LEC 16 Wednesday Feb 26 2/26/14

3.6 Determinants (Loose ends)

Ex Find the determinant of

$$A = \begin{bmatrix} 1 & 6 & 2 \\ 7 & 3 & 8 \\ 1 & 6 & 2 \end{bmatrix}$$

Sol $\det(A) = 0$

Reason: the elem row op $r_1 \leftrightarrow r_3$ changes the sign of $\det(A)$

But this row op leaves A unchanged

So $\det(A) = -\det(A)$

So $\det(A) = 0$

- For any $n \times n$ matrix A, if A has 2 identical rows (or 2 identical cols) then $\det(A) = 0$.

Ex Given 3 matrices

$$A = \begin{bmatrix} a & b & c \\ 2 & 8 & 5 \\ 9 & 7 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} \alpha & \beta & \gamma \\ 2 & 8 & 5 \\ 9 & 7 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} a+\alpha & b+\beta & c+\gamma \\ 2 & 8 & 5 \\ 9 & 7 & 3 \end{bmatrix}$$

Find $\det C$ in terms of $\det A$, $\det B$

Sol Use cofactor expansion along row 1:

$$\det C = (a+\alpha)\begin{vmatrix} 8 & 5 \\ 7 & 3 \end{vmatrix} - (b+\beta)\begin{vmatrix} 2 & 5 \\ 9 & 3 \end{vmatrix} + (c+\gamma)\begin{vmatrix} 2 & 8 \\ 9 & 7 \end{vmatrix}$$

$$\det A = a\begin{vmatrix} 8 & 5 \\ 7 & 3 \end{vmatrix} - b\begin{vmatrix} 2 & 5 \\ 9 & 3 \end{vmatrix} + c\begin{vmatrix} 2 & 8 \\ 9 & 7 \end{vmatrix}$$

$$\det B = \alpha\begin{vmatrix} 8 & 5 \\ 7 & 3 \end{vmatrix} - \beta\begin{vmatrix} 2 & 5 \\ 9 & 3 \end{vmatrix} + \gamma\begin{vmatrix} 2 & 8 \\ 9 & 7 \end{vmatrix}$$

so $\det C = \det A + \det B$

- Given $n \times n$ matrices A, B, C

  Given integer i $(1 \le i \le n)$

  Assume:

  (i) row i of A + row i of B = row i of C

  (ii) For $1 \le j \le n$, $j \ne i$

  row j of A = row j of B = row j of C

  Then

  $$\det A + \det B = \det C$$

  [sim result holds if replace rows by cols]

2/26/14

Ex Given 2x2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$$

Find the transpose $(AB)^t$ in terms of A^t, B^t

Sol

$$AB = \begin{bmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{bmatrix}$$

$$(AB)^t = \begin{bmatrix} ar+bt & cr+dt \\ as+bu & cs+du \end{bmatrix}$$

$$B^t = \begin{bmatrix} r & t \\ s & u \end{bmatrix} \qquad A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$B^tA^t = \begin{bmatrix} ra+tb & rc+td \\ sa+ub & sc+ud \end{bmatrix} = (AB)^t$$

- For $n \times n$ matrices A, B $\quad (AB)^t = B^tA^t$

Ex Given $n \times n$ matrices E, B with E elementary. Verify that

$$\det(EB) = \det(E)\det(B)$$

Sol E corresponds to an elem row operation, denoted ero.

Apply ero:

$$\mathbb{I} \xrightarrow{\text{ero}} E$$

$$B \xrightarrow{\text{ero}} EB$$

Recall the effect of ero on the determinant:

Cases

ero	$\det E$	$\det(EB)$
$r_i \leftrightarrow r_j$	$-\det I = -1$	$-\det B = \det E \det B$ ✓
$r_i' = k r_i$	$k \det I = k$	$k \det B = \det E \det B$ ✓
$r_i' = r_i + k r_j$	$\det I = 1$	$\det B = \det E \det B$ ✓

— o —

Note Given $n \times n$ matrices A, B one can prove

$$\det(AB) = \det(A)\det(B)$$

Using the above Ex and the fact that each invertible matrix is a product of elem matrices.

Ex Find the inverse of

$$A = \begin{bmatrix} -3 & -2 & 3 \\ 0 & 3 & 2 \\ 2 & 3 & -5 \end{bmatrix}$$

Sol 1 $\quad [A \mid I] \xrightarrow{GJ} [I \mid A^{-1}]$

→ Sol 2 use

$$A^{-1} = \frac{\operatorname{adj} A}{\det A}$$

Find $\det A$:

use cofactor down col 1

$$\det A = -3 \underbrace{\begin{vmatrix} 3 & 2 \\ 3 & -5 \end{vmatrix}}_{\substack{-15-6 \\ = \\ -21}} - 0 + 2 \underbrace{\begin{vmatrix} -2 & 3 \\ 3 & 2 \end{vmatrix}}_{\substack{= \\ -4-9 \\ = \\ -13}}$$

$$= 63 - 26$$

$$= 37$$

Find adj A

adj A =

$$
\begin{bmatrix}
\begin{gathered} \begin{vmatrix} 3 & 2 \\ 3 & -5 \end{vmatrix} \\ = -15-6 \\ = -21 \end{gathered} &
\begin{gathered} -\begin{vmatrix} 0 & 2 \\ 2 & -5 \end{vmatrix} \\ = 4 \end{gathered} &
\begin{gathered} \begin{vmatrix} 0 & 3 \\ 2 & 3 \end{vmatrix} \\ = -6 \end{gathered} \\
\begin{gathered} -\begin{vmatrix} -2 & 3 \\ 3 & -5 \end{vmatrix} \\ = -(10-9) \\ = -1 \end{gathered} &
\begin{gathered} \begin{vmatrix} -3 & 3 \\ 2 & -5 \end{vmatrix} \\ = 15-6 \\ = 9 \end{gathered} &
\begin{gathered} -\begin{vmatrix} -3 & -2 \\ 2 & 3 \end{vmatrix} \\ = -(-9+4) \\ = 5 \end{gathered} \\
\begin{gathered} \begin{vmatrix} -2 & 3 \\ 3 & 2 \end{vmatrix} \\ = -4-9 \\ = -13 \end{gathered} &
\begin{gathered} -\begin{vmatrix} -3 & 3 \\ 0 & 2 \end{vmatrix} \\ = -(-6) \\ = 6 \end{gathered} &
\begin{gathered} \begin{vmatrix} -3 & -2 \\ 0 & 3 \end{vmatrix} \\ = -9 \end{gathered}
\end{bmatrix}^{t}
$$

$$\operatorname{adj} A = \begin{bmatrix} -21 & -1 & -13 \\ 4 & 9 & 6 \\ -6 & 5 & -9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{37}\begin{bmatrix} -21 & -1 & -13 \\ 4 & 9 & 6 \\ -6 & 5 & -9 \end{bmatrix}$$

$\square$

Ex Find determinant of

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

"Vandermond matrix"

Sol Apply elem row operations

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad \begin{matrix} r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \end{matrix}$$

[col 1 cofactor expansion]

$$= \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

Recall $b^2-a^2 = (b-a)(b+a)$

$$r_1' = \frac{1}{b-a} r_1$$

$$= (b-a) \begin{vmatrix} 1 & b+a \\ c-a & c^2-a^2 \end{vmatrix}$$

$$= (b-a)(c-a) \underbrace{\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}}_{\substack{\shortparallel \\ c+a-(b+a) \\ \shortparallel \\ c-b}}$$

$$= (b-a)(c-a)(c-b)$$

$\square$

Ex For

$$A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \qquad a,b,c \in \mathbb{R}$$

For what values of a,b,c is A invertible?

Sol Recall A is invertible if and only if

$$\det(A) \neq 0$$

We saw $\det(A) = (b-a)(c-a)(c-b)$

A is invertible if and only if a,b,c are mutually distinct

$\square$

__Ex__ For an invertible $n \times n$ matrix A, find the determinant of A^{-1} in terms of $\det(A)$.

__Sol__

$$A A^{-1} = \mathbb{I}$$

$$\det(AA^{-1}) = \det(A)\det(A^{-1})$$

$$\det(AA^{-1}) = \det \mathbb{I} = 1$$

so

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$\square$

Ex For

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix}$$

Find the det of A^{-2}.

Sol obs

$$A^{-2} = (A^{-1})^2 = (A^2)^{-1}$$

$$\det(A^{-2}) = \frac{1}{\det(A^2)}$$

$$\det(A^2) = \det(A)\det(A) = (\det(A))^2$$

Find $\det(A)$:

A is Vandermonde

$$\det(A) = \underbrace{(3-2)}_{=1}\underbrace{(5-2)}_{=3}\underbrace{(5-3)}_{=2}$$

$$= 6$$

$$\det(A^{-2}) = \frac{1}{36}$$

$\square$

Ex Find det of

$$\begin{bmatrix} 3 & 2 & 5 \\ 6 & 6 & 6 \\ 9 & 4 & 25 \end{bmatrix}$$

Sol Reduce to Vandermonde

$$\begin{vmatrix} 3 & 2 & 5 \\ 6 & 6 & 6 \\ 9 & 4 & 25 \end{vmatrix}$$

$(r_2' = \frac{1}{6} r_2)$

$$= 6 \begin{vmatrix} 3 & 2 & 5 \\ 1 & 1 & 1 \\ 9 & 4 & 25 \end{vmatrix}$$

$(r_1 \leftrightarrow r_2)$

$$= -6 \begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 5 \\ 9 & 4 & 25 \end{vmatrix}$$

take trans

$$= -6 \begin{vmatrix} 1 & 3 & 9 \\ 1 & 2 & 4 \\ 1 & 5 & 25 \end{vmatrix}$$

$$\underbrace{(2-3)}_{-1}\underbrace{(5-3)}_{2}\underbrace{(5-2)}_{3} = -6$$

$= 36$

$\square$