Classical t-designs

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In this notes, I will introduce t-designs, a standard design in combinatorics to approximate the whole set.

0.1 Basics of designs

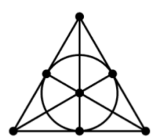
Definition 0.1 (t-(v, k, λ) Design)**.** Let $t, k, v, \lambda \in \mathbb{Z}_+$ with $t \leq k \leq v$. Consider a finite subset V consisting of v points and the set $V^{(k)}$ consisting of the k-element subsets of V. A pair $(V, \mathcal{B})$ of V and a subset $\mathcal{B}$ of $V^{(k)}$ is called a **t-(v, k, λ) design** (or simply a **t-design**) if there exists a $\lambda \geq 1$ such that for any $T \in V^{(t)}$, the following holds:

$$|\{B \in \mathcal{B} : T \subseteq B\}| = \lambda.$$

For a t-design $(V, \mathcal{B})$, an element of V is called a **point**, and an element of $\mathcal{B}$ is called a **block**. A design is also called a **block design**.

Remark 0.1. In other words, a t-(v, k, λ) design is a set of v points and a collection of blocks, each with k points, such that any t points occur together in exactly λ blocks.

Example 0.1. The following figure shows a 2-$(7, 3, 1)$ design, where V is a 7-element set and there are 7 blocks consisting of 3 edges and 3 medians of the triangle, and 1 circle inscribed in the triangle.



Remark 0.2. Note that $(V, V^{(k)})$ is a t-design, called the **trivial t-design**. By default we consider only designs with no repeated blocks, called **simple design**.

Definition 0.2 (Isomorphism of block designs)**.** Two t-designs $(V, \mathcal{B})$ and $(V', \mathcal{B}')$ are said to be isomorphic if there exists a bijection from V to V', and the bijection induces a bijection from $\mathcal{B}$ to $\mathcal{B}'$, and moreover $p \in B$ in $(V, \mathcal{B})$ implies $p^\sigma \in B^\sigma$ in $(V', \mathcal{B}')$, where σ denotes the bijection.

Proposition 0.1. *Let $(V, \mathcal{B})$ be a t-(v, k, λ) design. For any integer s with $0 \leq s \leq t$, $(V, \mathcal{B})$ is an s-design. Namely, if we let*

$$\lambda_s = \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}} \lambda$$

then λ_s is a natural number and $(V, \mathcal{B})$ is an s-(v, k, λ_s) design. (The concept of a 0-design has no special meaning but λ_0 can be regarded as the number $|\mathcal{B}|$ of blocks.)

Proof. For $S \in V^{(s)}$, let $\lambda(S) = |\{B \in \mathcal{B} \mid S \subseteq B\}|$. We prove that $\lambda(S)$ is independent of the choice of S as follows. By counting the number of pairs (T, B) of $T \in V^{(t)}$ and B such that $S \subseteq T \subseteq B \in \mathcal{B}$ in two ways, we obtain

$$\lambda(S) \binom{k-s}{t-s} = \binom{v-s}{t-s} \lambda.$$

(If we choose T first, we will get the right-hand side, and if we choose B first, we will get the left-hand side.) Therefore $\lambda(S) = \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}\lambda = \lambda_s$, and $\lambda(S)$ is independent of the choice of S. Hence λ_s is a natural number and $(V, \mathcal{B})$ is an s-(v, k, λ_s) design. □

Proposition 0.2. *A t-(v, k, λ) design exists only if λ_s is a natural number for any integer s with $0 \leq s \leq t$.*

Remark 0.3. Apply this to $t = 2$, let r be the number of blocks containing a point and $b = |\mathcal{B}|$ the number of blocks. Then $r = \lambda_1 = \frac{v-1}{k-1}\lambda$, $b = \lambda_0 = \frac{v(v-1)}{k(k-1)}\lambda$, so

$$\begin{cases} r(k-1) = (v-1)\lambda, \\ bk = vr. \end{cases} \tag{1}$$

Definition 0.3 (Incidence matrix of a design $(V, \mathcal{B})$)**.** For a design $(V, \mathcal{B})$, we define a matrix M whose rows are indexed by $\mathcal{B}$ and whose columns are indexed by V as follows. For $B \in \mathcal{B}, P \in V$, define the (B, P)-entry of M as

$$M(B, P) = \begin{cases} 1 & (\text{if } P \in B) \\ 0 & (\text{if } P \notin B) \end{cases}.$$

The matrix M is called the **incidence matrix** of a design $(V, \mathcal{B})$.

Example 0.2. The incidence matrix of the 2-$(7, 3, 1)$ design in 0.1 is given as follows:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Definition 0.4 (Complementary design)**.** For a t-(v, k, λ) design $(V, \mathcal{B})$, define $\mathcal{B}' = \{V - B : B \in \mathcal{B}\}$. Then $(V, \mathcal{B}')$ becomes a block design. Let t-(v, k', λ') be its parameters. Then $k' = v - k, \lambda' = \frac{\binom{v-k}{t}}{\binom{k}{t}}\lambda$ holds. The design $(V, \mathcal{B}')$ is called the **complementary design** of $(V, \mathcal{B})$.

Remark 0.4. Let M be the incidence matrix of a t-(v, k, λ) design $(V, \mathcal{B})$. In general, each row of M has exactly k 1's, and each column of M has exactly r 1's. In particular, if $(V, \mathcal{B})$ is a 2-design, for any two different columns of M, there are exactly λ rows in which both columns have 1's. Therefore

$$M^\dagger M = \begin{pmatrix} r & & & \lambda \\ & r & & \\ & & \ddots & \\ \lambda & & & r \end{pmatrix}$$

and that $\det\left(M^\dagger M\right) = (r + (v-1)\lambda)(r - \lambda)^{v-1}$ by induction on v.

Theorem 0.3 (Fisher type inequality)**.** *For a 2-(v, k, λ) design, assume $v > k$, then $b \geq v$.*

Proof. By the assumption $k < v$ and by (1), we have $r > \lambda$. Then the determinant $\det\left(M^\dagger M\right)$ is non-zero. Namely, $M^\dagger M$ is a non-singular matrix of size v, and this implies $b \geq v$. (In general, for matrices A and B, the rank of AB does not exceed the rank of A or B.) □

Definition 0.5 (Symmetric design)**.** A 2-(v, k, λ) design with $b = v$ is called a **symmetric** (v, k, λ) **design**.

Remark 0.5. By (1), $r = k$ for a symmetric (v, k, λ) design.

Remark 0.6. Let M be the incidence matrix of a symmetric (v,k,λ) design, then M is a square matrix and $MJ = JM = kJ$ holds. Besides, $M^\dagger M$ can be expressed as $(r-\lambda)I + \lambda J$. By the proof of Fisher type inequality, M is non-singular. Therefore we have $M^\dagger = \{(r-\lambda)I + \lambda J\}M^{-1}$. Now

$$MM^\dagger = M\{(r-\lambda)I + \lambda J\}M^{-1} = \{(r-\lambda)I + \lambda J\}MM^{-1} = \{(r-\lambda)I + \lambda J\} = \begin{pmatrix} r & & & \lambda \\ & r & & \\ & & \ddots & \\ \lambda & & & r \end{pmatrix}.$$

So B_i and B_j contain exactly λ common points.

Proposition 0.4. *Assume that a 2-(v,k,λ) design satisfies $v > k$. Then the following four conditions are equivalent:*

1. $b = v$;
2. $r = k$;
3. *any two blocks have exactly λ common points;*
4. *any two blocks have exactly m common points for a constant m.*

Proof. Since $bk = vr$, we have (1) $\implies$ (2); (2) $\implies$ (3) follows from the above discussion; and (3) $\implies$ (4) is obvious. We show (4) $\implies$ (1). We exchange the roles of points for those of blocks. Namely, the matrix $M^\dagger$ has exactly r 1's in each row, and $M^\dagger$ has exactly k 1's in each column. Besides, any two columns of $M^\dagger$ have exactly m 1's in common. Therefore $M^\dagger$ is the incidence matrix of a 2-(b,r,m) design. Each block of this design contains r points and each point is contained in exactly k blocks. Applying (1) to the given design, we have $bk = vr$. Moreover, since $v > k$, we have $b > r$. Next if we apply (1) to the 2-(b,r,m) design, we have $k(r-1) = (b-1)m$, and thus $k > m$. Note $\det\left(MM^\dagger\right) = (k + (b-1)m)(k-m)^{b-1} > 0$. Hence $MM^\dagger$ is a non-singular matrix of size b, and $b \le v$ holds. On the other hand, by applying the Fisher type inequality to the given design $(V,\mathcal{B})$, we have $b \ge v$, and thus $b = v$. □

Remark 0.7. If (4) holds for a 2-(v,k,λ) design $(V,\mathcal{B})$, the existence of a 2-(b,r,m) design is easily shown by exchanging the roles of points for those of blocks. The 2-(b,r,m) design is called the **dual structure** or the dual design of $(V,\mathcal{B})$.

Remark 0.8. To sum up, the incidence matrix M of a symmetric 2-(v,k,λ) design satisfies

$$MJ = JM = kJ = rJ, \quad M^\dagger M = MM^\dagger = (r-\lambda)I + \lambda J.$$

Remark 0.9. A symmetric 2-(v,k,λ) design $(V,\mathcal{B})$ has the same parameters as the dual design $(\mathcal{B},V)$, but they are not necessarily isomorphic.

0.2 Important theorems for designs

We know 0.2 is a necessary condition for the existence of a t-(v,k,λ) design. How strong is it? How close is it to a sufficient condition? It is not clear in general, but for the case of $t = 2$, we know it is very strong. For a 2-$(v,3,1)$ design, 0.2 is shown to also be a sufficient condition for the existence, which is equivalent to $v \equiv 1, 3 \pmod 6$. The following theorem provides a sufficient condition:

Theorem 0.5 (Wilson)**.** *Suppose that k, λ are given. There exists a number v_0 determined by k, λ such that there is a 2-(v,k,λ) design with $r = \frac{\lambda(v-1)}{k-1}$ and $b = \frac{\lambda v(v-1)}{k(k-1)}$ whenever $v \ge v_0$, $\lambda(v-1) \equiv 0 \pmod{k-1}$, $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.*

Remark 0.10. It would be desirable if we could show a similar result for $t \ge 3$; however, it is an open problem. If we allow repeated blocks, this necessary condition is known to be very close to a sufficient condition.

Regarding the necessary condition for the existence of a symmetric 2-(v, k, λ) design:

Theorem 0.6 (Bruck-Ryser-Chowla). *For a symmetric* 2-(v, k, λ) *design, if we let $n = k - \lambda$, the following hold:*

1. *If v is even, then n is a square*

2. *If v is odd, then $z^2 = nx^2 + (-1)^{\frac{v-1}{2}}\lambda y^2$ has a solution in integers x, y, z, not all of which are 0.*

Proof. (1): The left-hand side of $\det\left(M^\dagger M\right) = (r + (v-1)\lambda)(r-\lambda)^{v-1}$ is a square. By symmetry of the design, we have $r = k$. Moreover, by (1), we have $r + (v-1)\lambda = k^2$. Therefore $(r-\lambda)^{v-1}$ is a square, and if we note that $v - 1$ is odd, $n = k - \lambda$ must be a square. (2): Use theorem of Lagrange and proper representation for $MM^\dagger$, see 10.3 of M. Hall. $\square$