Lec 14 Friday Feb 21

3.5 (inverses) + 3.6 (determinants)

We continue to discuss elementary matrices

Recall

- Each elementary row operation for $n \times n$ matrices corresponds to an $n \times n$ elementary matrix. To get the elementary matrix, apply the row op to I

- Each elementary matrix is invertible, and its inverse is an elementary matrix.

- Multiplying a matrix A on the left by an elementary matrix has the same effect as applying the corresponding elem row operation to A.

thm Given an $n \times n$ matrix A, the following are equivalent:

(i) A is row-equivalent to the $n \times n$ identity matrix $\mathbb{I}$

(ii) A is invertible

(iii) the equation $\underbrace{A\mathbb{X} = \mathbb{O}}$ has only the trivial solution $\mathbb{X} = \mathbb{O}$

column n-vectors

pf (i) → (ii) We apply Gauss-Jordan elimination to A and get $\mathbb{I}$

$$A \xrightarrow{ero_1} A_1 \xrightarrow{ero_2} A_2 \longrightarrow \cdots \xrightarrow{ero_r} A_r = \mathbb{I}$$

For $1 \le i \le r$ let

$$E_i = \text{elementary matrix for } ero_i$$

then

$$E_1 A = A_1$$

$$E_2 A_1 = A_2$$

$$\vdots$$

$$E_r A_{r-1} = A_r = \mathbb{I}$$

So

$$E_r E_{r-1} \cdots E_2 E_1 \; A = \mathbb{I}$$

invertible

In this equation multiply each side by

$$E_1^{-1} E_2^{-1} \cdots E_r^{-1} \qquad \text{(on left)}$$

and simplify

Get

$$A = E_1^{-1} E_2^{-1} \cdots E_r^{-1}$$

(product of invertible matrices)

So

A is invertible ✓

(ii) → (iii) If $A\mathbb{X} = \mathbb{O}$ then

$$\mathbb{X} = \mathbb{I}\mathbb{X} = (A^{-1}A)\mathbb{X} = A^{-1}(A\mathbb{X}) = A^{-1}\mathbb{O} = \mathbb{O}$$

(iii) → (i) If

$$A \xrightarrow{GJ} \begin{bmatrix} \ddots & & \\ 0 \; 0 & \cdots & 0 \end{bmatrix}$$

then the equation $A\mathbb{X} = \mathbb{O}$ has a nonleading variable, hence ∞ sols, cont.

So $A \xrightarrow{GJ} \mathbb{I}$

We now give some more characterizations of invertible matrices.

<u>thm</u> Given an $n \times n$ matrix A, the following are equivalent:

(a) A is invertible

(b) For all column n-vectors b the equation $A\mathbb{X} = b$ has a unique solution.

(c) For all column n-vectors b the equation $A\mathbb{X} = b$ is consistent (ie at least one solution)

pf (a) $\to$ (b) $\mathbb{X} = A^{-1}b$ is a solution. It is the only one, since for any col n-vector $\mathbb{X}$ with $A\mathbb{X} = b$

$$\mathbb{X} = \mathbb{I}\mathbb{X} = (A^{-1}A)\mathbb{X} = A^{-1}(A\mathbb{X}) = A^{-1}b$$

(b) $\to$ (c) clear

(c) $\to$ (a) For $1 \le i \le n$ define

e_i = column i of the $n \times n$ identity matrix $\mathbb{I}$

So

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$$

By assumption there exists a column n-vector X_i such that $A X_i = e_i$

Define an $n \times n$ matrix B such that

$$\text{col } i \text{ of } B = X_i \qquad (1 \le i \le n)$$

By construction

$$AB = I$$

We now claim B is invertible. This is because the equation $BX = 0$ has only the triv sol $X = 0$

Indeed $BX = 0$ implies

$$X = IX = (AB)X = A(BX) = A0 = 0$$

Now B is invertible by prev thm.

Now

$$A = AI = A(BB^{-1}) = (AB)B^{-1} = IB^{-1} = B^{-1}$$

↑ inv matrix

A is invertible ✓

Determinants

Def For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant of A is

$$ad - bc$$

Write

$$\det(A) \quad a \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

for the determinant of A.

Recall A is invertible if and only if $\det(A) \neq 0$. In this case

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Consider a generic 2×2 system in the variables x, y:

$$ax + by = r \qquad\qquad a,b,c,d,\ r,s \in \mathbb{R}$$
$$cx + dy = s$$

Matrix form:

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}$$

Assume A is invertible, so the system has unique sol.

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} r \\ s \end{pmatrix}$$

$$= \frac{1}{ab-cd}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$$

$$x = \frac{dr - bs}{ad-bc} = \frac{\begin{vmatrix} r & b \\ s & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$y = \frac{as - cr}{ad-bc} = \frac{\begin{vmatrix} a & r \\ c & s \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

"Cramer's rule for 2×2 systems"

Ex Use Cramer's rule to solve the system

$$5x + 6y = 12$$
$$3x + 4y = 6$$

Sol view

$$\begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

$$\begin{vmatrix} 5 & 6 \\ 3 & 4 \end{vmatrix} = 20 - 18 = 2$$

$$x = \frac{\begin{vmatrix} 12 & 6 \\ 6 & 4 \end{vmatrix}}{2} = \frac{48 - 36}{2} = \frac{12}{2} = 6$$

$$y = \frac{\begin{vmatrix} 5 & 12 \\ 3 & 6 \end{vmatrix}}{2} = \frac{30 - 36}{2} = \frac{-6}{2} = -3$$

check

$$5(6) + 6(-3) = 12 \checkmark$$
$$3(6) + 4(-3) = 6 \checkmark$$

Shortly we will give Cramer's rule for general $n \times n$ systems.

The determinant of an $n \times n$ matrix A is defined recursively:

n	$\det(A)$
2	✓
3	uses the det of 2×2 matrices
4	- - - 3×3 --
$\vdots$	
n	- - - $(n-1) \times (n-1)$ --

<u>Details</u>

Def Given an $n \times n$ matrix A

For $1 \le i, j \le n$ the (i,j)-<u>minor</u> M_{ij} of A is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and col j of A

The (i,j)-cofactor A_{ij} is

$$A_{ij} = (-1)^{i+j} M_{ij}$$

The $n \times n$ matrix of minors has (i,j)-entry

M_{ij} for $1 \le i,j \le n$.

The $n \times n$ cofactor matrix has (i,j)-entry

A_{ij} for $1 \le i,j \le n$.

The $n \times n$ adjoint matrix $\mathrm{adj}(A)$

has (i,j)-entry A_{ji} for $1 \le i,j \le n$

"transpose of cofactor matrix"

Ex Fn

$$A = \begin{bmatrix} -5 & -2 & 2 \\ 1 & 5 & -3 \\ 5 & -3 & 1 \end{bmatrix}$$

Find M_{11}

$$\begin{bmatrix} \cancel{-5} & \cancel{-2} & \cancel{2} \\ \cancel{1} & 5 & -3 \\ \cancel{5} & -3 & 1 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 5 & -3 \\ -3 & 1 \end{vmatrix} = 5 \cdot 1 - (-3)^2$$
$$= 5 - 9$$
$$= -4$$

Find A_{11}

$$A_{11} = (-1)^{1+1} M_{11}$$
$$= M_{11}$$
$$= -4$$

Find M_{23}

$$\begin{bmatrix} -5 & -2 & 2 \\ 1 & 5 & -3 \\ 5 & -3 & 1 \end{bmatrix}$$

$$M_{23} = \begin{vmatrix} -5 & -2 \\ 5 & -3 \end{vmatrix} = (-5)(-3) - (-2)5$$

$$= 15 + 10$$

$$= 25$$

Find A_{23}

$$A_{23} = (-1)^{2+3} M_{23}$$

$$= -M_{23}$$

$$= -25$$

Find the matrix of minors

$$\begin{bmatrix} -4 & 16 & -28 \\ 4 & -15 & 25 \\ -4 & 13 & -23 \end{bmatrix}$$

Find the cofactor matrix

$$\begin{bmatrix} -4 & -16 & -28 \\ -4 & -15 & -25 \\ -4 & -13 & -23 \end{bmatrix}$$

Find the adjoint of A

$$\operatorname{adj}(A) = \begin{bmatrix} -4 & -4 & -4 \\ -16 & -15 & -13 \\ -28 & -25 & -23 \end{bmatrix}$$

Find the product $A \operatorname{adj}(A)$

$$\begin{bmatrix} -5 & -2 & 2 \\ 1 & 5 & -3 \\ 5 & -3 & 1 \end{bmatrix}\begin{bmatrix} -4 & -4 & -4 \\ -16 & -15 & -13 \\ -28 & -25 & -23 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= -4\,I$$

As we will see, for any $n \times n$ matrix A

$$A \operatorname{adj}(A) = \det(A)\, I$$