Math 475 **Text: Brualdi, Introductory Combinatorics 5th Ed.**
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Selected solutions for Chapter 3

4, 5, 6. Given integers $n \geq 1$ and $k \geq 2$ suppose that $n+1$ distinct elements are chosen from $\{1, 2, \ldots, kn\}$. We show that there exist two that differ by less than k. Partition $\{1, 2, \ldots, nk\} = \cup_{i=1}^{n} S_i$ where $S_i = \{ki, ki-1, ki-2, \ldots, ki-k+1\}$. Among our $n+1$ chosen elements, there exist two in the same S_i. These two differ by less than k.

9. Consider the set of 10 people. The number of subsets is $2^{10} = 1024$. For each subset consider the sum of the ages of its members. This sum is among $0, 1, \ldots, 600$. By the pigeonhole principle the 1024 sums are not distinct. The result follows. Now suppose we consider at set of 9 people. Then the number of subsets is $2^9 = 512 < 600$. Therefore we cannot invoke the pigeonhole principle.

10. For $1 \leq i \leq 49$ let b_i denote the number of hours the child watches TV on day i. Consider the numbers $\{b_1 + b_2 + \cdots + b_i + 20\}_{i=0}^{48} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{49}$. There are 98 numbers in the list, all among $1, 2, \ldots, 96$. By the pigeonhole principle the numbers $\{b_1 + b_2 + \cdots + b_i + 20\}_{i=0}^{48} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{49}$. are not distinct. Therefore there exist integers i, j $(0 \leq i < j \leq 49)$ such that $b_{i+1} + \cdots + b_j = 20$. During the days $i+1, \ldots, j$ the child watches TV for exactly 20 hours.

14. After n minutes we have removed n pieces of fruit from the bag. Suppose that among the removed fruit there are at most 11 pieces for each of the four kinds. Then our total n must be at most $4 \times 11 = 44$. After $n = 45$ minutes we will have picked at least a dozen pieces of fruit of the same kind.

16. Label the people $1, 2, \ldots, n$. For $1 \leq i \leq n$ let a_i denote the number of people aquainted with person i. By construction $0 \leq a_i \leq n-1$. Suppose the numbers $\{a_i\}_{i=1}^{n}$ are mutually distinct. Then for $0 \leq j \leq n-1$ there exists a unique integer i $(1 \leq i \leq n)$ such that $a_i = j$. Taking $j = 0$ and $j = n-1$, we see that there exists a person aquainted with nobody else, and a person aquainted with everybody else. These people are distinct since $n \geq 2$. These two people know each other and do not know each other, for a contradiction. Therefore the numbers $\{a_i\}_{i=1}^{n}$ are not mutually distinct.

18. Divide the 2×2 square into four 1×1 squares. By the pigeonhole principle there exists a 1×1 square that contains at least two of the five points. For these two points the distance apart is at most $\sqrt{2}$.

20. Color the edges of K_{17} red or blue or green. We show that there exists a K_3 subgraph of K_{17} that is red or blue or green. Pick a vertex x of K_{17}. In K_{17} there are 16 edges that contain x. By the pigeonhole principle, at least 6 of these are the same color (let us say red). Pick distinct vertices $\{x_i\}_{i=1}^{6}$ of K_{17} that are connected to x via a red edge. Consider the K_6 subgraph with vertices $\{x_i\}_{i=1}^{6}$. If this K_6 subgraph contains a red edge, then the two vertices involved together with x form the vertex set of a red K_3 subgraph. On the other

hand, if the K_6 subgraph does not contain a red edge, then since $r(3,3) = 6$, it contains a K_3 subgraph that is blue or green. We have shown that K_{17} has a K_3 subgraph that is red or blue or green.

27. Let $s_1, s_2, \ldots, s_k$ denote the subsets in the collection. By assumption these subsets are mutually distinct. Consider their complements $\overline{s_1}, \overline{s_2}, \ldots, \overline{s_k}$. These complements are mutually distinct. Also, none of these complements are in the collection. Therefore $s_1, s_2, \ldots, s_k$, $\overline{s_1}, \overline{s_2}, \ldots, \overline{s_k}$ are mutually distinct. Therefore $2k \leq 2^n$ so $k \leq 2^{n-1}$. There are at most 2^{n-1} subsets in the collection.

28. The answer is 1620. Note that $1620 = 81 \times 20$. First assume that $\sum_{i=1}^{100} a_i < 1620$. We show that no matter how the dance lists are selected, there exists a group of 20 men that cannot be paired with the 20 women. Let the dance lists be given. Label the women $1, 2, \ldots, 20$. For $1 \leq j \leq 20$ let b_j denote the number of men among the 100 that listed woman j. Note that $\sum_{j=1}^{20} b_j = \sum_{i=1}^{100} a_i$ so $(\sum_{j=1}^{20} b_j)/20 < 81$. By the pigeonhole principle there exists an integer j $(1 \leq j \leq 20)$ such that $b_j \leq 80$. We have $100 - b_j \geq 20$. Therefore there exist at least 20 men that did not list woman j. This group of 20 men cannot be paired with the 20 women.
Consider the following selection of dance lists. For $1 \leq i \leq 20$ man i lists woman i and no one else. For $21 \leq i \leq 100$ man i lists all 20 women. Thus $a_i = 1$ for $1 \leq i \leq 20$ and $a_i = 20$ for $21 \leq i \leq 100$. Note that $\sum_{i=1}^{100} a_i = 20 + 80 \times 20 = 1620$. Note also that every group of 20 men can be paired with the 20 women.