Lec 15 Monday Feb 24 2/24/14

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3.6 determinants

Given an $n \times n$ matrix A,

we defined some $n \times n$ matrices:

- the matrix of minors
- ... cofactors
- the adjoint of A

Ex $A = \begin{bmatrix} -5 & -2 & 2 \\ 1 & 5 & -3 \\ 5 & -3 & 1 \end{bmatrix}$

Find matrix of minors

to get (i,j)-entry M_{ij}, delete row i, col j of A and take det.

so $M_{31} = \begin{vmatrix} -2 & 2 \\ 5 & -3 \end{vmatrix} = 6 - 10 = -4$

$$\begin{bmatrix} -4 & 16 & -28 \\ 4 & -15 & 25 \\ -4 & 13 & -23 \end{bmatrix}$$

Find the matrix of cofactors

(i,j)-entry $A_{ij} = (-1)^{i+j} M_{ij}$

$$\begin{bmatrix} -4 & -16 & -28 \\ -4 & -15 & -25 \\ -4 & -13 & -23 \end{bmatrix}$$

Find adjoint of A

(i,j)-entry is A_{ji} "transpose of cofactor matrix"

$$\operatorname{adj}(A) = \begin{bmatrix} -4 & -4 & -4 \\ -16 & -15 & -13 \\ -28 & -25 & -23 \end{bmatrix}$$

Find the product $A \operatorname{adj}(A)$

$$\begin{bmatrix} -5 & -2 & 2 \\ 1 & 5 & -3 \\ 5 & -3 & 1 \end{bmatrix} \begin{bmatrix} -4 & -4 & -4 \\ -16 & -15 & -13 \\ -28 & -25 & -23 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= -4\mathbb{I}$$

"wow!"

Find the product $\operatorname{adj}(A) A$

the product is $-4\mathbb{I}$

— ∘ —

For any $n \times n$ matrix A, we now define the determinant $\det(A)$ so that both

$$A \operatorname{adj}(A) = \det(A)\mathbb{I}, \qquad (*)$$

$$\operatorname{adj}(A) A = \det(A)\mathbb{I} \qquad (**)$$

Use (*) to discover the def of det(A):

In each side of

$$\det(A)\,I = A \operatorname{adj}(A)$$

compute (1,1)-entry:

$$\det(A) = (\text{row } 1 \text{ of } A) \times (\text{col } 1 \text{ of } \operatorname{adj}(A))$$

[let a_{ij} = (i,j)-entry of A]

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{12} \\ \vdots \\ A_{1n} \end{bmatrix}$$

$$= a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}.$$

(def of det A)

Cofactor expansions of det A

Consider

$$\det(A)\,I = A\,\mathrm{adj}(A)$$

For $1 \le i \le n$ compute (i,i)-entry of each side.

Get

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

"cofactor expansion of det A along row i of A"

Consider

$$\det(A)\,I = \mathrm{adj}(A)\,A$$

For $1 \le j \le n$ compute (j,j)-entry of each side

Get

$$\det(A) = A_{1j}a_{1j} + A_{2j}a_{2j} + \cdots + A_{nj}a_{nj}$$

"cofactor expansion of det A along col j of A"

Using (*), (**) one finds

<u>thm</u> For an $n \times n$ matrix A

A is invertible if and only if $\det(A) \neq 0$.

In this case

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$$

Properties of determinants

We illustrate with 2×2 matrices.

take $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- Elementary row operations have nice effect on $\det A$:

(i) $\begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$ $r_1' = k r_1$ $\quad k \in \mathbb{R}$

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = (ka)d - (kb)c$$
$$= k(ad - bc)$$
$$= k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(ii) $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$ $r_1 \Leftrightarrow r_2$

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc)$$
$$= - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(iii)

$$\begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix} \quad r_1' = r_1 + kr_2$$

$$\begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} = (a+kc)d - (b+kd)c$$

$$= ad - bc$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- taking the transpose does not change the det:

$$A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

"transpose of A"

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- The det of a product equals the prod of the determinants.

take $B = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$

Show $\det(AB) = \det(A)\det(B)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{bmatrix}$$

$$\det(AB) = (ar+bt)(cs+du) - (as+bu)(cr+dt)$$

// one checks

$$\det A \det B = (ad-bc)(ru-st)$$

An $n \times n$ matrix A is called <u>upper triangular</u> whenever the (i,j)-entry is 0 for $1 \le j < i \le n$

ex

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} \qquad A = \begin{pmatrix} 3 & 1 & 6 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

A is called <u>lower triangular</u> whenever the transpose A^t is upper triangular

ex

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} \qquad A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 5 & 0 \\ 6 & 2 & 4 \end{pmatrix}$$

- For an $n \times n$ matrix A that is upper or lower triangular

$\det(A)$ = product of the entries on the main diagonal of A

Ex $n=3$

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

To find $\det A$, use cofactor expansion down col 1.

$$\det(A) = a \underbrace{\begin{vmatrix} d & e \\ 0 & f \end{vmatrix}}_{\substack{\shortparallel \\ df - 0e \\ \shortparallel \\ df}} + 0 + 0$$

$$= adf$$

Cramer's rule

Given $n \times n$ matrix A

Consider linear system

$$A\underline{X} = b \qquad (*)$$

where

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Assume A is invertible, so $(*)$ has unique sol.

Find the sol:

$$A\underline{X} = b$$

$$\underline{X} = A^{-1} b$$

$$= \frac{\operatorname{adj} A}{\det A} \, b$$

For $1 \le i \le n$

$$x_i = \frac{(\text{row } i \text{ adj}A) \times b}{\det(A)}$$

So

$$\det(A)\, x_i = (\text{row } i \text{ of adj}A) \times b$$

$$\det(A)\, x_i = \begin{bmatrix} A_{1i} & A_{2i} & \cdots & A_{ni} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}$$

= det of matrix obtained from A by replacing col i by b

so

$$x_i = \frac{\text{det of matrix obtained from } A \text{ by replacing col } i \text{ by } b}{\det(A)}$$

"Cramer's rule"

Ex Solve

$$\begin{bmatrix} 5 & 4 & -2 \\ 2 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$\overset{\shortparallel}{A} \qquad \overset{\shortparallel}{\underline{x}} \qquad \overset{\shortparallel}{b}$$

Using Cramer's rule

Find det(A) :

$$\begin{vmatrix} 5 & 4 & -2 \\ 2 & 0 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 13 & 0 & 2 \\ 2 & 0 & 3 \\ 2 & -1 & 1 \end{vmatrix} \quad r_1' = r_1 + 4r_3$$

$$= \begin{vmatrix} 13 & 2 \\ 2 & 3 \end{vmatrix} \qquad \text{(col 2 cofactor expansion)}$$

$$= 39 - 4$$

$$= 35$$

$$x = \frac{\begin{vmatrix} 4 & 4 & -2 \\ 2 & 0 & 3 \\ 1 & -1 & 1 \end{vmatrix}}{35} = \frac{20}{35} = \frac{4}{7}$$

$$y = \frac{\begin{vmatrix} 5 & 4 & -2 \\ 2 & 2 & 3 \\ 2 & 1 & 1 \end{vmatrix}}{35} = \frac{15}{35} = \frac{3}{7}$$

$$z = \frac{\begin{vmatrix} 5 & 4 & 4 \\ 2 & 0 & 2 \\ 2 & -1 & 1 \end{vmatrix}}{35} = \frac{10}{35} = \frac{2}{7}$$

Check:

$$\begin{bmatrix} 5 & 4 & -2 \\ 2 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \frac{1}{7} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \checkmark$$