Math 542 **Final Exam, Spring 2016**

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Your Name (please print) SOLUTIONS

NO CALCULATORS/ELECTRONIC DEVICES ALLOWED.

MAKE SURE YOUR CELL PHONE IS OFF.

Problem	Value	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. Let x denote an indeterminate, and consider the ring of polynomials $\mathbb{Z}[x]$ with integer coefficients. Determine if the following polynomial is irreducible in $\mathbb{Z}[x]$:

$$f(x) = x^6 + 30x^5 - 15x^3 + 6x - 120.$$

Solution. The polynomial $f(x)$ is monic. The prime 3 divides each nonleading coefficient, and 3^2 does not divide the constant coefficient. Therefore $f(x)$ is irreducible by the Eisenstein criterion.

2. Let $F = \mathbb{Z}/2\mathbb{Z}$ denote the field with just 2 elements. Let x denote an indeterminate, and consider the ring of polynomials $R = F[x]$. Consider the ideal J of R generated by $f(x) = x^4 + x^3 + x^2 + x + 1$. Viewing the quotient ring R/J as a vector space over F, find the dimension and prove that your answer is correct.

Solution. View R as a vector space over F. Let U denote the subspace of R with basis $1, x, x^2, x^3$. The dimension of U is 4. The sum $R = U + J$ is direct. Therefore the quotient map $R \to R/J, r \mapsto r + J$ induces a vector space isomorphism $U \to R/J$. Consequently the vector space R/J has dimension 4.

3. Let F denote a field, and consider a vector space V over F with dimension 5. Let $T : V \to V$ denote a linear transformation whose Jordan canonical form consists of a single Jordan block with eigenvalue 0. (i) Find the dimension of the subspace $U = \{v \in V | T^3 v = 0\}$. (ii) Find all the subspaces W of V such that $TW \subseteq W$ and the sum $V = U + W$ is direct. Prove that your answer is correct.

Solution. The vector space V has a basis v_i $(0 \le i \le 4)$ such that $Tv_0 = 0$ and $Tv_i = v_{i-1}$ for $1 \le i \le 4$. Pick any $v \in V$ and write $v = \sum_{i=0}^{4} \alpha_i v_i$ with $\alpha_i \in F$. Observe $T^3 v = \alpha_3 v_0 + \alpha_4 v_1$. So $v \in U$ iff each of α_3, α_4 is zero. Consequently U has basis v_0, v_1, v_2. The dimension of U is 3. Given a subspace W of V that meets the requirements of (ii), observe that W has dimension 2. Note that $T^2 V = U$, so $T^2 W \subseteq U$. Also $T^2 W \subseteq W$, so $T^2 W \subseteq U \cap W = 0$. Now $W \subseteq U$, so $W \subseteq U \cap W = 0$, for a contradiction. There is no such W.

4. Consider the field $\mathbb{R}$ of real numbers. Let x denote an indeterminate, and consider the vector space V over $\mathbb{R}$ consisting of polynomials in x that have coefficients in $\mathbb{R}$ and degree at most 3. Note that $1, x, x^2, x^3$ is a basis for V. Consider the map $\varphi : V \to \mathbb{R}$ that sends $f \mapsto f(2)$ for all $f \in V$. (i) Prove that φ is in the dual space V^*. (ii) Express φ as a linear combination of the dual basis.

Solution. (i) We check that φ is $\mathbb{R}$-linear. For $f, g \in V$ we have

$$\varphi(f+g) = (f+g)(2) = f(2) + g(2) = \varphi(f) + \varphi(g).$$

For $f \in V$ and $a \in \mathbb{R}$ we have

$$\varphi(af) = (af)(2) = af(2) = a\varphi(f).$$

(ii) Let φ_i $(0 \le i \le 3)$ denote the dual basis. So $\varphi_i(x^j) = \delta_{i,j}$ for $0 \le i, j \le 3$. Note that $\varphi(x^j) = 2^j$ for $0 \le j \le 3$. Therefore $\varphi = \sum_{j=0}^{3} 2^j \varphi_j$.

5. Let R denote an integral domain. Let W denote a finitely generated R-module that is nonzero and torsion. Prove that the Annihilator of W is nonzero.

Solution. Write $W = Rv_1 + Rv_2 + \cdots + Rv_n$. Each v_i is torsion, so there exists a nonzero $a_i \in R$ such that $a_i v_i = 0$. Define $a = a_1 a_2 \cdots a_n$. By construction $av_i = 0$ for $1 \leq i \leq n$, so $aW = 0$. Also $a \neq 0$ since R is an integral domain. By these comments the Annihilator of W is nonzero.

6. Over the field of real numbers, find the rational canonical form of the matrix

$$A = \begin{pmatrix} -1 & -9 & 0 \\ 1 & 5 & 0 \\ 1 & 3 & 2 \end{pmatrix}.$$

Solution. First find the determinant of $xI - A$ to get the characteristic polynomial $c(x) = (x-2)^3$. The eigenvalues of A are $2, 2, 2$. Next find the dimension for the eigenspace of A with eigenvalue 2. To do this, row reduce $A - 2I$. One finds that this eigenspace has dimension 2. Therefore A has Jordan canonical form

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

This shows that A has minimal polynomial $m(x) = (x-2)^2 = x^2 - 4x + 4$. The invariant factors of A must be $x - 2$ and $(x-2)^2$. By these comments A has rational canonical form

$$\begin{pmatrix} 0 & -4 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

7. Consider the field $F = \mathbb{Z}/7\mathbb{Z}$ of order 7. Consider the following matrix $A \in \mathrm{Mat}_7(F)$:

$$A : \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Find the Jordan canonical form J for A. Prove that your answer is correct.

Solution. The matrix A is a permutation matrix, and the permutation involved is a 7-cycle. Therefore $x^7 - 1$ is both the minimal and characteristic polynomial of A. In the field F we have $7 = 0$ so $x^7 - 1 = (x - 1)^7$. By these comments A has Jordan canonical form

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

8. Recall the ring of integers $\mathbb{Z}$ is a PID. Consider the $\mathbb{Z}$-module $\mathbb{Z}/100\mathbb{Z}$. (i) Find its invariant factor decomposition. (ii) Find its elementary divisor decomposition.

Solution. (i) The $\mathbb{Z}$-module $\mathbb{Z}/100\mathbb{Z}$ is cyclic, so it is its own invariant factor decomposition. (ii) We have $100 = 4 \times 25$ with $4, 25$ relatively prime. So there exists a $\mathbb{Z}$-module isomorphism

$$\mathbb{Z}/100\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z}.$$

The elements $4 = 2^2$ and $25 = 5^2$ are prime powers. Therefore the above decomposition is the elementary divisor decomposition.

9. Let $n = 1000$. Let G denote the group of units for the ring $\mathbb{Z}/n\mathbb{Z}$. Find $|G|$.

Solution. We have $|G| = 400$. The number of units in the ring $\mathbb{Z}/n\mathbb{Z}$ is equal to the number of positive integers up to n that are relatively prime to n. This is $\phi(n)$ where ϕ is the Euler function. We have $\phi(n) = \phi(10^3) = \phi(2^3 5^3) = \phi(2^3)\phi(5^3)$. Also $\phi(2^3) = 2^2 = 4$ and $\phi(5^3) = 5^3 - 5^2 = 100$. So $\phi(n) = 400$.

10. View the group G in Problem 9 as a $\mathbb{Z}$-module. For this module (i) find its invariant factor decomposition; (ii) find its elementary divisor decomposition.

Solution. Consider the prime factorization $1000 = 2^3 5^3$. The group G is a direct product $H \times K$, where H is the group of units for $\mathbb{Z}/2^3\mathbb{Z}$ and K is the group of units for $\mathbb{Z}/5^3\mathbb{Z}$. Recall that H is the direct product of two cyclic groups of order 2. The group K is cyclic of order $5^2 \times 4 = 100$. Now in the invariant factor decomposition, G is a direct product of two cyclic groups of order 2 and a cyclic group of order 100. In the elementary divisor decomposition, G is a direct product of two cyclic groups of order 2, a cyclic group of order 4, and a cyclic group of order 25.