Lec 17 Monday March 3 3/3/14

1

4.1 The vector space $\mathbb{R}^3$

We now turn our attention to vector spaces.

We start with the vector space $\mathbb{R}^3$.

Def the vector space $\mathbb{R}^3$ is the set

of all column 3-vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

together with the operations of

- scalar multiplication
- vector addition

the zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is often called the <u>origin</u>

often view a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ as an <u>arrow</u> from the

origin to the pt with coords x, y, z.

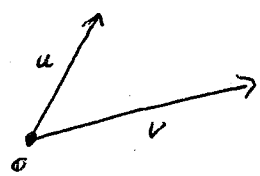
this vector has <u>length</u> (or <u>magnitude</u>) (or <u>norm</u>)

$$\sqrt{x^2 + y^2 + z^2}$$

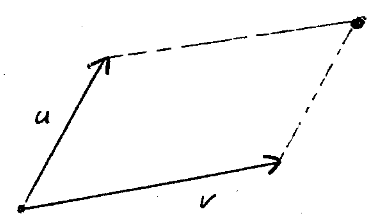
by Pythagorean thm

Geometric interpretation of vector addition

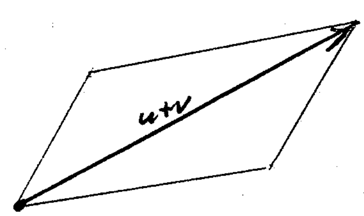
Given vectors $u, v \in \mathbb{R}^3$



To find $u+v$, consider parallelogram

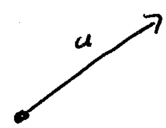


Then



Geometric interpretation of scalar multiplication

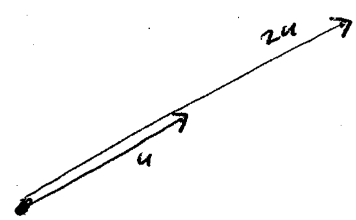
Given vector $u \in \mathbb{R}^3$



Given $\alpha \in \mathbb{R}$

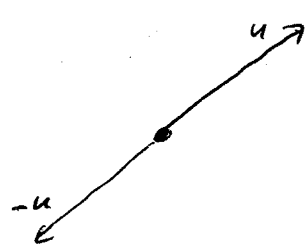
If $\alpha > 0$ the vector αu points in same direction with length α times the length of u

ex $\alpha = 2$



If $\alpha < 0$ the vector αu points in the opposite direction as u, with length $|\alpha|$ times the length of u.

ex $\alpha = -1$

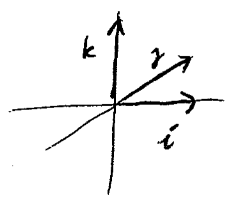


the vectors i, j, k

Def

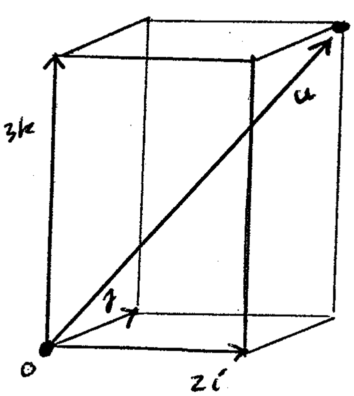
$$i = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad j = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad k = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\mathbb{R}^3$



Given any vector $u \in \mathbb{R}^3$, say $u = \begin{pmatrix}2\\1\\3\end{pmatrix}$

then $u = 2i + 1j + 3k$ "lin combination of i, j, k"



subspaces of $\mathbb{R}^3$

Consider these subsets of $\mathbb{R}^3$:

- the origin $\mathbb{O}$
- Any Line containing $\mathbb{O}$
- Any plane containing $\mathbb{O}$
- All of $\mathbb{R}^3$

What these subsets have in common:

(i) Given any vectors u, v in the subset, then $u+v$ is in the subset "closure under addition"

(ii) Given any vector u in the subset, and given any scalar c then cu is in the subset "closure under scalar mult"

Def A subspace of $\mathbb{R}^3$ is a nonempty subset of $\mathbb{R}^3$ that satisfies (i), (ii) above.

<u>Ex</u> Given $u, v \in \mathbb{R}^3$

$$u = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \qquad v = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$$

Define a subset

$$V = \left\{ au + bv \;\middle|\; a, b \in \mathbb{R} \right\}$$

"set of all linear combinations of u, v"
"span of u, v"

Show V is a subspace of $\mathbb{R}^3$

<u>Sol</u> (i) Show V is closed under $+$:

Given two vectors in V, say

$au+bv$, $a'u+b'v$

then

$$(au+bv) + (a'u+b'v) = (a+a')u + (b+b')v$$
$$= \text{linear comb of } u, v$$
$$\in V \quad \checkmark$$

(ii) Show V is closed under sc mult:

Given vector in V, say $au+bv$. Given scalar c

$$c(au+bv) = (ac)u + (bc)v$$
$$= \text{linear comb of } u, v$$
$$\in V \quad \checkmark$$

Obs V is the unique plane in $\mathbb{R}^3$

that contains the points

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\4\\3\end{pmatrix}, \quad \begin{pmatrix}-1\\-2\\2\end{pmatrix}$$

<u>Ex.</u> Referring to above Ex, is the vector

$$w = \begin{pmatrix}3\\10\\4\end{pmatrix}$$

contained in V?

<u>Sol</u> We seek $a, b \in \mathbb{R}$ such that

$$au + bv = w.$$

Require:

$$a\begin{pmatrix}1\\4\\3\end{pmatrix} + b\begin{pmatrix}-1\\-2\\2\end{pmatrix} = \begin{pmatrix}3\\10\\4\end{pmatrix}$$

Require:

$$\begin{aligned} a - b &= 3 \\ 4a - 2b &= 10 \\ 3a + 2b &= 4 \end{aligned}$$

Solve this system

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 4 & -2 & 10 \\ 3 & 2 & 4 \end{array}\right]$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 5 & -5 \end{bmatrix} \quad \begin{array}{l} r_2' = r_2 - 4r_1 \\ r_3' = r_3 - 3r_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 5 & -5 \end{bmatrix} \quad r_2' = \tfrac{1}{2} r_2$$

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \quad r_3' = r_3 - 5r_2$$

Backsolve

$$b = -1$$

$$a = 3 + b = 2$$

So $2u - v = w$

So $w \in V$ ✓

Ex Define V to be the set of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that

$$x+y+z=0$$

Show V is a subspace of $\mathbb{R}^3$.

Sol (i) Show V is closed under $+$:

Given two vectors in V, say

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$x+y+z=0$

$x'+y'+z'=0$

obs

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x+x' \\ y+y' \\ z+z' \end{pmatrix}$$

$$\begin{aligned} (x+x') + (y+y') + (z+z') &= (x+y+z) + (x'+y'+z') \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

so

$$\begin{pmatrix} x+x' \\ y+y' \\ z+z' \end{pmatrix} \in V \qquad \checkmark$$

3/3/14

10

(ii) Show V is closed under sc mult:

Given vector in V, say

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad\qquad x+y+z=0$$

Given scalar c

obs
$$c\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix}$$

$$\begin{aligned} cx + cy + cz &= c(x+y+z) \\ &= c(0) \\ &= 0 \end{aligned}$$

so $\quad c\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V \quad$ ✓

<u>Ex.</u> Referring to prev ex, find two vectors u, v that span V

<u>Sol.</u> Consider a generic vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V$$

$$x + y + z = 0$$

View

$$y = s, \qquad z = t \qquad\qquad s, t \text{ free}$$

$$x = -s - t$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{u} + t \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{v}$$

$$V = \left\{ su + tv \;\middle|\; s, t \in \mathbb{R} \right\}$$

"V is spanned by u, v"

Def Given three vectors u, v, w in $\mathbb{R}^3$.

Call them <u>linearly dependent</u> whenever

there exists scalars a, b, c (not all 0) such that

$$au + bv + cw = 0$$

Call them <u>linearly independent</u> whenever they

are not dependent.

<u>Geometric meaning of linear independence</u>: none of

u, v, w is in the subspace spanned by the other two

Ex Given $u, v, w \in \mathbb{R}^3$:

$$u = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \qquad v = \begin{pmatrix} -2 \\ -1 \\ 6 \end{pmatrix} \qquad w = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$$

Determine if u, v, w are lin dependent

Sol I We seek $a, b, c \in \mathbb{R}$, not all 0, such that

$$au + bv + cw = 0$$

Require
$$a\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + b\begin{pmatrix} -2 \\ -1 \\ 6 \end{pmatrix} + c\begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Require
$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 7 \\ -2 & 6 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (*)$$

Apply GJ:

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 7 \\ -2 & 6 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{bmatrix} \quad \begin{matrix} r_2' = r_2 - r_1 \\ r_3' = r_3 + 2r_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad r_3' = r_3 - 2r_2$$

Backsolve

$c = t$ t free

$b = -4t$

$a = 2b - 3c$

$= -11t$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = t\begin{pmatrix} -11 \\ -4 \\ 1 \end{pmatrix}$$

So

$$-11u - 4v + 1w = 0$$

u, v, w are linearly dependent

Sol II Find det

$$\begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 7 \\ -2 & 6 & 2 \end{vmatrix}$$

(the det is 0)

So (*) has nonO solution for $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

So u, v, w are lin dep.

— o —

Given $u, v, w \in \mathbb{R}^3$

the following are equivalent:

- u, v, w are lin indep.
- Each vector in $\mathbb{R}^3$ can be written as

$$au + bv + cw \qquad a, b, c \in \mathbb{R}$$

We call u, v, w a <u>basis</u> for $\mathbb{R}^3$ whenever these conditions hold.

Ex. The vectors i, j, k form a basis for $\mathbb{R}^3$.