Lec 10 Wednesday Feb 12

3.2 Matrices and Gaussian Elimination

In the previous lecture we solved systems of linear equations using elementary operations.

We now bring in matrices to do this more efficiently.

Consider the linear system

$$\left.\begin{aligned} 3x - 6y - 2z &= 1 \\ 2x - 4y + z &= 17 \\ x - 2y - 2z &= -9 \end{aligned}\right\} \ast$$

Define

$$A = \begin{bmatrix} 3 & -6 & -2 \\ 2 & -4 & 1 \\ 1 & -2 & -2 \end{bmatrix}$$

"coefficient matrix"

A has <u>shape</u> (or <u>size</u>) $\underset{\substack{\uparrow \\ \#\text{rows}}}{3} \times \underset{\substack{\uparrow \\ \#\text{cols}}}{3}$

For $1 \le i, j \le 3$ the <u>(i,j)-entry of A</u> is in row i and column j

So -6 is the $(1,2)$-entry of A

-4 ... $(2,2)$-entry of A

Define

$$b = \begin{bmatrix} 1 \\ 17 \\ -9 \end{bmatrix}$$

We view b as a 3×1 matrix or <u>column vector</u> of length 3

Combine A, b:

$$\left[\begin{array}{c|c} A & b \end{array}\right] = \begin{bmatrix} 3 & -6 & -2 & 1 \\ 2 & -4 & 1 & 17 \\ 1 & -2 & -2 & -9 \end{bmatrix}$$

"augmented matrix for $*$"

To solve $*$ we apply elem operations to put $*$ in triangular form.

For efficiency, we work with the corresponding augmented matrix.

Elementary operations become:

(i) multiply a row by a nonzero constant;

(ii) interchange two rows;

(iii) Add a constant multiple of a row to another row.

"elementary row operations"

$$\begin{bmatrix} 3 & -6 & -2 & 1 \\ 2 & -4 & 1 & 17 \\ 1 & -2 & -2 & -9 \end{bmatrix}$$

work on column 1

$$\begin{bmatrix} 1 & -2 & -2 & -9 \\ 2 & -4 & 1 & 17 \\ 3 & -6 & -2 & 1 \end{bmatrix} \quad r_1 \leftrightarrow r_3$$

$$\begin{bmatrix} 1 & -2 & -2 & -9 \\ 0 & 0 & 5 & 35 \\ 0 & 0 & 4 & 28 \end{bmatrix} \quad \begin{matrix} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 3r_1 \end{matrix}$$

"Gaussian elimination"

or

"row reduction"

column 1 ok

column 2: ok

colum 3:

$$\begin{bmatrix} 1 & -2 & -2 & -9 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 4 & 28 \end{bmatrix} \quad r_2' = \frac{1}{5} r_2$$

$$\begin{bmatrix} 1 & -2 & -2 & -9 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad r_3' = r_3 - 4r_2$$

this <u>echelon matrix</u> is the augmented matrix of a linear system in triangular form

Interpret as linear system:

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$$x \quad -2y \quad -2z = -9$$
$$z = 7$$

Backsolve:

$$z = 7$$
$$y = t \quad t \text{ free}$$
$$x = -9 + 2y + 2z$$
$$= -9 + 2t + 14$$
$$= 2t + 5$$

$*$ has sol set

$x = 2t+5, \quad y = t, \quad z = 7 \qquad t$ free

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Echelon matrices

Consider the echelon matrix

Leading entries

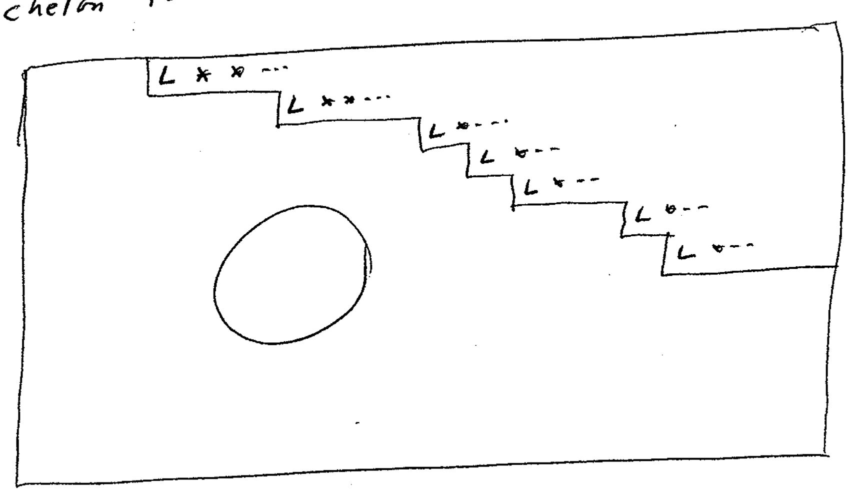
$$\begin{bmatrix} 1 & -2 & -2 & -9 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Features

- All zero rows are at the bottom
- For each nonzero row consider the left-most nonzero entry "leading entry"

Any leading entry below it lies strictly to its right.

Echelon form:



$*$: arbitrary

L : Nonzero

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Row equivalent matrices

Given $m \times n$ matrices A, B.

Call them row equivalent whenever we can get from one to the other by a sequence of elementary row operations.

View each of A, B as the augmented matrix of a linear system.

If A, B are row equivalent, then the corresponding linear systems have the same solution set.

Reason: An elementary row operation does not change the solution set of the corresp linear system.

Using Gaussian elimination, we see that each matrix is row equivalent to an echelon matrix

Ex Find an echelon matrix that is row equivalent to

$$\begin{bmatrix} 3 & 1 & -3 & 6 \\ 2 & 7 & 1 & -9 \\ 2 & 5 & 0 & -5 \end{bmatrix}$$

Sol. Apply Gaussian elimination

$$\begin{bmatrix} 1 & -6 & -4 & 15 \\ 2 & 7 & 1 & -9 \\ 2 & 5 & 0 & -5 \end{bmatrix} \quad r_1' = r_1 - r_2$$

work on col 1

$$\begin{bmatrix} 1 & -6 & -4 & 15 \\ 0 & 19 & 9 & -39 \\ 0 & 17 & 8 & -35 \end{bmatrix} \quad \begin{array}{l} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 2r_1 \end{array}$$

work on col 2

$$\begin{bmatrix} 1 & -6 & -4 & 15 \\ 0 & 2 & 1 & -4 \\ 0 & 17 & 8 & -35 \end{bmatrix} \quad r_2' = r_2 - r_3$$

$$\begin{bmatrix} 1 & -6 & -4 & 15 \\ 0 & 1 & \frac{1}{2} & -2 \\ 0 & 17 & 8 & -35 \end{bmatrix} \quad r_2' = \frac{1}{2} r_2$$

$$\begin{bmatrix} 1 & -6 & -4 & 15 \\ 0 & 1 & \frac{1}{2} & -2 \\ 0 & 0 & -\frac{1}{2} & -1 \end{bmatrix} \quad r_3' = r_3 - 17r_2$$

(echelon matrix)

We could also take another step

$$\begin{bmatrix} 1 & -6 & -4 & 15 \\ 0 & 1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad r_3' = -2r_3$$

(echelon matrix)

So the requested echelon matrix is not unique

Ex. Find the solution set for

$$\left.\begin{aligned} 2x + 8y + 3z &= 2 \\ x + 3y + 2z &= 5 \\ 2x + 7y + 4z &= 8 \end{aligned}\right\} *$$

Ans Put corresp augmented matrix in ech form

$$\begin{bmatrix} 2 & 8 & 3 & 2 \\ 1 & 3 & 2 & 5 \\ 2 & 7 & 4 & 8 \end{bmatrix}$$

work on col 1

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 2 & 8 & 3 & 2 \\ 2 & 7 & 4 & 8 \end{bmatrix} \quad r_1 \leftrightarrow r_2$$

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 2 & -1 & -8 \\ 0 & 1 & 0 & -2 \end{bmatrix} \quad \begin{aligned} r_2' &= r_2 - 2r_1 \\ r_3' &= r_3 - 2r_1 \end{aligned}$$

work on col 2

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 2 & -1 & -8 \end{bmatrix} \quad r_2 \leftrightarrow r_3$$

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -4 \end{bmatrix} \quad r_3' = r_3 - 2r_2$$

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad r_3' = -r_3$$

echelon form

Interpret as linear system:

$$\begin{aligned} x + 3y + 2z &= 5 \\ y &= -2 \\ z &= 4 \end{aligned}$$

Backsolve:

$$z = 4$$
$$y = -2$$
$$\begin{aligned} x &= 5 - 3y - 2z \\ &= 5 + 6 - 8 \\ &= 3 \end{aligned}$$

linear system $*$ has unique sol

$$x = 3, \quad y = -2, \quad z = 4$$

check:

$$2(3) + 8(-2) + 3(4) = 2 \quad \checkmark$$
$$1(3) + 3(-2) + 2(4) = 5 \quad \checkmark$$
$$2(3) + 7(-2) + 4(4) = 8 \quad \checkmark$$

Ex Pick $k \in \mathbb{R}$ and consider linear system

$$\left.\begin{aligned} 3x + 2y &= 11 \\ 6x + ky &= 21 \end{aligned}\right\} *$$

For what k does $*$ have

a) unique sol

b) No sol

c) ∞ many sols

Ans. Put corresp aug matrix in ech form

$$\begin{bmatrix} 3 & 2 & 11 \\ 6 & k & 21 \end{bmatrix}$$

col 1

$$\begin{bmatrix} 3 & 2 & 11 \\ 0 & k-4 & -1 \end{bmatrix} \quad r_2' = r_2 - 2r_1$$

<u>Case $k \neq 4$</u>

$$3x + 2y = 11$$
$$(k-4)y = -1$$

Backsolve
$$y = \frac{-1}{k-4}$$
$$x = \frac{1}{3}\left(11 + \frac{2}{k-4}\right)$$

unique sol

<u>Case $k = 4$</u>

ech matrix is

$$\begin{bmatrix} 3 & 2 & 11 \\ 0 & 0 & -1 \end{bmatrix}$$

$$3x + 2y = 11$$
$$0y = -1$$

No sol.

$\square$