Math 542 **Final Exam, Spring 2016**

Prof: Paul Terwilliger

Your Name (please print) ______________________________

NO CALCULATORS/ELECTRONIC DEVICES ALLOWED.

MAKE SURE YOUR CELL PHONE IS OFF.

Problem	Value	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. Let x denote an indeterminate, and consider the ring of polynomials $\mathbb{Z}[x]$ with integer coefficients. Determine if the following polynomial is irreducible in $\mathbb{Z}[x]$:

$$f(x) = x^6 + 30x^5 - 15x^3 + 6x - 120.$$

2. Let $F = \mathbb{Z}/2\mathbb{Z}$ denote the field with just 2 elements. Let x denote an indeterminate, and consider the ring of polynomials $R = F[x]$. Consider the ideal J of R generated by $f(x) = x^4 + x^3 + x^2 + x + 1$. Viewing the quotient ring R/J as a vector space over F, find the dimension and prove that your answer is correct.

3. Let F denote a field, and consider a vector space V over F with dimension 5. Let $T : V \to V$ denote a linear transformation whose Jordan canonical form consists of a single Jordan block with eigenvalue 0. (i) Find the dimension of the subspace $U = \{v \in V | T^3 v = 0\}$. (ii) Find all the subspaces W of V such that $TW \subseteq W$ and the sum $V = U + W$ is direct. Prove that your answer is correct.

4. Consider the field $\mathbb{R}$ of real numbers. Let x denote an indeterminate, and consider the vector space V over $\mathbb{R}$ consisting of polynomials in x that have coefficients in $\mathbb{R}$ and degree at most 3. Note that $1, x, x^2, x^3$ is a basis for V. Consider the map $\varphi : V \to \mathbb{R}$ that sends $f \mapsto f(2)$ for all $f \in V$. (i) Prove that φ is in the dual space V^*. (ii) Express φ as a linear combination of the dual basis.

5. Let R denote an integral domain. Let W denote a finitely generated R-module that is nonzero and torsion. Prove that the Annihilator of W is nonzero.

6. Over the field of real numbers, find the rational canonical form of the matrix

$$A = \begin{pmatrix} -1 & -9 & 0 \\ 1 & 5 & 0 \\ 1 & 3 & 2 \end{pmatrix}.$$

7. Consider the field $F = \mathbb{Z}/7\mathbb{Z}$ of order 7. Consider the following matrix $A \in \mathrm{Mat}_7(F)$:

$$A : \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Find the Jordan canonical form J for A. Prove that your answer is correct.

8. Recall the ring of integers $\mathbb{Z}$ is a PID. Consider the $\mathbb{Z}$-module $\mathbb{Z}/100\mathbb{Z}$. (i) Find its invariant factor decomposition. (ii) Find its elementary divisor decomposition.

9. Let $n = 1000$. Let G denote the group of units for the ring $\mathbb{Z}/n\mathbb{Z}$. Find $|G|$.

10. View the group G in Problem 9 as a $\mathbb{Z}$-module. For this module (i) find its invariant factor decomposition; (ii) find its elementary divisor decomposition.