

From $\mathcal{O}$ -Modules to $\boxtimes$ -Modules

Our goal in this lecture is to prove the following theorem.

Theorem (1.8)

Let V denote a finite dimensional irreducible $\mathcal{O}$ -module of type $(0,0)$. Then, there exists a unique $\boxtimes$ -module structure of V such that the standard generators X, Y act on V as X_{01}, X_{23} respectively. This $\boxtimes$ -module structure is irreducible.

Definition 7.1

Let V be a finite dimensional irreducible $\mathcal{O}$ -module of type $(0,0)$. Let A, A^* denote the actions of X, Y respectively. Let $\mathbb{I} = \{0, 1, 2, 3\}$ be a set of symbols. For each element of $\mathbb{I}$ we associate a flag of diameter d on V :

Element of $\mathbb{I}$	i^{th} component of assoc. flag
0	$V_A(-d) + \cdots + V_A(2i - d)$
1	$V_A(d) + \cdots + V_A(d - 2i)$
2	$V_{A^*}(-d) + \cdots + V_{A^*}(2i - d)$
3	$V_{A^*}(d) + \cdots + V_{A^*}(d - 2i)$

Lemma 7.2

The four flags from definition 7.1 are mutually opposite.

See *Leonard Pairs from 24 Points of View* by Paul Terwilliger (Theorem 7.3) for details on the above lemma.

Definition 7.3

For distinct $r, s \in \mathbb{I}$, let $x_{rs} : V \rightarrow V$ denote the unique linear map satisfying the following conditions. Let $U_0, \dots, U_d$ denote the induced decomposition of V by the flags associated with r, s as in Definition 7.1. For $0 \leq i \leq d$, U_i is an eigenspace for x_{rs} with eigenvalue $2i - d$.

Notation 7.4

We denote the λ -eigenspace of x_{rs} by $V_{rs}(\lambda)$.

Lemma 7.5

We have $x_{01} = A$, $x_{23} = A^*$

Proof. Immediate from Def. 7.3. □

Lemma 7.6

For $r \in \mathbb{I}$, consider the flag associated with r . Pick $s \in \mathbb{I}$ such that $r \neq s$. Then, for $0 \leq i \leq d$, the i^{th} component of the flag is

$$V_{rs}(-d) + \cdots + V_{rs}(2i - d).$$

Proof. The decomposition $V_{rs}(-d), \dots, V_{rs}(d)$ on V is induced by the flags associated with r and s (Def. 7.3). The result follows from the definitions of opposite and induced flags. $\square$

Lemma 7.7

For distinct $r, s \in \mathbb{I}$,

$$x_{rs} + x_{sr} = 0.$$

Proof. By Def. 7.3, we have $V_{rs}(2i - d) = V_{sr}(d - 2i)$. The result follows. $\square$

Lemma 7.8

For mutually distinct $r, s, t \in \mathbb{I}$, we have

$$[x_{rs}, x_{st}] = 2x_{rs} + 2x_{st}.$$

Proof. We consider 3 cases.

- (i) st is one of 01, 10, 23, 32,
- (ii) rs is one of 01, 10, 23, 32,
- (iii) Neither rs nor st is one of 01, 10, 23, 32.

Case (i): Lemma 3.1 tells us that

$$[x_{rs}, x_{st}] = 2x_{rs} + 2x_{st} \iff (x_{st} + (2i - d)I)V_{rs}(2i - d) \subseteq V_{st}(2i - d + 2) \quad (7.1)$$

for $0 \leq i \leq d$. Let $u \in \mathbb{I}$ be the unique element such that r, s, t, u are mutually distinct. Lemma 7.5 and 7.7 imply that x_{st}, x_{ur} are either $\epsilon A, \epsilon^* A^*$ or $\epsilon^* A^*, \epsilon A$ where $\epsilon, \epsilon^* \in \{1, -1\}$. It follows from this and Def. 7.1 that x_{st}, x_{ur} is a TD-pair on V with diameter d and $d, d - 2, \dots, -d$ for both the eigenvalue and dual eigenvalue sequences. It follows from Lemma 7.6 and facts about TD pairs that (7.1) holds for $0 \leq i \leq d$. The result follows.

Case (ii): Use (i) and Lemma 7.7.

Case (iii): We show

$$(x_{st} + (2i - d)I)V_{rs}(2i - d) \subseteq V_{rs}(2i - d + 2) \quad (7.2)$$

for $0 \leq i \leq d$. Observe that rt is one of 01, 10, 23, 32. By case (ii), $[x_{rt}, x_{ts}] = 2x_{rt} + 2x_{ts}$. Since $x_{rt} = -x_{tr}$, by Lem. 3.1 we have

$$(x_{st} + (2i - d)I)V_{tr}(2i - d) \subseteq V_{tr}(2i - d - 2) \quad (7.3)$$

for $0 \leq i \leq d$. Thus, for all i from 0 to d , we have

$$\begin{aligned}
 (x_{st} + (2i - d)I)V_{rs}(2i - d) &\subseteq (x_{st} + (2i - d)I) \sum_{n=0}^i V_{rs}(2n - d) \\
 &= (x_{st} + (2i - d)I) \sum_{n=0}^i V_{tr}(d - 2n) && \text{(Lem. 7.6)} \\
 &\subseteq \sum_{n=0}^{i+1} V_{tr}(d - 2n) && \text{(by (7.3))} \\
 &= \sum_{n=0}^{i+1} V_{rs}(2n - d) && \text{(Lem. 7.6)}
 \end{aligned}$$

and

$$\begin{aligned}
 (x_{st} + (2i - d)I)V_{rs}(2i - d) &\subseteq (x_{st} + (2i - d)I) \sum_{n=0}^i V_{rs}(2n - d) \\
 &= (x_{st} + (2i - d)I) \sum_{n=i}^d V_{st}(d - 2n) && \text{(Lem. 7.6)} \\
 &= \sum_{n=i+1}^d V_{st}(d - 2n) \\
 &= \sum_{n=i+1}^d V_{rs}(2n - d) && \text{(Lem. 7.6).}
 \end{aligned}$$

Combining these gives us that (7.2) holds for $0 \leq i \leq d$. The result follows by Lemma 3.1. $\square$

Lemma 7.9

For mutually distinct $r, s, t, u \in \mathbb{I}$, we have

$$[x_{rs}, [x_{rs}, [x_{rs}, x_{tu}]]] = 4[x_{rs}, x_{tu}].$$

Proof. By Lemma 2.1, it suffices to show

$$x_{tu}V_{rs}(2i - d) \subseteq V_{rs}(2i - d + 2) + V_{rs}(2i - d) + V_{rs}(2i - d - 2) \quad (7.4)$$

for $0 \leq i \leq d$. We have for $0 \leq i \leq d$ that

$$\begin{aligned}
x_{tu}V_{rs}(2i-d) &\subseteq x_{tu} \sum_{n=i}^d V_{rs}(2n-d) \\
&= x_{tu} \sum_{n=i}^d V_{st}(d-2n) && (\text{Lem. 7.6}) \\
&\subseteq \sum_{n=i-1}^d V_{st}(d-2n) && (\text{Lem. 7.8 \& 3.1}) \\
&= \sum_{n=i-1}^d V_{rs}(2n-d) && (\text{Lem. 7.6})
\end{aligned}$$

and

$$\begin{aligned}
x_{tu}V_{rs}(2i-d) &\subseteq x_{tu} \sum_{n=0}^i V_{rs}(2n-d) \\
&= x_{tu} \sum_{n=0}^i V_{ur}(d-2n) && (\text{Lem. 7.6}) \\
&\subseteq \sum_{n=0}^{i+1} V_{ur}(d-2n) && (\text{Lem. 7.8 \& 3.1}) \\
&= \sum_{n=0}^{i+1} V_{rs}(2n-d) && (\text{Lem. 7.6}).
\end{aligned}$$

The result follows. $\square$

The previous 3 lemmas give us a unique $\boxtimes$ -module structure on V , with X_{rs} acting on V as x_{rs} . This structure is unique because $\{X_{rs} : r, s \in \mathbb{I}, r \neq s\}$ is a generating set for $\boxtimes$. By Lemma 7.5 A and A^* are among the transforms x_{rs} , $r, s \in \mathbb{I}$. Since A, A^* is a TD pair, the $\boxtimes$ -module V is irreducible. Thus, we have the following theorem:

Theorem 7.10

There exists a unique $\boxtimes$ -module structure on V such that X_{rs} acts as x_{rs} for all distinct $r, s \in \mathbb{I}$. This $\boxtimes$ -module is irreducible.

We now have all the tools we need to prove Theorem 1.8.

Proof of Theorem 1.8. Existence and irreducibility are given by Theorem 7.10. We show uniqueness. Suppose we are given any $\boxtimes$ -module structure on V where X_{01}, X_{23} act on V as X, Y respectively. This module structure is irreducible since the $\mathcal{O}$ -module structure is irreducible. The action of each generator X_{rs} of $\boxtimes$ is determined by the decomposition of V associated with (r, s) . Lemma 5.7 gives us that this decomposition is determined by the flags on V associated with r and s . Thus, our $\boxtimes$ -module structure is determined by the four flags associated with the four elements of $\mathbb{I}$. By Corollary 5.5, these flags are determined by the actions of X_{01} and X_{23} on V . Therefore, the $\boxtimes$ -module structure is unique. $\square$