Lec 11 Friday Feb 14 2/14/14

3.3 Reduced Row-Echelon Matrices

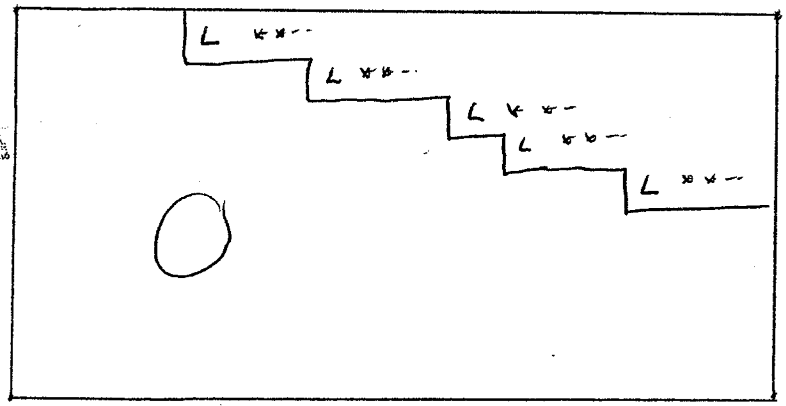
In the previous lecture we used Gaussian elimination to put a matrix in echelon form.

The echelon form is not unique.

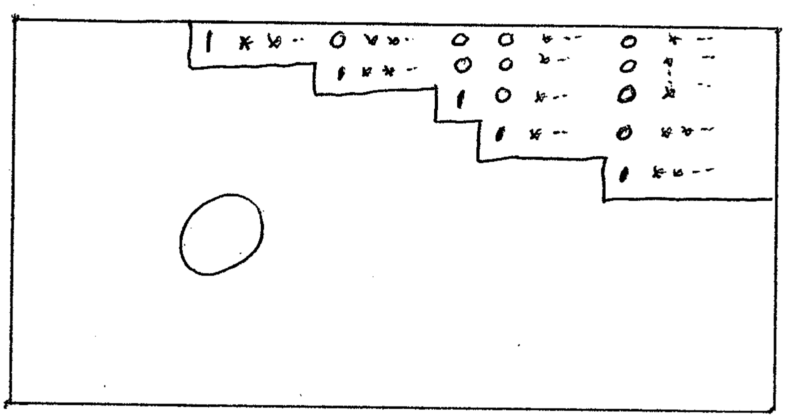
We now give an "improved" version of echelon form that is unique. This is the <u>reduced echelon form</u>

Comparison:

Echelon Form



Reduced Echelon Form



Echelon matrix is <u>reduced</u> whenever

- each leading entry is a $\underline{1}$
- For each column that contains a leading entry, all other entries are 0.

Ex Use elementary row operations put this echelon matrix in reduced form:

$$\begin{bmatrix} 1 & -4 & -7 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Sol

$$\begin{bmatrix} 1 & -4 & -7 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad r_3' = \frac{1}{5} r_3$$

$$\begin{bmatrix} 1 & 0 & 1 & 18 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad r_1' = r_1 + 4r_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{aligned} r_1' &= r_1 - 18 r_3 \\ r_2' &= r_2 - 3 r_3 \end{aligned}$$

(Reduced echelon form)

$\square$

To put a matrix in reduced echelon form:

"Gauss-Jordan elimination"

(i) Use Gaussian elimination to put it in echelon form.

(ii) Make each leading entry a "1" by multiplying its row by a constant

(iii) Use elem row ops to put 0's above each leading 1

Ex Use Gauss-Jordan elemination to put this matrix in reduced echelon form:

$$\begin{bmatrix} 1 & -2 & -4 & -13 & -8 \\ 1 & -2 & -2 & -5 & -2 \\ -2 & 4 & 8 & 26 & 16 \end{bmatrix}$$

Sol First put in ech form:

$$\begin{bmatrix} 1 & -2 & -4 & -13 & -8 \\ 0 & 0 & 2 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} r_2' = r_2 - r_1 \\ r_3' = r_3 + 2r_1 \end{matrix}$$

(ech form)

Make each leading entry a "1":

$$\begin{bmatrix} 1 & -2 & -4 & -13 & -8 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad r_2' = \tfrac{1}{2} r_2$$

put 0's above each leading 1:

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad r_1' = r_1 + 4r_2$$

(reduced echelon form)

□

Reduced ech form is useful for solving linear systems of equations

Ex Find the solution set for

$$\left.\begin{aligned} x - 2y - 4z - 13w &= -8 \\ x - 2y - 2z - 5w &= -2 \\ -2x + 4y + 8z + 26w &= 16 \end{aligned}\right\} \quad *$$

Sol. augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -2 & -4 & -13 & -8 \\ 1 & -2 & -2 & -5 & -2 \\ -2 & 4 & 8 & 26 & 16 \end{array}\right]$$

Put in reduced echelon form:

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cccc} x & y & z & w \\ \uparrow & & \uparrow & \end{array}$$

Leading variable.

A variable is <u>leading</u> if its column contains a leading entry

$$\begin{aligned} x - 2y \qquad\quad + 3w &= 4 \\ z + 4w &= 3 \end{aligned}$$

Backsolve

$w = t$ free

$z = 3 - 4t$

$y = s$ free

$x = 4 + 2s - 3t$

Sol set is:

$x = 4 + 2s - 3t$

$y = s$ $\qquad$ s, t free

$z = 3 - 4t$

$w = t$

Obs: • each non leading variable is free to take any value in $\mathbb{R}$

• Each leading variable is determined by the non leading variables.

the reduced ech form clarifies

why each linear system has either

unique sol

No sol

∞ many sols

Reason:

Given linear system

Apply Gauss Jordan elem to put augmented matrix in reduced ech form (REF)

3 cases

Case Every variable is leading

REF:

$$\left[\begin{array}{cccc|c} 1 & & & 0 & * \\ & 1 & & & * \\ & & \ddots & & \vdots \\ 0 & & & 1 & * \end{array}\right]$$

system has unique solution

Case rightmost column contains a leading 1

REF:

$$\left[\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & & & \vdots \\ & & & 0 \\ & & 0 & 1 \end{array}\right]$$

system has no solution

Case None of the above

The system is consistent.

There is at least one nonleading variable, which can take any value in $\mathbb{R}$.

System has ∞ many sols.

Homogeneous systems

An example of a homogeneous system:

$$\begin{aligned} x - 2y - 4z - 13w &= 0 \\ x - 2y - 2z - 5w &= 0 \\ -2x + 4y + 8z + 26w &= 0 \end{aligned}$$

For each equation, constant on right-hand side is 0

Obs A homogeneous system has at least one solution, for which all variables are 0

$x=0, \quad y=0, \quad z=0, \quad w=0$

thm A homogeneous system has a unique solution or ∞ many sols.

— ∘ —

Given a homogeneous system, with m eqs and n variables

Consider augmented matrix $m \times (n+1)$

Rightmost column has all entries 0

For efficiency, work with coef matrix A $(m \times n)$

Apply Gauss - Jordan elim to put A in REF

2 Cases

Case Every variable is leading

REF : $\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots & \\ & & & & 1 \end{bmatrix}$ "identity matrix"

system has unique sol, for which all variables are 0

In this case must have $m = n$

Case there exists a non leading variable

system has ∞ many sols

thm. Given a homogeneous system
with m equations and n variables

Assume $m < n$

then the system has ∞ many solutions

thm. Given a nonhomogeneous system
with m equations and n variables

Assume $m < n$.

then either

- system has no sol

or

- system has ∞ many sols

Reason: must have at least one nonleading variable.

Ex Find all values of c such that the homog system has ∞ many sols

$$(c+2)x + 3y = 0$$

$$2x + (c-3)y = 0$$

Sol Write coef matrix

$$\begin{bmatrix} c+2 & 3 \\ 2 & c-3 \end{bmatrix}$$

Apply Gauss-Jordan

$$\begin{bmatrix} 2 & c-3 \\ c+2 & 3 \end{bmatrix} \quad r_1 \leftrightarrow r_2$$

$$\begin{bmatrix} 1 & \frac{c-3}{2} \\ c+2 & 3 \end{bmatrix} \quad r_1' = \tfrac{1}{2} r_1$$

$$\begin{bmatrix} 1 & \frac{c-3}{2} \\ 0 & \frac{12+c-c^2}{2} \end{bmatrix} \quad r_2' = r_2 - (c+2) \cdot r_1$$

$$\frac{12+c-c^2}{2} = -\tfrac{1}{2}(c-4)(c+3)$$

Case $c = -3$ or 4 ∞ many sols

Case $c \neq -3$, $c \neq 4$ unique sol $x=0$, $y=0$

$\square$