Lec 13 Wednesday Feb 19 2/19/14

3.5 Inverses of Matrices

Given an $n \times n$ matrix A

__Observe__ there exists at most one $n \times n$ matrix B such that

$$AB = I \quad \text{and} \quad BA = I \qquad *$$

Reason: Suppose B, B' both satisfy $*$.

Then

$$B = BI = B(AB') = (BA)B' = IB' = B'$$

Def For an $n \times n$ matrix A

call A <u>invertible</u> whenever there exists an $n \times n$ matrix B such that

$$AB = \mathbb{I} \quad \text{and} \quad BA = \mathbb{I}$$

In this case, we call B the <u>inverse</u> of A

Write $$B = A^{-1}$$

By def.

$$AA^{-1} = \mathbb{I} \quad \text{and} \quad A^{-1}A = \mathbb{I}$$

<u>**Note**</u> Given $n \times n$ matrices A, B

Assume $$AB = \mathbb{I}$$

but don't assume in advance that $BA = \mathbb{I}$

then one can <u>prove</u>
$$BA = \mathbb{I}$$

so

A, B are invertible, and $A^{-1} = B$, $B^{-1} = A$

Ex Given

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$$

Find A^{-1} if it exists.

Sol. Find 2×2 matrix B such that

$$AB = I$$

(We did this in prev lecture)

$$B = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$$

Double check:

$$\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Also

$$\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

So

$$A^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$$

$\square$

For 2×2 matrices we now give a formula for the inverse.

Thm Given a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A is invertible if and only if

$$ad-bc \neq 0.$$

In this case

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Pf: First assume $ad-bc \neq 0$

Define
$$B = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

one checks
$$AB = I \quad \text{and} \quad BA = I.$$

Next assume $ad-bc = 0.$

Show A is not invertible:

Obs

$$\overset{B}{\overset{\parallel}{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}} \overset{A}{\overset{\parallel}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\substack{\parallel \\ \mathbb{O}}}$$

If A^{-1} exists then

$$\underbrace{BAA^{-1}}_{\substack{\parallel \\ B\mathbb{I} \\ \parallel \\ B}} = \underbrace{\mathbb{O}A^{-1}}_{\substack{\parallel \\ \mathbb{O}}}$$

$$B = \mathbb{O}$$

$$a = b = c = d = 0$$

$$A = \mathbb{O}$$

$$\begin{aligned} \mathbb{I} &= AA^{-1} \\ &= \mathbb{O}A^{-1} \\ &= \mathbb{O} \end{aligned}$$

Contradiction. So A^{-1} does not exist. $\square$

Given $n \times n$ matrix A

We now show how to find A^{-1} if it exists.

Illustrate with small example

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$$

Hunt for matrix B such that

$$AB = I$$

Write

$$B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Require

$$\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Require

$$3x + 4z = 1 \qquad 3y + 4w = 0$$
$$5x + 7z = 0 \qquad 5y + 7w = 1$$

For each system, put augmented matrix in Reduced Echelon Form:

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 5 & 7 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 3 & 4 & 0 \\ 5 & 7 & 1 \end{array}\right]$$

$\downarrow$ Gauss-Jordan $\downarrow$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -5 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 3 \end{array}\right]$$

For efficiency, do both reductions at once:

$$\left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 5 & 7 & 0 & 1 \end{array}\right]$$

$$\downarrow \quad \text{Gauss-Jordan}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -4 \\ 0 & 1 & -5 & 3 \end{array}\right]$$

In conclusion:

To find the inverse of an $n \times n$ matrix A,

consider the $n \times 2n$ matrix

$$\left[\, A \,\middle|\, I \,\right]$$

Apply Gauss-Jordan elimination to put this in REF

Either

Case I: $$REF = \left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & * \\ 0 & & 1 & \end{array}\right]$$

Case II: $$REF = \left[\begin{array}{cccc|c} & & & & \\ & & & & * \\ 0 & 0 & \cdots & 0 & \end{array}\right]$$

In Case I A is invertible and $REF = [\, I \,|\, A^{-1} \,]$

In Case II A is not invertible.

$\square$

<u>thm</u> Given invertible $n \times n$ matrices A, B

then the matrix AB is invertible, and

$$(AB)^{-1} = B^{-1} A^{-1}$$

pf One checks

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= A \, \mathbb{I} \, A^{-1} \\ &= (A \, \mathbb{I}) A^{-1} \\ &= A A^{-1} \\ &= \mathbb{I}\end{aligned}$$

and similarly

$$(B^{-1}A^{-1})(AB) = \mathbb{I}$$

$\square$

More generally

<u>thm</u> A product of invertible matrices is invertible.

We now consider a type of invertible matrix called an <u>elementary matrix</u>

<u>Def</u> An $n \times n$ matrix E is called <u>elementary</u> whenever it is obtained from the $n \times n$ identity matrix $\mathbb{I}$ by applying a single elem row operation

$$\mathbb{I} \xrightarrow{\text{elem row op}} E$$

Ex $n=3$

$$\mathbb{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Cases:

elem row op	E
$r_1' = 3r_1$	$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$r_1 \leftrightarrow r_2$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$r_2' = r_2 - 3r_1$	$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Obs Each elementary matrix is invertible, and its inverse is an elem matrix

ex

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

obs Multiplying a matrix A on the left by an elem matrix has the same effect as applying the corresp elem row operation to A

Ex $n = 3$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$A \xrightarrow{r_1' = 3r_1} \begin{bmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$A \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$A \xrightarrow{r_2' = r_2 - 3r_1} \begin{bmatrix} a & b & c \\ d-3a & e-3b & f-3c \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$