Lec 18 Wednesday March 5 3/5/14

4.2 The vector space $\mathbb{R}^n$ and subspaces

Earlier we discussed the vector space $\mathbb{R}^3$.

We now generalize

$$\mathbb{R}^3 \to \mathbb{R}^n$$

Def The vector space $\mathbb{R}^n$ consists of the set of column n-vectors $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ together with the operations of vector addition and scalar mult.

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Vector addition and scalar mult obey these axioms (routinely checked):

- $u + v = v + u$
- $u + (v + w) = (u + v) + w$
- $u + \mathbb{O} = \mathbb{O} + u = u$
- $u + (-u) = (-u) + u = \mathbb{O}$
- $a(u + v) = au + av$
- $(a + b)u = au + bu$
- $a(bu) = (ab)u$
- $(1)u = u$

For all u, v, w in $\mathbb{R}^n$ and scalars a, b

We use the above axioms to define an abstract vector space ...

Def A <u>vector space</u> (over $\mathbb{R}$)

is a set V (of vectors) together with a binary operation $+$ and a scalar mult, that obey the above axioms.

<u>Ex</u> Given pos integers m, n def

V = set of all $m \times n$ matrices with entries in $\mathbb{R}$.

Then V together with the usual matrix addition and sc mult is vector space.

<u>Ex</u> Let V = set of all functions $f: \mathbb{R} \to \mathbb{R}$ that are continuous everywhere.

Define $+$:

Given $f, g \in V$

the function $f+g$ sends $x \to f(x) + g(x)$

$f+g \in V$ since the sum of two contin functions is contin.

Def sc mult :

Given $f \in V$ Given scalar a

af sends $x \to a f(x)$

$af \in V$ since scalar mult of contin function is contin.

This gives Vector Space

<u>Ex</u> Given a vectorspace V

let W = a nonempty subset of V such that:

- Given any vectors u, v in W, then $u+v$ is in W.

  "closure under addition"

- Given any vector u in W, and given any scalar c, then cu is in W.

  "closure under scalar mult"

Then the set W, together with the addition operation and scalar mult inherited from V, form a vector space, called a <u>subspace of V</u>.

Examples of subspaces

Ex Given a vector space V.

Define

W = subset of V consisting of single element 0

Then W is a subspace of V

"the zero subspace"

pf check W is closed under $+$:

$0 + 0 = 0$ ✓

check W is closed under sc mult:

$c(0) = 0$ for all $c \in \mathbb{R}$

Ex Given a vector space V

Define

$W = V$

Then W is a subspace of V ✓

Ex Given a vector space V

Given a vector $u \in V$

Define $W = \{ cu \mid c \in \mathbb{R} \}$ "set of all scalar multiples of u"

Then W is a subspace of V

pf check W is closed under $+$:

Given two vectors in W, say

$cu, \quad c'u$

then

$$cu + c'u = (c+c')u$$
$$= \text{sc multiple of } u$$
$$\in W \quad ✓$$

check W is closed under sc mult:

Given vector in W, say

cu

Given scalar a

$$a(cu) = (ac)u$$
$$= \text{scalar multiple of } u$$
$$\in W \quad ✓$$

Ex Given a vector space V

Given two vectors u, v in V

Define

$$W = \left\{ au + bv \;\middle|\; a, b \in \mathbb{R} \right\}$$

"set of all linear combinations of u and v"

"span of u and v"

Then W is a subspace of V

pf. Check W is closed under $+$:

Given two vectors in W, say

$au+bv$, $\quad a'u+b'v$

Then

$$(au+bv) + (a'u+b'v) = (a+a')u + (b+b')v$$
$$= \text{linear comb of } u, v$$
$$\in W \quad \checkmark$$

Check W is closed under scalar mult:

Given vector in W, say $au+bv$. Given scalar c

$$c(au+bv) = (ac)u + (bc)v$$
$$= \text{lin comb of } u, v$$
$$\in W \quad \checkmark$$

Ex. Given $m \times n$ matrix A.

Define

W = set of all solutions to $AX = 0$

Then W is a subspace of the vector space $\mathbb{R}^n$
"the solution space of A"
"the null space of A"

pf Check W is closed under $+$

Given two vectors in W, say u, v

So $Au = 0$, $Av = 0$

then

$$A(u+v) = \underbrace{Au}_{=0} + \underbrace{Av}_{=0}$$

$$= 0$$

so $u + v \in W$

Check W is closed under sc mult:

Given $u \in W$ so $Au = 0$

Given scalar c

$$A(cu) = c\underbrace{(Au)}_{=0}$$

$$= 0$$

so $cu \in W$ ✓

$\square$

Caution Given $m \times n$ matrix A

Given a NONZERO $b \in \mathbb{R}^m$

Define

$$W = \text{set of all solutions to } \quad AX = b$$

then W is NOT a subspace of $\mathbb{R}^n$.

Indeed W is not closed under $+$:

Given $u, v \in W$, so

$Au = b \qquad Av = b$

then $$A(u+v) = \underset{\overset{\parallel}{b}}{Au} + \underset{\overset{\parallel}{b}}{Av}$$

$$= 2b$$

$$\neq b$$

so $u+v \notin W$

Also W is not closed under sc mult.

Ex Describe the nullspace of

$$A = \begin{bmatrix} 1 & -4 & 1 & -4 \\ 1 & 2 & 1 & 8 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

Sol. Find all the sols to

$$A\mathbf{X} = \mathbf{0} \qquad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Apply GJ

$$\begin{bmatrix} 1 & -4 & 1 & -4 \\ 1 & 2 & 1 & 8 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 1 & -4 \\ 0 & 6 & 0 & 12 \\ 0 & 5 & 0 & 10 \end{bmatrix} \begin{matrix} \\ r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -4 & 1 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 5 & 0 & 10 \end{bmatrix} \quad r_2' = \frac{1}{6} r_2$$

$$\begin{bmatrix} 1 & -4 & 1 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad r_3' = r_3 - 5r_2$$

Backsolve

Leading vars : x, y

$z = s \qquad w = t \qquad s, t$ free

$y = -2t$

$x = 4y - z + 4w$

$= -s - 4t$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{= u} + t \underbrace{\begin{bmatrix} -4 \\ -2 \\ 0 \\ 1 \end{bmatrix}}_{= v}$$

Null space of $A = \{ su + tv \mid s, t \in \mathbb{R} \}$

= set of all lin combinations of u, v

Ex

Given a vector space V

Given two subspaces of V, denoted U, W

Show that the intersection

$$U \cap W$$

is a subspace of V

Sol. • Check $U \cap W$ is closed under $+$

Given two vectors in $U \cap W$, denoted u, v

So $u \in U$, $v \in U$, $u \in W$, $v \in W$

Show $u+v \in U \cap W$:

$u+v \in U$ (since U is closed under $+$)

$u+v \in W$ (.. W ..)

So $u+v \in U \cap W$ ✓

• Check $U \cap W$ is closed under sc mult

Given vector u in $U \cap W$, so

$u \in U$ $u \in W$

Given scalar c. Show

$cu \in U \cap W$:

$cu \in U$ (since U is closed under sc mult)

$cu \in W$ (.. W ..)

So $cu \in U \cap W$ ✓

Ex Describe the null space of

$$A = \begin{bmatrix} 1 & 5 & 1 & -8 \\ 2 & 5 & 0 & -5 \\ 2 & 7 & 1 & -9 \end{bmatrix}$$

Sol Find all the sols to

$$A\mathbb{X} = \mathbb{0} \qquad \mathbb{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Apply GJ

$$\begin{bmatrix} 1 & 5 & 1 & -8 \\ 2 & 5 & 0 & -5 \\ 2 & 7 & 1 & -9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 1 & -8 \\ 0 & -5 & -2 & 11 \\ 0 & -3 & -1 & 7 \end{bmatrix} \quad \begin{matrix} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 2r_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 5 & 1 & -8 \\ 0 & 1 & 0 & -3 \\ 0 & -3 & -1 & 7 \end{bmatrix} \quad r_2' = r_2 - 2r_3$$

$$\begin{bmatrix} 1 & 5 & 1 & -8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix} \quad r_3' = r_3 + 3r_2$$

$$\begin{bmatrix} 1 & 5 & 1 & -8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad r_3' = -r_3$$

Backsolve

x, y, z leading vars

$w = t$ $\quad$ t free

$z = -2t$

$y = 3t$

$x = -5y - z + 8w$

$= -5t$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t \underbrace{\begin{bmatrix} -5 \\ 3 \\ -2 \\ 1 \end{bmatrix}}_{u}$$

Null space of $A = \{ tu \mid t \in \mathbb{R} \}$

= set of scalar multiples of u

$\square$