Ramanujan Graphs

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Spectral Graph Theory Review

Rayleigh Quotients

Let A be an $n \times n$ symmetric matrix. Fix an orthonormal eigenbasis $(b_0, \ldots, b_{n-1})$ of A, ordered such that the associated eigenvalues λ_i are decreasing. Writing a non-zero vector $v \in \mathbb{R}^n$ as $v = t_0 b_0 + \ldots + t_{n-1} b_{n-1}$, we see:

$$R_A(v) = \frac{v^T A v}{v^T v} = \frac{t_0^2 \lambda_0 + \ldots + t_{n-1}^2 \lambda_{n-1}}{t_0^2 + \ldots + t_{n-1}^2} \leq \lambda_0$$

Spectra of k-regular graphs

In particular, if A is the adjacency matrix of a graph $\Gamma = (X, \mathcal{R})$:

$$\lambda_0 \geq R_A(\mathbf{1}_n) = \frac{1}{n} \sum_{0 \leq i,j \leq n-1} a_{ij} = \frac{1}{n} \sum_{x \in X} \deg(x)$$

Now, let v be a λ_0-eigenvector of A, and pick j such that $|v_j| \geq |v_i|$ for all i. Then we have:

$$\lambda_0 |v_j| = |\lambda_0 v_j| = |(\lambda_0 v)_j| = |(Av)_j| = \left| \sum_{x_i \sim x_j} v_i \right| \leq \sum_{x_i \sim x_j} |v_i| \leq \sum_{x_i \sim x_j} |v_j| = \deg(x_j)|v_j|$$

Thus, $\lambda_0 \leq \deg(x_j)$, so the largest eigenvalue of a graph is between its average degree and maximum degree. In particular, for k-regular graphs, we have $\lambda_0 = k$. Notice in this case that we have equality all the way through, so $|v_i| = |v_j|$ for $x_i \sim x_j$. Applying this argument to the vertices of distance 2 from x_j, and so on, we get that $|v_i| = |v_j|$ for all i such that x_i is in the connected component C of x_j. Then $v - |v_j| \mathbf{1}_{x_i \in C}$ is another k-eigenvector of A with more zeroes than v, so we apply this argument again to eventually write v as a linear combination of the $\mathbf{1}_{x_i \in C}$ for C a connected component of Γ. Since those vectors are orthogonal, we get that the multiplicity of $k = \lambda_0$ as an eigenvalue of A is the number of connected components of Γ.

Proposition 1. *A k-regular graph $\Gamma = (X, \mathcal{R})$ with spectrum $\lambda_0 \geq \ldots \geq \lambda_{n-1}$ is bipartite if and only if $\lambda_{n-1} = -k$.*

The Laplacian

Closely related to the adjacency matrix of a graph is its Laplacian:

$$L = \operatorname{diag}(\deg(x_1), \ldots, \deg(x_n)) - A$$

If Γ is k-regular, this is equal to $kI - A$, so its spectrum is $0 \leq k - \lambda_1 \leq \ldots \leq k - \lambda_{n-1}$ (note that it is therefore *positive semi-definite*). We may also think about it as an averaging operator on $L^2(X)$, and to random walks on the graph.

1 Expander Graphs and the Spectral Gap

We wish to have a notion of "highly connected" graphs. To this end, given a subset W of the vertices, we let $\mathcal{E}(W)$ be the set of edges which are incident on both W and its complement. This allows us to define the *Cheeger constant* of a graph as follows:

$$h(\Gamma) = \min_{|W| \leq n/2} \frac{|\mathcal{E}(W)|}{|W|}$$

This is inspired by the notion of the Cheeger constant of a Riemannian manifold. The following proposition shows how to view how well-connected a graph is in terms of its Cheeger constant.

Proposition 2. *$h(\Gamma) > 0$ if and only if Γ is connected, and if $W \subset X$ with $|W|/n = \delta \leq 1/2$, then we must delete at least $\delta n h(\Gamma)$ edges to disconnect W from the rest of the graph.*

Proof. To disconnect W from the rest of the graph, we must delete $\mathcal{E}(W)$, which has $\geq |W|h(\Gamma) = \delta n h(\Gamma)$ edges. □

A *family of expander graphs* is a family of connected k-regular graphs (at least we want a global upper bound on the maximum degree) for which the number of vertices goes to infinity and for which the Cheeger constant is bounded below. The interesting thing is that we can characterize this property in terms of the *spectrum* of the graph.

Proposition 3. *Let Γ be a connected k-regular graph and let $\lambda_1(\Gamma)$ be the smallest eigenvalue of its Laplacian. Then:*

$$\frac{h(\Gamma)^2}{2k} \leq \lambda_1(\Gamma) \leq 2h(\Gamma)$$

Thus, a family $\Gamma_{n,k}$ of connected k-regular graphs with $n \to \infty$ is an expander family if and only if the $\lambda_1(\Gamma_{n,k})$ are bounded below.
Thinking in terms of the adjacency matrix, we are trying to bound the second largest eigenvalue above. More generally, we can bound the second largest eigenvalue *in absolute value* of a graph. Writing the spectrum of the adjacency matrix as $\lambda_0 \geq \ldots \geq \lambda_{n-1}$, this would be $\max(|\lambda_1|, |\lambda_{n-2}|)$ if Γ is bipartite, and $\max(|\lambda_1|, |\lambda_{n-1}|)$ otherwise. We denote this value by $\lambda(\Gamma)$. This quantity controls combinatorial properties of the graph such as diameter, independence number, and chromatic number. The best we can do in that regard was shown by Alon and Boppana:

Theorem 4. *Let $\Gamma_{n,k}$ be a family of k-regular graphs with $n \to \infty$. Then*

$$\lim_{n \to \infty} \lambda(\Gamma_{n,k}) \geq 2\sqrt{k-1}$$

This bound is derived by considering the universal cover of the $\Gamma_{n,k}$, namely the (infinite) k-regular tree. This motivates the following definition:

Definition 5. A k-regular graph $\Gamma_{n,k}$ is *Ramanujan* if $\lambda(X) \leq 2\sqrt{k-1}$.

We see that Ramanujan graphs are the best possible expanders. I will now outline their construction, following Lubotzky, Phillips, and Sarnak.

Constructing Ramanujan Graphs

Definition 6. Let G be a group and let $S \subset G$ be *symmetric*, i.e. $S = S^{-1}$. The *Cayley graph* $C(G, S)$ is a graph whose vertex set is G, and for which two vertices g and g' are connected if there is some $s \in S$ such that $gs = g'$.

For a quadratic form Q, we write $r_Q(n) = \#\{Qx = n\}$. For $Q(x) = x_0^2 + x_1^2 + x_2^2 + x_3^2$, we may obtain an explicit formula by writing down the function $\sum_n r_Q(n)e^{2\pi i n z}$, which is a *modular form*, in terms of a basis of modular forms with known Fourier coefficients (known as Eisenstein series):

$$r_Q(n) = 8 \sum_{4 \nmid d | n} d$$

In particular, let p and q be distinct primes which are congruent to 1 modulo 4. The above formula states that $r_Q(p) = 8(p+1)$, of which $p+1$ are such that x_0 is positive and odd. Given such a solution $\alpha = (a_0, a_1, \alpha_2, \alpha_3)$, we associate the matrix:

$$\underline{\alpha} = \begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{pmatrix}$$

where i is an integer for which i^2 is congruent to -1 modulo 4. If p is a square mod q. Let $S = \{\underline{\alpha}\}$. Then $|S| = p + 1$, so $C(PSL_2(\mathbb{F}_q, S)$ is a Ramanujan graph. Otherwise, w consider $C(PGL_2(\mathbb{F}_q), S)$. The ideas involved in proving that these graphs have the desired properties involve realizing this graph explicitly as a quotient of the k-regular tree using quatnerion algebras, and then doing harmonic analysis on the tree. The key input in this argument, which is where the name "Ramanujan graph" comes from, is an asymptotic formula for $r_Q(p^k)$ where

$$Q(x) = x_0^2 + 4q^2x_1^2 + 4q^2x_2^2 + 4q^2x_3^2$$

As before, we may attempt to write the exponential generating function $\sum_n r_Q(n)e^{2\pi i n z}$ as a modular form of a certain type, however the space of such modular forms no longer has a basis purely in terms of Eisenstein series. We may also encounter "cusp form", whose Fourier coefficients we do not know well. The Ramanujan conjecture, in its most naive form, gives a bound on the Fourier coefficients of cusp forms (this naive form is now a theorem, proven by Deligne, however a much more general representation-theoretic statement is still unknown).