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4.3 Linear combinations and Independence of vectors

Given a vector space V

Given vectors $v_1, v_2, \ldots, v_k$ in V

By a <u>Linear combination</u> of $v_1, v_2, \ldots, v_k$ we mean

a vector of the form

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \qquad c_1, c_2, \ldots, c_k \text{ scalars}$$

Def

$$\text{Span}(v_1, \ldots, v_k) = \text{set of all linear combinations of } v_1, v_2, \ldots, v_k$$

"the span of $v_1, \ldots, v_k$"

thm For $v_1, \dots, v_k$ as above

$$\underbrace{\mathrm{Span}(v_1, \dots v_k)}_{= W} \text{ is a subspace of } V$$

pf (i) Check W is closed under $+$

Given two vectors in W, say

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k,$$

$$c_1' v_1 + c_2' v_2 + \cdots + c_k' v_k$$

Obs

$$(c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) + (c_1' v_1 + c_2' v_2 + \cdots + c_k' v_k)$$

$$= (c_1 + c_1') v_1 + (c_2 + c_2') v_2 + \cdots + (c_k + c_k') v_k$$

$=$ lin comb of $v_1, v_2, \dots, v_k$

$\in W$

(ii) Check W is closed under sc mult

Given a vector in W, say

$$c_1 v_1 + \cdots + c_k v_k$$

Given scalar a

then $a(c_1 v_1 + \cdots + c_k v_k) = (ac_1) v_1 + (ac_2) v_2 + \cdots + (ac_k) v_k$

$=$ lin comb of $v_1, v_2, \dots, v_k$ $\in W$ ✓

Ex For the vector space $\mathbb{R}^n$

Define vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

"standard unit vectors"

Describe $\operatorname{Span}(e_1, e_2, \dots, e_n)$

Sol $\operatorname{Span}(e_1, e_2, \dots, e_n) = \mathbb{R}^n$

Since for any vector in $\mathbb{R}^n$ is a linear combin of $e_1, e_2, \dots, e_n$:

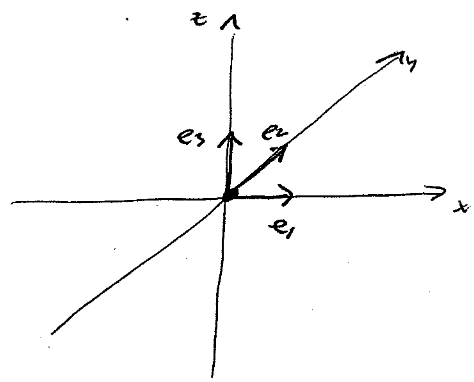
$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n$$

Ex Consider vector space $\mathbb{R}^3$

Consider

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

View



Describe $\quad \text{Span}(e_1, e_2)$

Sol For $v \in \mathbb{R}^3$

$v \in \text{Span}(e_1, e_2)$

$\iff$

$v = ae_1 + be_2 \qquad a, b \in \mathbb{R}$

$\iff$ $v = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$

$\iff$ coord 3 of v is 0

$\iff$ v is in xy-plane

$\text{Span}(e_1, e_2) = xy\text{-plane}$ $\square$

<u>Ex</u> Consider vector space $\mathbb{R}^4$

Define

$$v_1 = \begin{pmatrix} 7 \\ -6 \\ 4 \\ 5 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 3 \\ -3 \\ 2 \\ 3 \end{pmatrix}$$

Is

$$\underset{w}{\overset{\shortparallel}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}}} \quad \text{in } \operatorname{Span}(v_1, v_2) \quad ?$$

<u>Sol</u> Seek $a, b \in \mathbb{R}$ such that

$$av_1 + bv_2 = w$$

Require

$$a\begin{pmatrix} 7 \\ -6 \\ 4 \\ 5 \end{pmatrix} + b\begin{pmatrix} 3 \\ -3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Require

$$\begin{aligned} 7a + 3b &= 1 \\ -6a - 3b &= 0 \\ 4a + 2b &= 0 \\ 5a + 3b &= -1 \end{aligned}$$

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Aug matrix:

$$\left[\begin{array}{cc|c} 7 & 3 & 1 \\ -6 & -3 & 0 \\ 4 & 2 & 0 \\ 5 & 3 & 7 \end{array}\right]$$

GJ $\downarrow$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$a = 1, \quad b = -2$

So

$v_1 - 2v_2 = w$

$w \in \text{Span}(v_1, v_2)$ $\square$

Another view

LEM Given vectors $v_1, v_2, \dots, v_k$ in $\mathbb{R}^n$

Define a $n \times k$ matrix A:

For $1 \le i \le k$

$$\text{col } i \text{ of } A = v_i$$

Then for all $b \in \mathbb{R}^n$ the following are equiv:

(i) $b \in \text{Span}(v_1, v_2, \dots, v_k)$

(ii) the equation $A\mathbf{x} = b$ has at least one sol

pf obs $b \in \text{Span}(v_1, \dots, v_k)$

$\iff$ $b = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$ $\qquad c_1 \dots c_k \in \mathbb{R}$

$\iff$ $A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = b$ $\qquad c_1, c_2 \dots c_k \in \mathbb{R}$

$\square$

Linear dependence

Given a vector space V

Given vectors $v_1, v_2, \ldots, v_k$ in V

these vectors are called linearly dependent whenever

there exists scalars $c_1, c_2, \ldots, c_k$, (not all 0) such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

The vectors are called linearly independent whenever

they are not dependent.

<u>Ex</u> Recall standard unit vectors in $\mathbb{R}^n$

$$e_1, e_2, \ldots, e_n$$

Show these are lin indep.

<u>Sol</u> Given scalars $c_1, c_2, \ldots, c_n$ such that

$$c_1 e_1 + c_2 e_2 + \cdots + c_n e_n = \mathbb{0}$$

Show $c_1 = c_2 = \cdots = c_n = 0$

Obs

$$\mathbb{0} = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n$$

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} + \cdots + c_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

So $c_i = 0 \quad 1 \le i \le n$ ✓

<u>Ex</u> For vector space $\mathbb{R}^3$

Define
$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \quad v_3 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$$

Are v_1, v_2, v_3 lin indep?

<u>Sol</u> Seek scalars a, b, c (not all 0) such that

$$a v_1 + b v_2 + c v_3 = 0$$

Require
$$a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + c \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad a, b, c$$

Require
$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & 5 \\ 1 & 4 & 2 \end{bmatrix}}_{= A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad *$$

One checks $\det A = -7 \neq 0$ so

A is invertible

So only sol to $*$ is

$a = 0, \; b = 0, \; c = 0$

v_1, v_2, v_3 are lin indep ✓

$\square$

thm Given n vectors in $\mathbb{R}^n$, say

$v_1, v_2, \ldots, v_n$.

Define an $n \times n$ matrix A such that

for $1 \le i \le n$

$$\text{col } i \text{ of } A = v_i$$

Then the following are equivalent:

(i) $v_1, v_2, \ldots, v_n$ are lin indep

(ii) A is invertible

(iii) $\text{Span}(v_1, v_2, \ldots, v_n) = \mathbb{R}^n$

pf (i) $\Leftrightarrow$ (ii):

$v_1, v_2, \ldots, v_n$ lin indep

$\Longleftrightarrow$

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ has only triv sol $c_1 = c_2 = \cdots = c_n = 0$

$\Longleftrightarrow$ $A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$ has only triv sol $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$

$\Longleftrightarrow$ A is invertible

(ii) $\Leftrightarrow$ (iii)

A invertible

$\Longleftrightarrow$

the equation $A\underline{x} = b$ is consistent for all $b \in \mathbb{R}^n$

$\Longleftrightarrow$ $b \in \mathrm{Span}(v_1, v_2, \ldots, v_n)$ for all $b \in \mathbb{R}^n$

$\Longleftrightarrow$ $\mathrm{Span}(v_1, v_2, \ldots, v_n) = \mathbb{R}^n$

$\square$

thm Given a vector space V

Given linearly independent vectors in V:

$$v_1, v_2, \dots, v_k \qquad *$$

then any subset of $*$ is a lin indep set.

pf. Without loss of generality subset "

$$v_1, v_2, \dots, v_\ell \qquad (\ell \le k)$$

Suppose $v_1, v_2 \dots v_\ell$ lin dep

there exists scalars $c_1, c_2, \dots, c_\ell$ (not all 0) such that

$$c_1 v_1 + \dots + c_\ell v_\ell = 0$$

Define

$$c_i = 0 \qquad \ell+1 \le i \le k$$

obs

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0$$

and $c_1, c_2, \dots, c_k$ not all 0

so $*$ is lin dep, contradiction

therefore $v_1, v_2, \dots, v_\ell$ lin indep $\square$

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<u>thm</u> Given a vector space V

Given linearly dependent vectors in V: $\qquad *$

$v_1, v_2, \ldots, v_k$

Then any set of vectors in V that contains $*$

is linearly dependent.

pf very sim to prev thm $\square$

<u>Ex</u> Given vector space V

Given lin indep vectors in V,

$$v_1, v_2, v_3$$

Define

$$u_1 = v_1$$
$$u_2 = v_1 + 2v_2$$
$$u_3 = v_1 + 2v_2 + 3v_3$$

Det if u_1, u_2, u_3 are lin dep ?

<u>Sol</u> Seek scalars a, b, c (not all 0) such that

$$au_1 + bu_2 + cu_3 = 0$$

Require

$$av_1 + b(v_1 + 2v_2) + c(v_1 + 2v_2 + 3v_3) = 0$$

Require $(a + b + c)v_1 + (2b + 2c)v_2 + (3c)v_3 = 0$

Require

$$a + b + c = 0$$
$$2b + 2c = 0$$
$$3c = 0$$

(since v_1, v_2, v_3 are lin indep)

Require

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

↑ invertible

only sol is $a = b = c = 0$ so u_1, u_2, u_3 lin indep □