Row}(A)$
- # nonzero rows for E = row rank E = row rank A

Given $m \times n$ matrix A

For $1 \le i \le n$ let

$c_i =$ ith column of A "ith column vector for A"

$c_i \in \mathbb{R}^m$

Ex $m = 2$ $n = 3$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$c_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad c_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \qquad c_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Def For an $m \times n$ matrix A

$\mathrm{Col}(A)$ = subspace of $\mathbb{R}^m$ spanned by the col vectors $c_1, c_2, \dots, c_n$ of A

"column space of A"

The <u>column rank</u> of A is

the dimension of $\mathrm{Col}(A)$

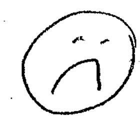
Next Goal

Given $m \times n$ matrix A

Find a basis for $\mathrm{Col}(A)$

Find column rank of A.

Sol I We could apply GJ to A^t to get a basis for $\mathrm{Row}(A^t) = \mathrm{Col}(A)$



Sol II We apply GJ to A

[this will show how col rank of A is related to row rank of A]

Ex Find a basis for the column space of

$$A = \begin{bmatrix} 2 & 5 & 2 \\ 1 & 5 & -9 \\ 1 & 2 & 3 \end{bmatrix}$$

Sol Apply GJ to put A in ech form

$$A = \begin{bmatrix} 2 & 5 & 2 \\ 1 & 5 & -9 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\Big\downarrow \text{GJ}$$

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Recall

$Null(A) = $ sol space of A

$= $ set of vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$ such that $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Recall elem row ops do not change the sol space

$Null(A) = Null(E)$ $\quad *$

"cols of A and cols of E satisfy the same linear dependencies"

Find basis for $*$

Backsolve

x, y leading variables

$z = t \qquad t$ free

$y = 4t$

$x = -2y - 3z$

$= -11t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -11 \\ 4 \\ 1 \end{bmatrix}$$

basis for $Null(A) = Null(E)$

$$E\begin{bmatrix} -11 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad A\begin{bmatrix} -11 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For E,

$$-11(\text{col } 1) + 4(\text{col } 2) + 1(\text{col } 3) = 0$$

col 1, col 2 are lin indep

So for A,

$$-11(\text{col } 1) + 4(\text{col } 2) + 1(\text{col } 3) = 0$$

col 1, col 2 are lin indep.

Therefore

col 1 of A, col 2 of A

is a basis for Col (A)

A has col rank 2

Comments

3/12/14

Consider

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \quad \nearrow$

pivot
cols

A column of E is called a pivot whenever it contains a leading 1.

the pivot cols are lin indep

the non pivot cols are linear combinations of the pivot cols

Cols of A and cols of E sat same linear dependencies

Col (A) has basis consisting of

cols of A that corresp to pivot cols of E

Given $m \times n$ matrix A

To find basis for $Col(A)$:

$$A \xrightarrow{GJ} E \quad \text{(ech form)}$$

$Col(A)$ has basis consisting of cols of A

that corresp to pivot cols of E

thm For an $m \times n$ matrix (A)

Row Rank of A = Col Rank of A

pf $A \xrightarrow{GJ} E$

Col Rank of A = $\dim Col(A)$

= # pivot cols for E

= # leading 1's for E

= # nonzero rows of E

= $\dim Row(E) = Row(A)$

= Row Rank A

$\square$

Ex For

$$A = \begin{bmatrix} 1 & 1 & -1 & 7 \\ 1 & 4 & 5 & 16 \\ 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \end{bmatrix}$$

Find bases for

Row (A), Null (A), Col (A)

Sol Apply GJ to A

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 3 & 6 & 9 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \begin{matrix} \\ r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \\ r_4' = r_4 - 2r_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \quad r_2' = \frac{1}{3} r_2$$

$$E = \begin{bmatrix} 1 & 1 & -1 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} r_3' = r_3 - 2r_2 \\ r_4' = r_4 - 3r_2 \end{array}$$

$\uparrow \quad \uparrow$

pivots

Row(A) = Row(E) :

basis consists of nonzero rows of E :

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 7 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

Null(A) = Null(E) :

Solve $A\underline{X} = 0$ $\qquad \underline{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$

x, y leading vars

$z = s \qquad w = t \qquad s, t$ free

$y = -2s - 3t$

$x = -y + z - 7w = 3s - 4t$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

basis for Null(A) = Null(E)

Col(A):

For A and E,

$3(\text{col}1) - 2(\text{col}2) + 1(\text{col}3) + 0(\text{col}4) = 0$

$-4(\text{col}1) - 3(\text{col}2) + 0(\text{col}3) + 1(\text{col}4) = 0$

col1, col2 lin indep.

So col1 of A, col2 of A is basis for Col(A)

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 4 \\ 3 \\ 5 \end{bmatrix}$$ is basis for Col(A)

$\square$