Lec 20 Monday March 10 3/10/14

4.4 Bases and Dimension for vector spaces

Given a vector space V

Given vectors in V:

$$v_1, v_2, \ldots, v_n$$

these vectors form a basis for V provided both

(i) $v_1, \ldots, v_n$ are linearly independent

(ii) $\text{Span}(v_1, \ldots, v_n) = V$

Example Recall the standard unit vectors in $\mathbb{R}^n$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow i \qquad 1 \le i \le n$$

then $e_1, e_2, \ldots, e_n$ form a basis for $\mathbb{R}^n$

"standard basis for $\mathbb{R}^n$"

the standard basis is not the only basis for $\mathbb{R}^n$. To see this, we extend a thm from Section 4.3.

<u>thm</u> Given n vectors in $\mathbb{R}^n$, say

$$v_1, v_2, \dots, v_n$$

Define an $n \times n$ matrix A such that for $1 \le i \le n$,

$$\text{col } i \text{ of } A = v_i$$

then the following are equivalent:

(i) $v_1, v_2, \dots, v_n$ are lin indep

(ii) A is invertible

(iii) $\text{span}(v_1, \dots, v_n) = \mathbb{R}^n$

(iv) $v_1, v_2, \dots, v_n$ is a basis for $\mathbb{R}^n$

pf (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) shown in Sec 4.3

(i), (iii) $\rightarrow$ (iv) this is def of basis

(iv) $\rightarrow$ (i) By def of basis

$\square$

<u>Ex</u> For vector space $\mathbb{R}^3$ define

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

Determine if v_1, v_2, v_3 form a basis for $\mathbb{R}^3$

<u>Sol</u> Define matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\det A = 1 \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + 0$$

(col 1 cofactor exp)

$$= -2 + 4$$

$$= 2$$

$$\neq 0$$

A is invertible, so v_1, v_2, v_3 form basis for $\mathbb{R}^3$.

Ex Let V = subspace of $\mathbb{R}^3$

consisting of all vectors

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad x - 2y + 5z = 0$$

Find a basis for V.

Sol Given $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V$.

$$x - 2y + 5z = 0$$

View y, z as free

$$y = s \qquad z = t \qquad s, t \text{ free}$$

$$x = 2s - 5t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{= u} + t \underbrace{\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}}_{= v}$$

$\text{Span}(u, v) = V$

u, v lin indep since one is not a scalar mult of the other

So u, v is a basis for V

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Next goal: Given vector space V

show any two bases for V have the

same number of vectors.

<u>thm</u> 1 Given vector space V.

Given a basis for V, say

$$v_1, v_2, \ldots, v_n$$

then any set of more than n vectors in V is linearly dep.

<u>pf</u> Given vectors in V

$$w_1, w_2, \ldots, w_m \qquad m > n$$

Show these are lin dep

Find scalars $c_1, c_2, \ldots, c_m$ (not all 0) such that

$$c_1 w_1 + c_2 w_2 + \cdots + c_m w_m = 0 \qquad (*)$$

Write each w_i in terms of $v_1, \ldots, v_n$

$w_1 = a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_n$

$w_2 = a_{12}v_1 + a_{22}v_2 + \cdots + a_{n2}v_n$

$\cdots$

$w_m = a_{1m}v_1 + a_{2m}v_2 + \cdots + a_{nm}v_n$

$(*)$ becomes

$$\begin{aligned} &(a_{11}c_1 + a_{12}c_2 + \cdots \qquad)v_1 \\ + &(a_{21}c_1 + a_{22}c_2 + \cdots \qquad)v_2 \\ + & \\ &\cdots \\ + &(a_{n1}c_1 + a_{n2}c_2 + \cdots \qquad)v_n \end{aligned}$$

$$= 0$$

But $v_1 \ldots v_n$ is lin indep so each coef is 0:

$$\left.\begin{aligned} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1m}c_m &= 0 \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2m}c_m &= 0 \\ \cdots \\ a_{n1}c_1 + a_{n2}c_2 + \cdots + a_{nm}c_m &= 0 \end{aligned}\right\} \quad (**)$$

Linear system $(**)$ has more variables than equations, so there exists a nontrivial sol for $c_1, c_2, \ldots, c_m$ ✓

$\square$

<u>thm</u> 2 Given vectorspace V.
Then any two bases for V have the same number
of vectors.

<u>pf</u> Call the bases

$v_1, v_2, \ldots, v_n$

and

$w_1, w_2, \ldots, v_m$,

Show $n = m$.
Interchanging the bases if necc, WLOG $n \leq m$.
Suppose $n < m$. Then $w_1, \ldots, w_m$ are lin dep by prev thm,
cont.

So $n = m$ $\square$

Def Given a vector space V

By the <u>dimension</u> of V, we mean the number of vectors in any basis for V

Ex the dimension of $\mathbb{R}^n$ is n.

<u>Caution</u> For some vector spaces the dimension $= \infty$

Ex let V denote the vector space of all polynomials in the variable x

[So V contains

$1 + x - x^2$, $2x - x^3$, $1 + x^{10} + x^{20}$, ...]

One checks that

$$1, x, x^2, x^3, \ldots \qquad (*)$$

is a basis for V.

$(*)$ has ∞ vectors so $\dim V = \infty$

Ex Given linear system

$$x_1 - 3x_2 + 2x_3 - 4x_4 = 0$$

$$2x_1 - 5x_2 + 7x_3 - 3x_4 = 0$$

Find a basis for the solution space.

Sol Solve the system

$$\begin{bmatrix} 1 & -3 & 2 & -4 \\ 2 & -5 & 7 & -3 \end{bmatrix}$$ coef matrix

Apply GJ

$\downarrow$

$$\begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 3 & 5 \end{bmatrix}$$ $r_2' = r_2 - 2r_1$

$$\begin{bmatrix} 1 & 0 & 11 & 11 \\ 0 & 1 & 3 & 5 \end{bmatrix}$$ $r_1' = r_1 + 3r_2$

Back solve:

$x_3 = s$ $\quad x_4 = t$ $\quad s, t$ free

$$x_2 = -3s - 5t$$

$$x_1 = -11s - 11t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \underbrace{\begin{bmatrix} -11 \\ -3 \\ 1 \\ 0 \end{bmatrix}}_{= u} + t \underbrace{\begin{bmatrix} -11 \\ -5 \\ 0 \\ 1 \end{bmatrix}}_{= v}$$

u, v span sol space ✓

u, v lin indep ✓

u, v is basis for sol space

□

Thm 3 Given a finite dimensional vector space V

Given a spanning set for V:

$$v_1, v_2, \ldots, v_n \qquad (*)$$

Then there exists a subset of $(*)$ that forms a basis for V

pf Suppose $(*)$ is lin indep. Then $(*)$ is basis for V done ✓

Suppose $(*)$ is lin dep.

So there exist scalars $c_1, c_2, \ldots, c_n$ (not all 0)

such that
$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$

There exists i $(1 \le i \le n)$ such that $c_i \neq 0$

Now
$$v_i \in \mathrm{Span}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$$

Now remove v_i from List $*$. Modified List

still spans V.

Iterate – procedure yields a basis for V $\square$

Ex For $\mathbb{R}^3$ define

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad v_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

let

$$V = \text{Span}(v_1, v_2, v_3, v_4)$$

Find a basis for V

Sol Find the linear dependencies among v_1, v_2, v_3, v_4:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

GJ $\downarrow$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{bmatrix} \quad \begin{matrix} r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad r_3' = r_3 - 2r_2$$

Backsolve

$c_3 = s \qquad c_4 = t \qquad s, t$ free

$c_2 = -s - t$

$c_1 = -s - 2t$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = s \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

each vector gives a dependency among v_1, v_2, v_3, v_4

$-v_1 - v_2 + v_3 = 0$ "discard v_3"

$-2v_1 - v_2 + v_4 = 0$ "discard v_4"

- Span$(v_1, v_2) = V$
- v_1, v_2 lin indep

v_1, v_2 basis for V $\square$

Thm Given vector space V with finite dim n

Given n vectors in V:

$v_1, \dots, v_n$.

Then the following are equiv

(i) $v_1, v_2, \dots, v_n$ are lin indep

(ii) $\mathrm{Span}(v_1, \dots, v_n) = V$

(iii) $v_1, \dots, v_n$ is a basis for V

pf (i) → (ii) Suppose $\mathrm{Span}(v_1, \dots, v_n) \neq V$

then $\exists$ vector in V that is not in $\mathrm{Span}(v_1, \dots, v_n)$

call it v_{n+1}

then $v_1, \dots, v_{n+1}$

are lin indep. This contradicts thm 1.

(ii) → (i) By thm 3 there exists

a subset of $v_1, \dots, v_n$ that is a basis for V. By thm 2

this subset consists of all of $v_1, v_2, \dots, v_n$.

(i), (ii) → (iii) def of basis

(iii) → (i) def of basis

$\square$

thm Given a finite dim'l vector space V

Given lin indep vectors in V: (*)

$$v_1, v_2, \ldots, v_n$$

then there exists a basis for V that contains (*)

pf Suppose (*) spans V. Then (*) is a basis for V, done.

Suppose (*) does not span V. Then there exists a vector in V that is not in

$$\mathrm{Span}(v_1, v_2, \ldots, v_n)$$

Call this vector v_{n+1}.

So $v_1, v_2, \ldots, v_{n+1}$ are lin indep.

Add v_{n+1} to *

Repeat (process ends by thm 1) to get a basis for V that contains * $\square$