

Mariana Grymksi Lecture

$\mathcal{O}$ -modules & Tri-diagonal Pairs

①

- V a f.d. v.s. over K , pos. finite dimension.
- A a linear transformation, a K -linear map on $V \rightarrow V$.

Let $\lambda \in K$. Then

Def: $V_A(\lambda) := \{v \in V \mid Av = \lambda v\}$.

λ is an eigenvalue of A iff $V_A(\lambda) \neq 0$,
and $V_A(\lambda) \neq 0$ is the eigenspace of A .

- $\sum_{\lambda \in K} V_A(\lambda)$ is direct

Def: A is diagonalizable if

$$V = \sum_{\lambda \in K} V_A(\lambda).$$

Lemma 2.1: Let V, A as above, and A^* be another linear transformation on V .

Then TFAB:

$$(*) \quad [A, [A, [A, A^*]]] - 4[A, A^*]$$

vanishes on $V_A(\lambda)$

$$(ii) A^* V_A(\lambda) \subseteq V_A(\lambda+2) + V_A(\lambda) + V_A(\lambda-2) \quad (2)$$

Proof: Let $\Phi := [A, [A, [A, A^*]]] - 4[A, A^*]$.

~~Multiplying out~~ We calculate:

$$\Phi = A^3 A^* - 3A^2 A^* A + 3A A^* A^2 - A^* A^3 - 4A A^* + 4A^* A.$$

Let $v \in V_A(\lambda)$; we have $Av = \lambda v$, so $\Phi v =$

$$(A^3 A^* - 3\lambda A^2 A^* + 3\lambda^2 A A^* - \lambda^3 A^* - 4\lambda A A^* +$$

$$4\lambda A^*)v =$$

$$(A - (\lambda-2)I)(A - \lambda I)(A - (\lambda+2)I)A^* v. \quad (1)$$

So if this is 0, $A^* v \in V_A(\lambda) \oplus V_A(\lambda-2) \oplus V_A(\lambda+2)$,

since $\text{char } K = 0$, so $\lambda, \lambda-2, \lambda+2$ are distinct.

Now if $A^* v$ is in this direct sum,

$$A^* v = a_{\lambda} + a_{\lambda-2} + a_{\lambda+2}, \text{ another}$$

each factor of (1) kills its respective

a_i , so the product kills $A^* v$, so

$$\Phi v = 0. \quad \square$$

Def: Let V a v.s. over K of positive

(3)

finite dimension. A tridiagonal pair

is an ordered pair A, A^* of linear transformations such that:

- A, A^* are diagonalizable.
- $\exists$ an ordering $V_0, \dots, V_d$ of the eigenspaces of A s.t. $A^* V_i \subseteq V_{i+1} + V_i + V_{i-1}$ $0 \leq i \leq d$,
w/ $V_{-1} = 0$ and $V_{d+1} = 0$
- $\exists$ an ordering $V_0^*, \dots, V_\delta^*$ of the eigenspaces of V^* s.t. $A V_i^* \subseteq V_{i+1}^* + V_i^* + V_{i-1}^*$ $0 \leq i \leq \delta$, $V_{-1}^* = V_{\delta+1}^* = 0$. (2)
- $\nexists W \subseteq V$ s.t. $AW \subseteq W, A^*W \subseteq W, W \neq 0, W \neq V$.

By a thm from paper 10 in the references, $d = \delta$. Orderings of eigenspaces of A and (A^*) are called standard when they satisfy the second (and third) bullets above.

An ^(column) eigenvale sequence is an ordering of eigenvalues ③

f.o. A, A^* s.t. the corresponding eigenspace ordering is standard.

Def: Let $\theta_0, \dots, \theta_d$ scalars in $\mathbb{K}$. This sequence is arithmetic w/ finite difference 2 whenever

$$\theta_{i-1} - \theta_i = 2 \quad \text{f.o. } 1 \leq i \leq d.$$

Thm 2.4

Let V a v.s. over $\mathbb{K}$ w/ pos. finite dimension.
Let A, A^* linear transformations on V . Then

TFAE.

(i) A, A^* is a tridiagonal pair on V whose eigenvalue sequence and dual eigenvalue sequence are both arithmetic w/ common difference 2.

(ii) $\exists$ an irreducible $\mathcal{O}$ -module structure on V s.t. the standard generators X, Y act on V as A, A^* respectively.

Proof: (i) $\Rightarrow$ (ii):

We'll first show $[A, [A, [A, A^*]]] = 4[AA^*]$. (3).

(5)

By assumption, AA^* has an eigenvalue sequence arithmetic w/ common difference 2. Let $v_0, \dots, v_d$ be the corresponding ordering of the eigenspaces of A .

This ordering is standard by construction, so it satisfies (2). Thus, $\forall \lambda \in \mathbb{K}$, it satisfies (1).

Then by Lemma 2.1 and since A is diagonalizable on V , A and A^* satisfy (3). We can reverse the roles of A and A^* to find

$$[A^*, [A^*, [A^*A]]] = 4[A^*, A] \quad (4)$$

By (3) and (4), $\exists$ an $\mathcal{O}$ -module structure on V s.t. X, Y act as AA^* , and by the 4th bullet of triangular pair, this $\mathcal{O}$ -module structure is irreducible.

(i) $\Rightarrow$ (i)

By reference to ex 6, since $\mathbb{K}$ is algebraically closed, all eigenvalues for A are in $\mathbb{K}$. Since $\dim(V)$ is pos & finite, A has @ least 1 eigenvalue λ . Since $\text{char } \mathbb{K} \neq 0$, $\lambda, \lambda+2, \dots$ are mutually distinct, and cannot all be eigenvalues of A . Thus, $\exists$ an eigenvalue θ s.t. $\theta+2$ isn't an eigenvalue for A . Similarly, since $\theta, \theta-2, \dots$ are distinct $\exists d \geq 0$ s.t. $\theta-2i$ is an eigenvalue for A , but for $0 \leq i \leq d$, but not for $i = d+1$. Let $V_A(\theta-2i) := V_i$, $0 \leq i \leq d$.

Since $\sum_{i=0}^d V_i$ is A -invariant since

$\theta-2i$ are all eigenvalues of A .

By Lemma 2.1, $A^* V_i \subseteq V_{i+1} + V_i + V_{i-1}$, $0 \leq i \leq d$, w/ $V_{-1} = 0$, $V_{d+1} = 0$. $\therefore$

$\sum_{i=0}^d V_i$ is A^* -invariant. Since our $\mathcal{O}$ -module structure is irred., and since $\sum_{i=0}^d V_i \neq 0$, $\sum_{i=0}^d V_i = V$. So, A is diagonalizable and

and our 3rd bullet above is satisfied
Since V is viewed as an $\mathcal{O}$ -module, we
have the 4th bullet. So, A, A^* is a
tridiagonal pair on V .

The ordering $v_0, \dots, v_d$ is standard, so
 $\{\mathcal{O} - 2i\}_{i=0}^d$ is an eigenvalue sequence
for A, A^* , and it's arithmetic w/
common difference 2. If we reverse
the roles of ~~A, A^*~~ A, A^* , we'll
have that A, A^* has an arithmetic
dual eigenvalue seq. w/ common
difference 2. $\square$.

Def: Let V be a f.d. irreducible O -module. (8)

Using Thm 2.4, $\exists d \geq 0, d \in \mathbb{Z}$, and scalars $\alpha, \alpha^* \in \mathbb{K}$ s.t. the set of distinct eigenvalues of X (resp. Y) on V is $\{d - 2i + \alpha \mid 0 \leq i \leq d\}$ (resp. $\{d - 2i + \alpha^* \mid 0 \leq i \leq d\}$). d is the diameter of V . (α, α^*) is the type of V .

Let V a f.d. irred. O -module of type (α, α^*) . Replace $X \rightarrow X - \alpha I, Y \rightarrow Y - \alpha^* I$, the type becomes $(0, 0)$.

Now restrict thm 2.4 to type $(0, 0)$.

We obtain:

Corollary: Let V a v.s. over $\mathbb{K}$ w/ + finite dimension. Let A, A^* linear transformations on V . Then TFAE:

(i) A, A^* is a tridiagonal pair on V and $d - 2i$ $0 \leq i \leq d$ is both an eigenvalue sequence and a dual eigenvalue sequence for A, A^* , d being the diameter

(10) $\exists$ an irred. σ -module structure on V of type $(0,0)$ s.t. X, Y act on V as A, A^* respectively. (9)

Follows from Thm 2.4 & def of type. $\blacksquare$