Lec 12 Monday Feb 17 2/17/14

1

3.4 Matrix operations

We now consider the matrix operations of addition, scaler mult, matrix mult

and their properties

Given two $m \times n$ matrices A & B

Write
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & & & \vdots \\ \vdots & & & \\ b_{m1} & & \cdots & b_{mn} \end{bmatrix}$$

Saying $A = B$ means

$a_{ij} = b_{ij}$ for $1 \le i \le m$, $1 \le j \le n$

By definition $A + B$ is the $m \times n$ matrix with (i,j)-entry

$a_{ij} + b_{ij}$

for $1 \le i \le m$ and $1 \le j \le n$

Note

$$A+B=B+A$$

By a <u>scalar</u> we mean an element of $\mathbb{R}$.

For a scalar $c \in \mathbb{R}$,

cA is the $m \times n$ matrix with (i,j)-entry

$$ca_{ij}$$

for $1 \le i \le m$, $1 \le j \le n$

A <u>linear combination</u> of A, B is any matrix of the form

$$c_1 A + c_2 B$$

where c_1, c_2 are scalars

Ex Given

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Is $\begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$ a linear combination of A, B ?

Sol. Hunt for scalars x, y such that

$$xA + yB = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$$

Require

$$x\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + y\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$$

$$\begin{bmatrix} x+y & 2x-y \\ 3x & 4x+y \end{bmatrix}$$

Require

$$x+y = 1$$
$$2x-y = 5$$
$$3x = 6$$
$$4x+y = 7$$

So

$$x = 2$$
$$y = 1 - x = -1$$

check

$$2 \cdot 2 - (-1) = 5$$
$$4 \cdot 2 - 1 = 7$$

So

$$\begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} = 2A - B$$

is a linear combin of A and B. $\square$

Matrix multiplication

For an integer $n \geq 1$,

a row vector of length n (a row n-vector) is

a $1 \times n$ matrix

$$A = (a_1, a_2, \dots, a_n)$$

A column vector of length n (a column n-vector) is

a $n \times 1$ matrix.

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

For A, B as above, then *product* AB

is the scalar

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Ex the product of $[1, 2, 3]$ and $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

is

$$1 \cdot 4 + 2 \cdot 1 + 3 \cdot 2 = 12$$

$\square$

Def

Given any matrices A, B

Requires:

$$\# \text{cols of } A = \# \text{rows of } B$$

(say A is $r \times s$, B is $s \times t$)

We define the matrix product AB to be

the $r \times t$ matrix with (i,j)-entry

$$(\text{row } i \text{ of } A) \text{ times } (\text{col } j \text{ of } B)$$

for $1 \le i \le r$ and $1 \le j \le t$

Ex Take

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 2x+3z & 2y+3w \\ 4x+5z & 4y+5w \end{bmatrix}$$

Ex For above A, B find BA

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} =$$

$$\begin{bmatrix} 2x+4y & 3x+5y \\ 2z+4w & 3z+5w \end{bmatrix}$$

Observe $AB \neq BA$ in general!

Ex Given

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Find all the 2×2 matrices B such that

$$AB = BA$$

Sol. Write

$$B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Find AB

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x+2z & y+2w \\ z & w \end{bmatrix}$$

)) require

Find BA

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & 2x+y \\ z & 2z+w \end{bmatrix}$$

Require

$$x + 2z = x$$

$$y + 2w = 2x+y$$

$$z = z$$

$$w = 2z + w$$

So $z = 0$, $w = x$

B must have form

$$\begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \qquad x, y \in \mathbb{R}$$

$\square$

Def the <u>zero matrix</u> (of any dimension) has all entries 0. Write $\mathbb{O}$ for zero matrix

Ex: Some zero matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>Note</u> for any matrix A,

$$A + \mathbb{O} = A.$$

Ex Given

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Find all the 2×2 matrices B such that

$$AB = \mathbb{O}$$

Sol Write

$$B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Require

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{c|c} z & w \\ \hline 0 & 0 \end{array}\right]$$

Require $z = 0$ and $w = 0$

B has form

$$B = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \qquad x, y \in \mathbb{R}$$

Ex Find all the vectors

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

such that

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Sol Require

$$x - 3y + 6w = 0$$
$$z + 9w = 0$$

Solve this system

x, z are leading vars

Write

$$y = s$$
$$w = t$$

s, t free

Backsolve

$$z = -9t$$

$$x = 3s - 6t$$

Write sol in vector form

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{= u} + t \underbrace{\begin{bmatrix} -6 \\ 0 \\ -9 \\ 1 \end{bmatrix}}_{= v}$$

The sol set consists of the linear combinations of u, v. $\square$

For all integers $n \geq 1$, recall $n \times n$ identity matrix

$$\mathbb{I} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

For any matrix A

$$A\,\mathbb{I} = A \qquad (\text{if } A \text{ has } n \text{ cols})$$

$$\mathbb{I}A = A \qquad (\text{if } A \text{ has } n \text{ rows})$$

Ex Given $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$

Find all the 2×2 matrices B such that

$$AB = I \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Sol Write

$$B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Require

$$\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3x+4z & 3y+4w \\ 5x+7z & 5y+7w \end{bmatrix}$$

Require

$$3x + 4z = 1 \qquad\qquad 3y + 4w = 0$$

$$5x + 7z = 0 \qquad\qquad 5y + 7w = 1$$

(solve this any method) (solve this any method)

$x = 7,\ y = -5$ $\qquad$ $y = -4$, $w = 3$

Unique sol is

$$B = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$$

$\square$

— o —

For an $n \times n$ matrix A and integer $r \geq 0$

define

$$A^r = \underbrace{A A \cdots A}_{r \text{ copies}}$$

So

$$A^2 = A A$$

$$A^1 = A$$

interp $A^0 = \mathbb{I}$ ($n \times n$ identity)

Ex Find all the 2×2 matrices of form

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

such that

$$A^2 = \mathbb{I}$$

Sol Require

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a^2 & ab+bc \\ 0 & c^2 \end{bmatrix}$$

Require $a^2 = 1$

$c^2 = 1$

$\underbrace{ab+bc}_{= b(a+c)} = 0$

Require $a = 1$ or -1

$c = 1$ or -1

$a = -c$ or $b = 0$

Sols for A are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix} \quad b \in \mathbb{R}$$

$\square$

Ex For a 2x2 matrix A

express A^2 as a linear combination of

A, I

Sol Write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

So

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{bmatrix}$$

Hunt for scalars x, y such that

$$A^2 = xA + yI$$

Require

$$\begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{bmatrix} = x\begin{bmatrix} a & b \\ c & d \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Take

$$x = a+d$$

$$y = bc-ad$$

$a+d$ is called the <u>trace</u> of A

$ad-bc$ is called the <u>determinant</u> of A

so for a 2×2 matrix A

$$A^2 = \mathrm{tr}(A)\,A - \det(A)\,\mathbb{I}$$ $\square$