

**Q-POLYNOMIAL STRUCTURES FOR
ASSOCIATION SCHEMES
AND DISTANCE-REGULAR GRAPHS**

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Abstract

We study symmetric association schemes and distance-regular graphs which admit at least one $\mathbb{Q}$ -polynomial structure. Our main result is the classification of the distance-regular graphs with diameter $d \geq 5$ and valency $k \geq 3$ which admit more than one $\mathbb{Q}$ -polynomial structure. As a corollary to this classification, we find that the eigenvalues of a $\mathbb{Q}$ -polynomial distance-regular graph with diameter $d \geq 5$ and valency $k \geq 3$ are all integers. The bounds on d and k in this result are tight. We also prove several independent results, which are used in the proof of the main Theorem described above. These include classifications for dual bipartite and almost dual imprimitive distance-regular graphs, and new identities on the Krein parameters of a $\mathbb{Q}$ -polynomial symmetric association scheme.

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This thesis is for my wife, Sandra.

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Chapter 1

Introduction

In this thesis we study symmetric association schemes and distance-regular graphs which admit at least one Q-polynomial structure. Our central results are the following:

- classification of the distance-regular graphs with diameter $d \geq 5$ and valency $k \geq 3$ admitting more than one Q-polynomial structure (Theorem 8.1.2);
- integrality of the eigenvalues of Q-polynomial distance-regular graphs with diameter $d \geq 5$ and valency $k \geq 3$ (Corollary 8.1.3).

We also obtain the following results, of independent interest:

- classification of the distance-regular graphs with diameter $d \geq 6$ and valency $k \geq 3$ admitting a *dual bipartite* Q-polynomial structure (Theorem 2.1.2);
- classification of the *almost dual imprimitive* distance-regular graphs with diameter $d \geq 4$ and valency $k \geq 3$ (Theorem 3.1.4);
- the implication $q_{11}^1 \neq 0 \Rightarrow q_{1i}^i \neq 0$, $1 \leq i \leq d-1$ for the Krein parameters of a Q-polynomial symmetric association scheme (Theorem 4.1.1);
- the implication $p_{11}^1 = 0 \Rightarrow p_{1i}^i = 0$, $1 \leq i \leq d-1$ for the intersection numbers of a 1-thin P-polynomial symmetric association scheme (Theorem 5.1.1);
- the implication $q_{11}^1 = 0 \Rightarrow q_{1i}^i = 0$, $1 \leq i \leq d-1$ for the Krein parameters of a dual 1-thin Q-polynomial symmetric association scheme (Theorem 5.1.2);
- local strong regularity of distance-regular graphs admitting more than one Q-polynomial structure (Theorem 6.2.3);
- thin-ness of distance-regular graphs admitting more than one Q-polynomial structure (Theorem 7.1.1).

The results of Chapters 2–7 are proved independently; we apply these results in Chapter 8. Chapters 2 and 5 present joint work with P. Terwilliger.

For the remainder of this introduction we define terminology and notation used throughout the thesis, quoting from [1], [2], and [20]. In Chapter 9 we describe each of the graphs which are mentioned elsewhere in the thesis.

1.1 Distance-Regular Graphs

Throughout, $\Gamma = (X, R)$ will denote a finite, connected, undirected graph with vertex set X , edge set R , path-length distance function ∂ , and diameter $d = \max\{\partial(x, y) \mid x, y \in X\}$. The graph Γ is said to be *distance-regular* if there exist integers p_{ij}^h so that for all integers h with $0 \leq h \leq d$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$,

$$p_{ij}^h = |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|. \quad 0 \leq i, j \leq d \quad (1.1)$$

The integers p_{ij}^h are known as the *intersection numbers* for Γ , and are abbreviated $a_i = p_{1i}^1$, $b_i = p_{1,i+1}^1$, $c_i = p_{1,i-1}^1$, $k_i = p_{ii}^0$, and $k = k_1 = b_0$, with $b_d = 0$, $c_0 = 0$. It is immediate from the definition that $b_i \neq 0$ for $0 \leq i \leq d-1$, $c_i \neq 0$ for $1 \leq i \leq d$, and $k_i \neq 0$ for $0 \leq i \leq d$. We refer to k as the *valency* of Γ .

Note that the distance-regular graphs with valency $k = 2$ are precisely the ordinary n -gons.

1.2 Symmetric Association Schemes

By a d -class *symmetric association scheme* we mean a pair $Y = (X, \{R_i\}_{0 \leq i \leq d})$, where X is a non-empty finite set, and where:

- (i) $\{R_i\}_{0 \leq i \leq d}$ is a partition of $X \times X$;
- (ii) $R_0 = \{xx \mid x \in X\}$;
- (iii) $R_i = R_i^t$ for $0 \leq i \leq d$, where $R_i^t = \{yx \mid xy \in R_i\}$;
- (iv) there exist integers p_{ij}^h such that for all $x, y \in X$ with $xy \in R_h$, the number of $z \in X$ with $xz \in R_i$ and $yz \in R_j$ is p_{ij}^h .

We refer to X as the *vertex set* of Y , and to the integers p_{ij}^h as the *intersection numbers* of Y . For the remainder of the thesis, we abbreviate *association scheme* for symmetric association scheme.

Note that any distance-regular graph $\Gamma = (X, R)$ determines an association scheme $Y = (X, \{R_i\}_{0 \leq i \leq d})$, with $xy \in R_i$ if and only if $\partial(x, y) = i$ in Γ . In fact, the association schemes obtained in this way are precisely those with the P-polynomial property; see Section 1.3. Therefore, the remainder of our discussion of association schemes applies equally to distance-regular graphs.

Let $Y = (X, \{R_i\}_{0 \leq i \leq d})$ denote an association scheme. Let $\text{Mat}_X(\mathbf{R})$ denote the algebra of matrices over $\mathbf{R}$ with rows and columns indexed by X . The *associate matrices* for Y are the matrices $A_i \in \text{Mat}_X(\mathbf{R})$ defined for $0 \leq i \leq d$ by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } xy \in R_i, \\ 0 & \text{otherwise.} \end{cases} \quad x, y \in X \quad (1.2)$$

From (i)–(iv) above we obtain

$$A_0 + \cdots + A_d = J, \quad (1.3)$$

$$A_i \circ A_j = \delta_{ij} A_i, \quad 0 \leq i, j \leq d \quad (1.4)$$

$$A_0 = I, \quad (1.5)$$

$$A_i = A_i^t, \quad 0 \leq i \leq d \quad (1.6)$$

$$A_i A_j = \sum_{h=0}^d p_{ij}^h A_h, \quad 0 \leq i, j \leq d \quad (1.7)$$

where J is the all-1s matrix and $\circ$ denotes the entry-wise matrix product.

By the *Bose-Mesner algebra* for Y we mean the sub-algebra M of $\text{Mat}_X(\mathbb{R})$ generated by the associate matrices $A_0, \dots, A_d$. Observe by (1.7) that the associate matrices form a basis for M . In particular, M is symmetric and closed under $\circ$.

The algebra M has a second basis $E_0, \dots, E_d$ such that

$$E_0 + \dots + E_d = I, \quad (1.8)$$

$$E_i E_j = \delta_{ij} E_i, \quad 0 \leq i, j \leq d \quad (1.9)$$

$$E_0 = \frac{1}{|X|} J, \quad (1.10)$$

$$E_i = E_i^t \quad 0 \leq i \leq d \quad (1.11)$$

[2, Theorem 2.6.1]. We refer to $E_0, \dots, E_d$ as the *primitive idempotents* of Y . Since M is closed under $\circ$, there exist real numbers q_{ij}^h with

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^d q_{ij}^h E_h, \quad 0 \leq i, j \leq d \quad (1.12)$$

The numbers q_{ij}^h are the *Krein parameters* for Y with respect to the ordering $E_0, \dots, E_d$ of the primitive idempotents. It is immediate that

$$q_{ij}^h = q_{ji}^h, \quad 0 \leq h, i, j \leq d \quad (1.13)$$

Furthermore, it is known that

$$q_{ij}^h \geq 0 \quad 0 \leq h, i, j \leq d \quad (1.14)$$

[2, Theorem 2.3.2]. The inequalities (1.14) are referred to as the *Krein conditions*. It will be useful to note that

$$\text{rank}(E_i) = \text{tr}(E_i) = k_i^* \quad 0 \leq i \leq d \quad (1.15)$$

[2, p. 45]. By (1.4), (1.5) and the fact that $A_0, \dots, A_d$ is a basis for M , the primitive idempotents have constant diagonal; from (1.15) we obtain

$$(E_i)_{xx} = \frac{k_i^*}{|X|}, \quad 0 \leq i \leq d, \quad x \in X \quad (1.16)$$

1.3 P-Polynomial Structures

Lemma 1.3.1 *Let Y denote a d -class association scheme, with associate matrices $A_0, \dots, A_d$ and intersection numbers p_{ij}^h . Then the following are equivalent:*

(i) for $0 \leq i, j \leq d$,

$$p_{1j}^i = 0, \quad j > i + 1 \quad (1.17)$$

$$p_{1j}^i \neq 0; \quad j = i + 1 \quad (1.18)$$

(ii) for $0 \leq h, i, j \leq d$,

$$p_{ij}^h = 0 \text{ if one of } h, i, j \text{ is greater than the sum of the other two,} \quad (1.19)$$

$$p_{ij}^h \neq 0 \text{ if one of } h, i, j \text{ is equal to the sum of the other two.} \quad (1.20)$$

If (i), (ii) hold, then $A_0, \dots, A_d$ is said to be a P-polynomial structure for Y . In this case, we abbreviate $a_i = p_{1i}^i$, $b_i = p_{1,i+1}^i$, $c_i = p_{1,i-1}^i$, $k_i = p_{ii}^0$, and $k = k_1 = b_0$.

Proof of (i) $\Rightarrow$ (ii). Immediate from [2, Lemma 2.1.1] and [2, Lemma 4.1.7]. $\square$

Proof of (ii) $\Rightarrow$ (i). Immediate. $\square$

Let Y denote a d -class association scheme. We say that Y is *P-polynomial* if there exists at least one P-polynomial structure for Y .

Suppose Y is P-polynomial, and fix $A_0, \dots, A_d$ a P-polynomial structure for Y . We refer to A_1 as the *adjacency matrix* of the structure. Observe by (1.7), (1.17), and (1.18) that A_i may be written as a polynomial in A_1 of degree exactly i , for $0 \leq i \leq d$ [20, Lemma 3.8]. It follows that A_1 generates M , that every P-polynomial structure has $A_0 = I$, and that distinct P-polynomial structures have distinct adjacency matrices.

Suppose Y is P-polynomial, and fix $A_0, \dots, A_d$ a P-polynomial structure for Y . Observe that $\Gamma = (X, R_1)$ is a distance-regular graph, with $\partial(x, y) = i$ in Γ if and only if $xy \in R_i$. Since $E_0, \dots, E_d$ forms a basis for M , there exist real numbers $\theta_0, \dots, \theta_d$ satisfying

$$A_1 = \sum_{h=0}^d \theta_h E_h. \quad (1.21)$$

Observe that $\theta_0, \dots, \theta_d$ are eigenvalues of A_1 . We refer to $\theta_0, \dots, \theta_d$ as the *eigenvalues* of the structure, or as the *eigenvalues* of the graph Γ . Since A_1 generates M , the eigenvalues $\theta_0, \dots, \theta_d$ are distinct [20, Lemma 3.8]. If $\theta_0 > \dots > \theta_d$, then we say that $E_0, \dots, E_d$ is the *natural ordering* of the primitive idempotents.

1.4 Q-Polynomial Structures

Lemma 1.4.1 *Let Y denote a d -class association scheme, with primitive idempotents $E_0, \dots, E_d$ and Krein parameters q_{ij}^h . Then the following are equivalent:*

$$(i) \text{ for } 0 \leq i, j \leq d, \quad q_{ij}^i = 0, \quad j > i + 1 \quad (1.22)$$

$$q_{ij}^i \neq 0; \quad j = i + 1. \quad (1.23)$$

(ii) for $0 \leq h, i, j \leq d$,

$$q_{ij}^h = 0 \text{ if one of } h, i, j \text{ is greater than the sum of the other two,} \quad (1.24)$$

$$q_{ij}^h \neq 0 \text{ if one of } h, i, j \text{ is equal to the sum of the other two.} \quad (1.25)$$

If (i), (ii) hold, then $E_0, \dots, E_d$ is said to be a Q-polynomial structure for Y . In this case, we abbreviate $a_i^* = q_{1i}^i$, $b_i^* = q_{1,i+1}^i$, $c_i^* = q_{1,i-1}^i$, $k_i^* = q_{ii}^0$, and $k^* = k_1^* = b_0^*$.

Proof of (i) $\Rightarrow$ (ii). Immediate from [2, Lemma 2.3.1] and recurrences dual to [2, Lemma 4.1.7]. $\square$

Proof of (ii) $\Rightarrow$ (i). Immediate. $\square$

Let Y denote a d -class association scheme. We say that Y is *Q-polynomial* if there exists at least one Q-polynomial structure for Y .

Suppose Y is Q-polynomial, and fix $E_0, \dots, E_d$ a Q-polynomial structure for Y . We refer to E_1 as the *primary idempotent* of the structure, and to the corresponding eigenvalue θ_1 of A_1 as the *primary eigenvalue* of the structure. Observe by (1.12), (1.22), and (1.23) that E_i may be written as an entry-wise polynomial in E_1 of degree exactly i , for $0 \leq i \leq d$ [20, Lemma 3.11]. It follows that E_1 generates M by entry-wise multiplication, that every Q-polynomial structure has $E_0 = J/|X|$, and that distinct Q-polynomial structures have distinct primary idempotents and primary eigenvalues.

Suppose Y is Q-polynomial, and fix $E_0, \dots, E_d$ a Q-polynomial structure for Y . Since $A_0, \dots, A_d$ forms a basis for M , there exist real numbers $\theta_0^*, \dots, \theta_d^*$ satisfying

$$E_1 = \sum_{h=0}^d \theta_h^* A_h. \quad (1.26)$$

We refer to $\theta_0^*, \dots, \theta_d^*$ as the *dual eigenvalues* of the structure. Since E_1 generates M under entry-wise multiplication, the dual eigenvalues $\theta_0^*, \dots, \theta_d^*$ are distinct [20, Lemma 3.11].

Chapter 2

Dual Bipartite Distance-Regular Graphs

2.1 Introduction

In this chapter we recall the notion of a *dual bipartite* Q -polynomial structure for a distance-regular graph. In Theorem 2.1.2 we classify the distance-regular graphs with diameter $d \geq 6$ and valency $k \geq 3$ which admit such structures. This chapter presents joint work with P. Terwilliger.

Lemma 2.1.1 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Suppose $E_0, \dots, E_d$ is a Q -polynomial structure for Γ , with Krein parameters q_{ij}^h . Then the following are equivalent:*

$$(i) \quad q_{ii}^i = 0; \quad 1 \leq i \leq d \quad (2.1)$$

$$(ii) \text{ for } 0 \leq h, i, j \leq d, \quad q_{ij}^h = 0. \quad h + i + j \text{ odd} \quad (2.2)$$

If (i), (ii) hold, then the Q -polynomial structure is said to be *dual bipartite*; if Γ admits at least one dual bipartite Q -polynomial structure, then Γ is said to be *dual bipartite*.

Proof of (i) $\Rightarrow$ (ii). Immediate from recurrences dual to [2, Lemma 4.1.7]. $\square$

Proof of (ii) $\Rightarrow$ (i). Immediate. $\square$

Our main result is the following Theorem:

Theorem 2.1.2 *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and valency $k \geq 3$. Then Γ has a dual bipartite Q -polynomial structure if and only if $b_i = c_{d-i}$ for $0 \leq i \leq d-1$ and the array $\{c_1, \dots, c_d\}$ for Γ is one of:*

- (i) $\{1, 2, 3, \dots, k\}$;
- (ii) $\{1, 6, 15, \dots, 2d^2 - d\}$;
- (iii) $\{1, 4, 9, \dots, d^2\}$;
- (iv) $\{1, k - a_1 - 1, k\}$;
- (v) $\{1, 2\gamma, 4\gamma - 1, 4\gamma\}$, where $\gamma \geq 1$ is an integer;
- (vi) $\{1, \beta\eta, (\beta^2 - 1)(2\eta - \beta + 1), \beta(2\eta + 2\eta\beta - \beta^2)\}$, where $\beta \geq 3$, $\eta \geq 3\beta/4$ are integers and η divides $\beta^2(\beta^2 - 1)/2$;

(vii) $\{1, c, k - c, k - 1, k\}$, where $k = \gamma(\gamma^2 + 3\gamma + 1)$, $c = \gamma(\gamma + 1)$, and $\gamma \geq 2$ is an integer.

The arrays (i), (ii), and (iii) are uniquely realized by the cube $H(d, 2)$, the half-cube $\frac{1}{2}H(2d, 2)$, and the Johnson graph $J(d, 2d)$, respectively. The graphs with array (iv) are uniquely realized by the complement of $K_{k+1} \times K_2$ when $a_1 = 0$; otherwise these are the Taylor graphs. The graphs with array (v) are the Hadamard graphs of order 4γ . We know of no examples with array (vi). The array (vii) is uniquely realized for $\gamma = 2$ by the double cover of the Higman-Sims graph. We know of no examples with $\gamma \geq 3$.

For more information on the graphs in Theorem 2.1.2, see Chapter 9. We prove Theorem 2.1.2 in Section 2.4.

2.2 Preliminaries

Lemma 2.2.1 (Terwilliger [19]) *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, and suppose Γ has a dual bipartite Q -polynomial structure with dual eigenvalues $\theta_0^*, \dots, \theta_d^*$.*

Further suppose Γ is neither the cube $H(d, 2)$, the half-cube $\frac{1}{2}H(2d, 2)$, the Johnson graph $J(d, 2d)$, nor the Gosset graph.

Then $b_i = c_{d-i}$ for $0 \leq i \leq d - 1$, and there exist complex numbers $q \neq 0$ and s with

$$q^j \neq 1, \quad 1 \leq j \leq d \quad (2.3)$$

$$q^j \neq -1, \quad 1 \leq j \leq d - 1 \quad (2.4)$$

$$sq^j \neq 1, \quad 2 \leq j \leq 2d \quad (2.5)$$

such that

$$a_i = -\frac{(q^{2i} - 1)(q^{2d} - q^{2i})(q^d + q^2)(sq^{d+1} + 1)}{(q - 1)(q^{2i} + q^{d+1})(q^{2i+1} + q^d)(sq^{2d} - 1)}, \quad 1 \leq i \leq d - 1 \quad (2.6)$$

$$c_i = \frac{q^{2i-1}(q^d + q)(q^d + q^2)(q^{2i} - 1)(sq^{2d+2-2i} - 1)}{(q^2 - 1)(q^{2i} + q^d)(q^{2i} + q^{d+1})(sq^{2d} - 1)}, \quad 1 \leq i \leq d - 1 \quad (2.7)$$

$$c_d = \frac{(q^d - 1)(q^d + q^2)(sq^2 - 1)}{q(q^2 - 1)(sq^{2d} - 1)}, \quad (2.8)$$

$$\theta_i^* = \frac{q^d(q^{-i} - q^{i-d})(sq^3 - 1)}{(q - 1)(sq^{d+2} - 1)}, \quad 0 \leq i \leq d \quad (2.9)$$

Proof. By the proof of [19, Theorem 3] and in the notation of [1, Theorem 3.5.1], Γ is of type (I) with the restrictions $r_2 = -r_1$, $r_3 = q^{-1-d}$, $s = r_1^2$, $s^* = -q^{-1-d}$. This gives the equations (2.6)–(2.9). The inequalities (2.3), (2.5) are from [20, Note 2.2(iii)], and (2.4) follows from the inequality

$$s^*q^i \neq 1 \quad 2 \leq i \leq 2d \quad (2.10)$$

of the same Note. □

Lemma 2.2.2 *Let Γ be as in Lemma 2.2.1. Then:*

$$(i) \quad \theta_i^* \neq \theta_j^*; \quad 0 \leq i, j \leq d, \quad i \neq j \quad (2.11)$$

$$(ii) \quad \theta_i^* \neq -\theta_j^*. \quad 0 \leq i, j \leq d, \quad i + j \neq d \quad (2.12)$$

Proof of (i). The dual eigenvalues of any Q-polynomial structure are distinct [20, Lemma 3.11(iii)]. □

Proof of (ii). Fix integers i and j with $0 \leq i, j \leq d$ and $i + j \neq d$. Observe $\theta_i^* = -\theta_{d-i}^*$ by (2.9). In particular, if $\theta_i^* = -\theta_j^*$ then we have $\theta_j^* = \theta_{d-i}^*$, and $j = d - i$ by (i) for a contradiction. Thus $\theta_i^* \neq -\theta_j^*$, as desired. □

Let $\Gamma = (X, R)$ denote a distance-regular graph. By the *standard module* for Γ we mean the column vector space $V = \mathbb{R}^X$, with inner product

$$\langle u, v \rangle = u^t v. \quad u, v \in V \quad (2.13)$$

For each vertex $x \in X$, let $\hat{x}$ denote the vector in V with a 1 in the x coordinate and 0's elsewhere. Observe that $\{\hat{x} \mid x \in X\}$ forms an orthonormal basis for V .

Suppose Γ has a Q-polynomial structure with primary idempotent E and dual eigenvalues $\theta_0^*, \dots, \theta_d^*$. For arbitrary $x, y \in X$ it can be seen from (1.26), (2.13) that

$$\langle E\hat{x}, \hat{y} \rangle = \frac{1}{|X|} \theta_h^*, \quad (2.14)$$

where $h = \partial(x, y)$.

Definition 2.2.3 Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter d . For arbitrary integers i, j with $0 \leq i, j \leq d$ and arbitrary vertices $x, y \in X$, define:

$$(i) \quad \Delta_{ij}(x, y) = \{w \in X \mid \partial(x, w) = i, \partial(y, w) = j\}; \quad (2.15)$$

$$(ii) \quad \delta_{ij}(x, y) = \sum_{w \in \Delta_{ij}(x, y)} \hat{w}. \quad (2.16)$$

Lemma 2.2.4 (Terwilliger [19]) *Let $\Gamma = (X, R)$ be as in Lemma 2.2.1. Fix integers i, j with $0 \leq i, j \leq d$. Fix distinct vertices $x, y \in X$ with $\partial(x, y) \leq d - 1$, and set $h = \partial(x, y)$. Then:*

$$(i) \quad 0 = \langle EV, \delta_{ij}(x, y) - p_{ij}^h \frac{\theta_j^* \theta_h^* - \theta_i^* \theta_0^*}{\theta_h^{*2} - \theta_0^{*2}} \hat{x} - p_{ij}^h \frac{\theta_i^* \theta_h^* - \theta_j^* \theta_0^*}{\theta_h^{*2} - \theta_0^{*2}} \hat{y} \rangle; \quad (2.17)$$

(ii)

$$0 = \langle EV, \delta_{ii}(x, y) - p_{ii}^h \frac{\theta_i^*}{\theta_h^* + \theta_0^*} (\hat{x} + \hat{y}) \rangle. \quad (2.18)$$

Proof of (i). By [19, Theorem 3(ii)] there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ with

$$0 = \langle EV, \delta_{ij}(x, y) - \alpha_1 \hat{x} - \alpha_2 \hat{y} \rangle. \quad (2.19)$$

In particular,

$$0 = \langle E\hat{x}, \delta_{ij}(x, y) - \alpha_1 \hat{x} - \alpha_2 \hat{y} \rangle, \quad (2.20)$$

$$0 = \langle E\hat{y}, \delta_{ij}(x, y) - \alpha_1 \hat{x} - \alpha_2 \hat{y} \rangle. \quad (2.21)$$

Evaluating (2.20), (2.21) using (2.14),

$$0 = p_{ij}^h \theta_i^* - \alpha_1 \theta_0^* - \alpha_2 \theta_h^*, \quad (2.22)$$

$$0 = p_{ij}^h \theta_j^* - \alpha_1 \theta_h^* - \alpha_2 \theta_0^*. \quad (2.23)$$

Solving (2.22), (2.23) for α_1, α_2 , we obtain

$$\alpha_1 = p_{ij}^h \frac{\theta_j^* \theta_h^* - \theta_i^* \theta_0^*}{\theta_h^{*2} - \theta_0^{*2}}, \quad (2.24)$$

$$\alpha_2 = p_{ij}^h \frac{\theta_i^* \theta_h^* - \theta_j^* \theta_0^*}{\theta_h^{*2} - \theta_0^{*2}}, \quad (2.25)$$

with $\theta_h^{*2} - \theta_0^{*2} \neq 0$ by Lemma 2.2.2 and the assumption $h \neq d$. Evaluating (2.19) using (2.24), (2.25), we obtain (2.17) as desired. $\square$

Proof of (ii). Setting $j = i$ in (2.17) and simplifying, we obtain (2.18). $\square$

2.3 A Diameter Bound

Lemma 2.3.1 *Let Γ be as in Lemma 2.2.1. Then the following are equivalent:*

(i) $a_1 = 0$;

(ii) $sq^{d+1} = -1$;

(iii) $a_i = 0$ for $0 \leq i \leq d$.

Proof. Immediate from (2.3), (2.4), and (2.6). $\square$

Definition 2.3.2 A graph Γ is said to be *strongly regular* with parameters (ν, k, λ, μ) if Γ has ν vertices and is regular of degree k , adjacent vertices of Γ have λ common neighbors, and non-adjacent vertices of Γ have μ common neighbors.

Lemma 2.3.3 Let $\Gamma = (X, R)$ be as in Lemma 2.2.1, and suppose $a_1 \neq 0$. Define γ_1, γ_2 by

$$\gamma_1 = -1 - \frac{q^{d+1} + q^2}{q^d + q^3}, \quad (2.26)$$

$$\gamma_2 = -1 + \frac{q^{d-1}(q^d + q)(sq^d - 1)}{(q^d + q^3)(sq^{2d} - 1)}. \quad (2.27)$$

Fix a vertex $x \in X$, and let Γ_1 denote the induced graph on the vertices adjacent to x in Γ . Then:

(i) Γ_1 is strongly regular with parameters (k, a_1, λ, μ) , for $k = b_0 = c_d$ as in (2.8), a_1 as in (2.6),

$$\lambda = \mu + \gamma_1 + \gamma_2, \quad (2.28)$$

and

$$\mu = -\frac{q^d(q^2 - 1)(q + 1)(q^d + q^2)(sq^{d+1} + 1)}{(q^d + q^3)^2(sq^{2d} - 1)}, \quad (2.29)$$

(ii) Γ_1 is connected and is not a clique;

(iii) the distinct eigenvalues of Γ_1 are a_1, γ_1, γ_2 ;

(iv) if $d \geq 4$, then $\gamma_1, \gamma_2 \in \mathbb{Z}$.

Proof of (i). Clearly Γ_1 has k vertices and is regular of degree a_1 .

Pick distinct vertices $y, z \in X$ with $\partial(x, y) = 1$, $\partial(x, z) = 1$, and let $h = \partial(y, z)$. Let $\sigma = \sigma(x, y, z)$ denote the number of vertices $w \in \Delta_{11}(y, z)$ with $\partial(x, w) = 1$. Applying Lemma 2.2.4(ii) and (2.14),

$$0 = \langle E\hat{x}, \delta_{11}(y, z) - p_{11}^h \frac{\theta_1^*}{\theta_h^* + \theta_0^*} (\hat{y} + \hat{z}) \rangle \quad (2.30)$$

$$= \theta_0^* + \sigma\theta_1^* + (p_{11}^h - \sigma - 1)\theta_2^* - p_{11}^h \frac{2\theta_1^{*2}}{\theta_h^* + \theta_0^*}. \quad (2.31)$$

Solving for σ , we have

$$\sigma = -\frac{(\theta_0^* - \theta_2^*)(\theta_h^* + \theta_0^*) + p_{11}^h \theta_2^*(\theta_h^* + \theta_0^*) - 2p_{11}^h \theta_1^{*2}}{(\theta_1^* - \theta_2^*)(\theta_h^* + \theta_0^*)}. \quad (2.32)$$

In particular, $\sigma = \sigma(x, y, z)$ depends only on $h = \partial(y, z)$. The possible values for h are 1 and 2; both occur, since $a_1 \neq 0$ and $c_2 \neq 0$. We write σ_1, σ_2 for these constants, and observe that Γ_1 is strongly regular with parameters (k, a_1, λ, μ) , where $\lambda = \sigma_1$ and $\mu = \sigma_2$. Computing λ, μ from (2.32) using (2.6), (2.7), and (2.9), we obtain (2.28), (2.29). $\square$

Proof of (ii). Observe $a_1 = k - b_1 - 1 < k - 1$, so that Γ_1 is not a clique. Furthermore, no factor in the numerator of (2.29) is zero by (2.3), (2.4), and Lemma 2.3.1, so that $\mu \neq 0$. It follows that Γ_1 is connected. $\square$

Proof of (iii). By (i), (ii), and [2, Theorem 1.3.1], Γ has exactly three distinct eigenvalues, and these are a_1 and the roots of the quadratic

$$x^2 + (\mu - \lambda)x + (\mu - a_1) = 0. \quad (2.33)$$

Using (2.6) and (2.26)–(2.29) we may check that γ_1 and γ_2 are roots of (2.33). It is easily shown that $\gamma_1 \neq \gamma_2$, by (2.4) and (2.5). Therefore the distinct eigenvalues of Γ_1 are a_1, γ_1, γ_2 as desired. $\square$

Proof of (iv). Suppose the eigenvalues of Γ_1 are not all integers. Then Γ_1 is a conference graph, and in particular $a_1 = 2\mu$ [2, Theorem 1.3.1]. By (2.6), (2.29),

$$-\frac{(q+1)(q^d-q)(q^d+q^2)(sq^{d+1}+1)}{q(q^d+q^3)(sq^{2d}-1)} = 2\frac{q^d(q^2-1)(q+1)(q^d+q^2)(sq^{d+1}+1)}{(q^d+q^3)^2(sq^{2d}-1)}. \quad (2.34)$$

Neither side is zero since $a_1 \neq 0$, so (2.34) reduces to

$$(q^d - q)(q^d + q^3) = 2q^{d+1}(q^2 - 1). \quad (2.35)$$

Collecting terms and factoring,

$$(q^d + q)(q^d - q^3) = 0. \quad (2.36)$$

Now either $q^{d-1} = -1$ or $q^{d-3} = 1$. Since $d \geq 4$, this contradicts either (2.4) or (2.3). Thus Γ_1 is not a conference graph, and the eigenvalues of Γ_1 are integers as desired. $\square$

Lemma 2.3.4 *Let Γ be as in Lemma 2.2.1, and suppose $a_1 \neq 0$. Then*

$$p_{2,i-1}^i \neq 0. \quad 2 \leq i \leq d-1 \quad (2.37)$$

Proof. Fix i with $2 \leq i \leq d-1$. By a counting argument [2, Lemma 4.1.7] and (2.6),

$$p_{2,i-1}^i = \frac{c_i}{c_2}(a_i + a_{i-1} - a_1) \quad (2.38)$$

$$= \frac{c_i(q^d + q^2)(q^d + q)(q^2 + 1)(q^{2i-2} - 1)(q^{2i} - q^{2d})(sq^{d+1} + 1)}{c_2(q^d + q^3)(q - 1)(q^{d+1} + q^{2i-2})(q^d + q^{2i+1})(sq^{2d} - 1)}. \quad (2.39)$$

No factor in the numerator of (2.39) is zero by (2.3), (2.4), and Lemma 2.3.1, so $p_{2,i-1}^i \neq 0$ as desired. $\square$

Lemma 2.3.5 *Let $\Gamma = (X, R)$ be as in Lemma 2.2.1, and suppose $a_1 \neq 0$. Fix an integer i with $2 \leq i \leq d-1$ and vertices $x, y, z \in X$ with $\partial(x, y) = i$, $\partial(x, z) = i-1$, and $\partial(y, z) = 2$. (Such x, y, z exist by Lemma 2.3.4.) Define L_i, H_i by*

$$L_i = |\{w \in \Delta_{11}(y, z) \mid \partial(x, w) = i-1\}|, \quad (2.40)$$

$$H_i = |\{w \in \Delta_{11}(y, z) \mid \partial(x, w) = i\}|. \quad (2.41)$$

Then:

$$(i) \quad c_2 = L_i + H_i; \quad (2.42)$$

$$(ii) \quad L_i = \frac{(q^d + q)(q^d + q^2)(q^{2i+1} + q^d)(sq^{2d-2} - 1)q^3}{(q^d + q^3)(q^d + q^4)(q^{2i} + q^{d+1})(sq^{2d} - 1)}, \quad (2.43)$$

$$H_i = \frac{(q^d + q)(q^d + q^2)(q^{2i-3} + q^d)(sq^{2d-2} - 1)q^5}{(q^d + q^3)(q^d + q^4)(q^{2i} + q^{d+1})(sq^{2d} - 1)}. \quad (2.44)$$

In particular, L_i and H_i are constants independent of x, y, z .

Proof of (i). Observe

$$L_i + H_i = |\Delta_{11}(y, z)| \quad (2.45)$$

$$= c_2. \quad (2.46)$$

□

Proof of (ii). By Lemma 2.2.4(ii) and (2.14),

$$0 = \langle E\hat{x}, \delta_{11}(y, z) - c_2 \frac{\theta_1^*}{\theta_2^* + \theta_0^*}(\hat{y} + \hat{z}) \rangle \quad (2.47)$$

$$= L_i \theta_{i-1}^* + H_i \theta_i^* - c_2 \frac{\theta_1^*(\theta_i^* + \theta_{i-1}^*)}{\theta_2^* + \theta_0^*}. \quad (2.48)$$

Eliminating H_i in (2.48) using (2.42) and solving for L_i we obtain

$$L_i = c_2 \frac{\theta_1^*(\theta_i^* + \theta_{i-1}^*) - \theta_i^*(\theta_2^* + \theta_0^*)}{(\theta_{i-1}^* - \theta_i^*)(\theta_2^* + \theta_0^*)}. \quad (2.49)$$

Applying (2.7) and (2.9) gives (2.43). A similar argument gives (2.44). □

Lemma 2.3.6 Let Γ be as in Lemma 2.2.1, and suppose $a_1 \neq 0$. Let L_i, H_i be as in Lemma 2.3.5. Then:

(i) $L_i \neq 0$ and $H_i \neq 0$ for $2 \leq i \leq d-1$;

(ii) if $d \geq 4$ then $L_2 \leq L_3$.

Proof of (i). Each term in the numerator of (2.43), (2.44) is non-zero by (2.4), (2.5). □

Proof of (ii). Suppose $d \geq 4$. Then by Lemma 2.3.4 there exist vertices $x, y, z \in X$ with $\partial(x, y) = 3$, $\partial(x, z) = 2$, and $\partial(y, z) = 2$. Since $L_3 \neq 0$, there exists a vertex $u \in \Delta_{11}(y, z)$ with $\partial(x, u) = 2$. Observe that any vertex w adjacent to both x and u has $\partial(y, w) = 2$, since $\partial(x, y) = 3$ and $\partial(u, y) = 1$. In particular,

$$L_2 = |\{w \in \Delta_{11}(x, z) \mid \partial(u, w) = 1\}| \quad (2.50)$$

$$\leq |\{w \in \Delta_{11}(x, z) \mid \partial(y, w) = 2\}| \quad (2.51)$$

$$= L_3, \quad (2.52)$$

as desired. $\square$

Lemma 2.3.7 *Let Γ be as in Lemma 2.2.1, and suppose $a_1 \neq 0$ and $d \geq 4$. Let $\beta = q + q^{-1}$. Then:*

- (i) $\beta^2 L_2 H_3 = c_2^2$, where L_2, H_3 are as in Lemma 2.3.5;
- (ii) $\beta^2 \in \mathbb{Q}$;
- (iii) $\beta^2 > 4$;
- (iv) $q \in \mathbb{R}$.

Proof of (i). Compute $\beta^2 L_2 H_3$ using (2.43) and (2.44), and c_2^2 using (2.7). The expressions obtained are equal. $\square$

Proof of (ii). Immediate from (i) and Lemma 2.3.6(i), since c_2, L_2 , and L_3 are integers. $\square$

Proof of (iii). By (i), (2.42), and Lemma 2.3.6(ii),

$$\beta^2 = \frac{c_2^2}{L_2 H_3} \quad (2.53)$$

$$= \frac{c_2^2}{L_2(c_2 - L_3)} \quad (2.54)$$

$$\geq \frac{c_2^2}{L_2(c_2 - L_2)} \quad (2.55)$$

$$\geq 4. \quad (2.56)$$

If $\beta^2 = 4$ then $q = \pm 1$, contradicting (2.3) or (2.4). Thus $\beta^2 > 4$, as desired. $\square$

Proof of (iv). Solving $\beta = q + q^{-1}$ for q we find $q \in \mathbb{R}$ by (iii). $\square$

Lemma 2.3.8 *Let Γ be as in Lemma 2.2.1, and suppose $a_1 \neq 0$ and $d \geq 4$. Let $\beta = q + q^{-1}$, and define*

$$\chi = \frac{q^{d+1} + q^2}{q^d + q^3}. \quad (2.57)$$

Then:

- (i) $\chi \in \mathbb{Z}$;
- (ii) $\beta \in \mathbb{Z}$;
- (iii) $d = 4$.

Proof of (i). Let γ_1 be as in (2.26). Then $\chi = -1 - \gamma_1$, and χ is an integer by Lemma 2.3.3(iv). $\square$

Proof of (ii). We split the proof into two cases, according to whether d is odd or even. In each case we show β is an integer by finding that β is an algebraic integer and that β is rational.

Case d odd. Let $m = (d-1)/2$, and observe $m \geq 1$. By (2.57),

$$\chi = \frac{q^m + q^{-m}}{q^{m-1} + q^{1-m}}. \quad (2.58)$$

Define polynomials $T_0, T_1, \dots$ in the indeterminant x by

$$T_0 = 2, \quad T_1 = x, \quad T_{i+1} = xT_i - T_{i-1}. \quad (2.59)$$

Easy induction arguments show that for $i \geq 0$,

$$T_i \in \mathbb{Z}[x], \quad (2.60)$$

$$T_i \text{ is monic of degree } i, \quad (2.61)$$

$$T_i(-x) = (-1)^i T_i(x), \quad (2.62)$$

$$T_i(\beta) = q^i + q^{-i}. \quad (2.63)$$

Furthermore, by (2.4) and (2.63)

$$T_i(\beta) \neq 0. \quad 0 \leq i \leq m \quad (2.64)$$

By (2.58) and (2.63),

$$T_m(\beta) - \chi T_{m-1}(\beta) = 0. \quad (2.65)$$

Now β is an algebraic integer by (2.60), (2.61), and (i). It remains to be shown that β is rational.

By (2.60), (2.62), there exist polynomials T'_i with

$$T'_i(x) \in \mathbb{Z}[x], \quad (2.66)$$

$$T_i(x) = \begin{cases} T'_i(x^2) & i \text{ even,} \\ xT'_i(x^2) & i \text{ odd.} \end{cases} \quad (2.67)$$

If m is odd, then evaluating (2.65) using (2.67),

$$\beta T'_m(\beta^2) - \chi T'_{m-1}(\beta^2) = 0. \quad (2.68)$$

Observe $\beta T'_m(\beta^2) = T_m(\beta) \neq 0$ by (2.67), (2.64), and

$$\beta = \frac{\chi T'_{m-1}(\beta^2)}{T'_m(\beta^2)}. \quad (2.69)$$

Now $\beta \in \mathbb{Q}$ by (i), (2.66), and Lemma 2.3.7(ii).

If m is even, then evaluating (2.65) using (2.67),

$$T'_m(\beta^2) - \chi \beta T'_{m-1}(\beta^2) = 0. \quad (2.70)$$

Here $T'_m(\beta^2) = T_m(\beta) \neq 0$ by (2.67), (2.64), so $\chi T'_{m-1}(\beta^2) \neq 0$ by (2.70), and

$$\beta = \frac{T'_m(\beta^2)}{\chi T'_{m-1}(\beta^2)}. \quad (2.71)$$

Now $\beta \in \mathbb{Q}$ by (i), (2.66), and Lemma 2.3.7(ii).

Case d even. The argument is similar. Let $n = d/2$, and observe $n \geq 2$. By (2.57),

$$\chi = \frac{(q^n - q^{-n}) - (q^{n-1} - q^{1-n})}{(q^{n-1} - q^{1-n}) - (q^{n-2} - q^{2-n})}. \quad (2.72)$$

Define polynomials $U_{-1}, U_0, \dots$ in the indeterminant x by

$$U_{-1} = 0, \quad U_0 = 1, \quad U_{i+1} = xU_i - U_{i-1}. \quad (2.73)$$

Easy induction arguments show that for $i \geq -1$,

$$U_i \in \mathbb{Z}[x], \quad (2.74)$$

$$U_i \text{ is monic of degree } i, \quad (2.75)$$

$$U_i(-x) = (-1)^i U_i(x), \quad (2.76)$$

$$U_i(\beta) = \frac{q^{i+1} - q^{-i-1}}{q - q^{-1}}. \quad (2.77)$$

Furthermore, by (2.3) and (2.77)

$$U_i(\beta) \neq 0, \quad 0 \leq i \leq n-1 \quad (2.78)$$

By (2.72) and (2.77),

$$U_{n-1}(\beta) - (\chi + 1)U_{n-2}(\beta) + \chi U_{n-3}(\beta) = 0. \quad (2.79)$$

Now β is an algebraic integer by (2.74), (2.75), and (i). It remains to be shown that β is rational.

By (2.74), (2.76), there exist polynomials U'_i with

$$U'_i(x) \in \mathbb{Z}[x], \quad (2.80)$$

$$U_i(x) = \begin{cases} U'_i(x^2) & i \text{ even,} \\ xU'_i(x^2) & i \text{ odd.} \end{cases} \quad (2.81)$$

If n is odd, then evaluating (2.79) using (2.81),

$$U'_{n-1}(\beta^2) - (\chi + 1)\beta U'_{n-2}(\beta^2) + \chi U'_{n-3}(\beta^2) = 0. \quad (2.82)$$

Observe $\chi + 1 \neq 0$; otherwise by (2.57) we have

$$0 = \frac{q^{d+1} + q^2}{q^d + q^3} + 1 \quad (2.83)$$

$$= \frac{(q^d + q^2)(q + 1)}{q^d + q^3}, \quad (2.84)$$

contradicting (2.4). Furthermore $\beta U'_{n-2}(\beta^2) = U_{n-2}(\beta) \neq 0$ by (2.81), (2.78). Now by (2.82),

$$\beta = \frac{U'_{n-1}(\beta^2) + \chi U'_{n-3}(\beta^2)}{(\chi + 1)U'_{n-2}(\beta^2)}, \quad (2.85)$$

and $\beta \in \mathbb{Q}$ by (i), (2.80), and Lemma 2.3.7(ii).

If n is even, then evaluating (2.79) using (2.81),

$$\beta U'_{n-1}(\beta^2) - (\chi + 1)U'_{n-2}(\beta^2) + \chi\beta U'_{n-3}(\beta^2) = 0. \quad (2.86)$$

Again $\chi + 1 \neq 0$, and $U'_{n-2}(\beta^2) = U_{n-2}(\beta) \neq 0$ by (2.81), (2.78). This gives $U'_{n-1}(\beta^2) + \chi U'_{n-3}(\beta^2) \neq 0$ by (2.86), and

$$\beta = \frac{(\chi + 1)U'_{n-2}(\beta^2)}{U'_{n-1}(\beta^2) + \chi U'_{n-3}(\beta^2)}. \quad (2.87)$$

Now $\beta \in \mathbb{Q}$ by (i), (2.80), and Lemma 2.3.7(ii). □

Proof of (iii). Observe by (2.57) that

$$\beta - \chi = \frac{q^{d-1} + q^4}{q^d + q^3}, \quad (2.88)$$

and note $q \in \mathbb{R}$ by Lemma 2.3.7(iv). Suppose for a contradiction that $d \geq 5$. Observe that the right hand side of (2.88) is unchanged when q is replaced by q^{-1} , so that we may assume $|q| > 1$. In each of the three cases $q < -1$ and d even, $q < -1$ and d odd, $1 < q$, a simple argument now shows that $|\beta - \chi| < 1$. For example, suppose for a contradiction that $1 < q$ and $|\beta - \chi| \geq 1$. It is clear from (2.88) and $1 < q$ that $\beta - \chi$ is non-negative, and

$$\frac{q^{d-1} + q^4}{q^d + q^3} \geq 1. \quad (2.89)$$

We also have $q^d + q^3 > 0$, so

$$q^{d-1} + q^4 \geq q^d + q^3. \quad (2.90)$$

Collecting terms and factoring,

$$0 \geq (q - 1)(q^{d-1} - q^3). \quad (2.91)$$

Each of the terms in the right hand side of (2.91) is positive by (2.3), $d \geq 5$, and $1 < q$. This gives a contradiction, so $|\beta - \chi| < 1$ in the case $1 < q$. The other cases are similar.

Now $\beta - \chi = 0$ by (i), (ii), and $q^{d-5} = -1$ by (2.88). This contradicts (2.4), so $d = 4$ as desired. □

Note 2.3.9 The polynomials T_j, U_j of (2.59), (2.73) are essentially the Tchebichev polynomials of the first and second kind [14, p. 3].

We need one more technical lemma, used in the bipartite case:

Lemma 2.3.10 *Let Γ be as in Lemma 2.2.1, and suppose $a_1 = 0$. Fix an integer i with $2 \leq i \leq d - 1$, and suppose there exist vertices $x, y, z \in X$ with $\partial(x, y) = i$, $\partial(x, z) = i$, and $\partial(y, z) = 2$. Define the integer D_i by*

$$D_i = |\{w \in \Delta_{11}(y, z) \mid \partial(x, w) = i - 1\}|. \quad (2.92)$$

Then

$$D_i = -c_2 \frac{\theta_0^* \theta_{i+1}^* - 2\theta_1^* \theta_i^* + \theta_2^* \theta_{i+1}^*}{(\theta_{i-1}^* - \theta_{i+1}^*)(\theta_2^* + \theta_0^*)}. \quad (2.93)$$

In particular, D_i is a constant independent of x, y, z .

Proof. Observe $a_i = 0$ by Lemma 2.3.1. By Lemma 2.2.4(ii) and (2.14),

$$0 = (E\hat{x}, \delta_{11}(y, z) - c_2 \frac{\theta_1^*}{\theta_2^* + \theta_0^*}(\hat{y} + \hat{z})) \quad (2.94)$$

$$= D_i \theta_{i-1}^* + (c_2 - D_i) \theta_{i+1}^* - c_2 \frac{2\theta_1^* \theta_i^*}{\theta_2^* + \theta_0^*} \quad (2.95)$$

$$= D_i(\theta_{i-1}^* - \theta_{i+1}^*) + c_2 \frac{\theta_0^* \theta_{i+1}^* - 2\theta_1^* \theta_i^* + \theta_2^* \theta_{i+1}^*}{(\theta_{i-1}^* - \theta_{i+1}^*)(\theta_2^* + \theta_0^*)}. \quad (2.96)$$

Solving for D_i , we obtain (2.93) as desired. $\square$

2.4 Proof of Theorem 2.1.2

Proof of Theorem 2.1.2. Let Γ denote a distance-regular graph with diameter $d \geq 3$ and valency $k \geq 3$, and suppose Γ has a dual bipartite Q-polynomial structure. Further suppose Γ is neither the cube $H(d, 2)$, the half cube $\frac{1}{2}H(2d, 2)$, the Johnson graph $J(d, 2d)$, nor the Gosset graph (which has array (iv) with $k = 27$, $a_1 = 16$). Then Lemma 2.2.1 applies. In particular, $b_i = c_{d-i}$ for $0 \leq i \leq d-1$. If $d = 3$, then the restrictions $c_2 = b_1 = k - a_1 - 1$, $c_3 = b_0 = k$ give Γ array (iv), so we may assume $d \geq 4$.

Suppose $a_1 = 0$. By Lemma 2.3.1, we have $a_i = 0$ for $1 \leq i \leq d$. By Lemma 2.3.10, Γ is 2-homogeneous in the sense of Nomura. In particular, Nomura's classification of bipartite and almost bipartite 2-homogeneous distance-regular graphs applies [13]. The only possibilities with diameter $d \geq 4$, valency $k \geq 3$, and $a_d = 0$ are the graphs with intersection arrays (v), (vii).

Suppose $a_1 \neq 0$, and let $\beta = q + q^{-1}$. By Lemma 2.3.8 we have $\beta \in \mathbb{Z}$ and $d = 4$. Setting $\eta = L_2$, we compute from (2.7), (2.8), (2.43) that Γ has array (vi). Recall $\eta = L_2 > 0$ by Lemma 2.3.6 and $c_2 = \beta\eta$, so that $\beta > 0$. By Lemma 2.3.7(iii) we have $\beta^2 > 4$, so that $\beta > 2$; by $\beta \in \mathbb{Z}$, we have $\beta \geq 3$. We may calculate from (2.28), (2.43) that $\lambda = 4\eta - 3\beta$, so that by $\lambda \geq 0$ we have $\eta \geq 3\beta/4$. Finally, we compute from the intersection array using the formulas of [2, Section 4.1] that the eigenvalues of Γ are $k, k/\beta, a_2/\beta^2, -\beta, -\beta^2$ with the multiplicity f of k/β given by

$$f = -\frac{\beta(\beta^3 - 2\eta\beta^2 - 2\eta\beta - \beta + 2\eta)}{2\eta}. \quad (2.97)$$

From $f \in \mathbb{Z}$ we conclude η divides $\beta^2(\beta^2 - 1)/2$. Thus Γ has intersection array (vi).

Uniqueness of the graphs with arrays (i)–(iii) is by [16], [18]. The Q-polynomial structures on (i)–(v) are well-known; see Chapter 9. The graphs with arrays (v), (vii) have a dual bipartite Q-polynomial structure by computation of the Krein parameters, using the formulas of [2, Section 4.1]. $\square$

Chapter 3

Almost Dual Imprimitve Distance-Regular Graphs

3.1 Introduction

In this chapter we introduce the notions of *almost dual bipartite* and *almost dual antipodal* Q -polynomial structures for a distance-regular graph, and show that such structures co-exist. A distance-regular graph with such structures is said to be *almost dual imprimitive*. In Theorem 3.1.4 we classify the almost dual imprimitive distance-regular graphs with diameter $d \geq 4$ and valency $k \geq 3$.

Lemma 3.1.1 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Suppose $E_0, \dots, E_d$ is a Q -polynomial structure for Γ , with Krein parameters q_{ij}^h . Then the following are equivalent:*

$$(i) \quad q_{1i}^i = 0, \quad 1 \leq i \leq d-1 \quad (3.1)$$

$$q_{1d}^d \neq 0; \quad (3.2)$$

(ii) for $0 \leq i, j \leq d$,

$$q_{hj}^i = 0, \quad i+j = h+1, \quad 0 \leq h \leq d-1 \quad (3.3)$$

$$q_{dj}^i \neq 0, \quad i+j = d+1. \quad (3.4)$$

If (i), (ii) hold, then the Q -polynomial structure is said to be almost dual bipartite.

Proof of (i) $\Rightarrow$ (ii). Immediate from recurrences dual to [2, Lemma 4.1.7]. $\square$

Proof of (ii) $\Rightarrow$ (i). Immediate from (1.13). $\square$

Lemma 3.1.2 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Suppose $E_0, \dots, E_d$ is a Q -polynomial structure for Γ , with Krein parameters q_{ij}^h . Then the following are equivalent:*

$$(i) \quad q_{id}^d = 0, \quad 2 \leq i \leq d \quad (3.5)$$

$$q_{1d}^d \neq 0; \quad (3.6)$$

$$(ii) \text{ for } 0 \leq i, j \leq d, \quad q_{dj}^i = 0, \quad i+j > d+1 \quad (3.7)$$

$$q_{dj}^i \neq 0, \quad i+j = d+1 \quad (3.8)$$

If (i), (ii) hold, then the Q -polynomial structure is said to be almost dual antipodal.

Proof of (i) $\Rightarrow$ (ii). Fix n maximal such that there exist integers i, j with $q_{dj}^i \neq 0$, $i + j = n$, and $0 \leq i, j \leq d$. By the maximality of n , (1.13), [2, Lemma 2.3.1(iv)], and recurrences dual to [2, Lemma 4.1.7],

$$q_{d,j+1}^{i-1} = \frac{b_j^* k_i^*}{b_{j-1}^* k_{i-1}^*} q_{dj}^i, \quad (3.9)$$

$$q_{d,j-1}^{i+1} = \frac{b_i^* k_i^*}{b_{j-1}^* k_{i+1}^*} q_{dj}^i. \quad (3.10)$$

In particular, $q_{d,n-d}^d \neq 0$, and by (1.13), (3.5) we have $n \leq d + 1$. By (1.13), (3.6) we have $n = d + 1$. Now (3.7), (3.8) are immediate from (3.9), (3.10), and the maximality of n . $\square$

Proof of (ii) $\Rightarrow$ (i). Immediate from (1.13). $\square$

Lemma 3.1.3 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. For each ordering $E_0, \dots, E_d$ of the primitive idempotents of Γ , the following are equivalent:*

- (i) $E_0, \dots, E_d$ is an almost dual bipartite Q -polynomial structure for Γ ;
- (ii) $E_0, E_2, E_4, \dots, E_5, E_3, E_1$ is an almost dual antipodal Q -polynomial structure for Γ .

If (i), (ii) hold for some ordering $E_0, \dots, E_d$ of the primitive idempotents of Γ , then Γ is said to be almost dual imprimitive.

Proof of (i) $\Rightarrow$ (ii). Equations (1.24), (1.25), (3.3), (3.4) with respect to the ordering $E_0, \dots, E_d$ imply (1.22), (1.23), (3.5), (3.6) with respect to the ordering $E_0, E_2, E_4, \dots, E_5, E_3, E_1$. $\square$

Proof of (ii) $\Rightarrow$ (i). Equations (1.24), (1.25), (3.7), (3.8) with respect to the ordering $E_0, E_2, E_4, \dots, E_5, E_3, E_1$ imply (1.22), (1.23), (3.1), (3.2) with respect to the ordering $E_0, \dots, E_d$. $\square$

Our main result is the following Theorem:

Theorem 3.1.4 *Let Γ denote a distance-regular graph with diameter $d \geq 4$ and valency $k \geq 3$. Then Γ is almost dual imprimitive if and only if Γ is one of:*

- (i) the half-cube $\frac{1}{2}H(2d+1, 2)$;
- (ii) the antipodal quotient of the cube $\tilde{H}(2d+1, 2)$;
- (iii) the dual polar graph on $[{}^2A_{2d-1}(r)]$, $r \geq 2$ a prime power.

In particular, if Γ is almost dual imprimitive then Γ has precisely one almost dual bipartite Q -polynomial structure and precisely one almost dual antipodal Q -polynomial structure. For the graph (i), the natural ordering of the primitive idempotents is the almost dual bipartite Q -polynomial structure. For the graphs (ii) and (iii), the natural ordering of the primitive idempotents is the almost dual antipodal Q -polynomial structure.

For more information on the graphs in Theorem 3.1.4, see Chapter 9. We prove Theorem 3.1.4 in Section 3.6.

3.2 Preliminaries

Lemma 3.2.1 (Terwilliger [19]) *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, and suppose Γ has an almost dual bipartite Q -polynomial structure with eigenvalues $\theta_0, \dots, \theta_d$, dual eigenvalues $\theta_0^*, \dots, \theta_d^*$, and primary idempotent E .*

Further suppose Γ is neither the half-cube $\frac{1}{2}H(2d+1, 2)$ nor the antipodal quotient of the cube $\tilde{H}(2d+1, 2)$.

Then there exist complex numbers q and s^* with

$$|q| \geq 1, \quad (3.11)$$

$$q^j \neq 1, \quad 1 \leq j \leq 2d \quad (3.12)$$

$$s^* q^j \neq 1, \quad 2 \leq j \leq 2d \quad (3.13)$$

$$s^* q^j \neq -1, \quad 1 \leq j \leq 2d+1 \quad (3.14)$$

such that

$$a_i = -\frac{(q^{2i}-1)(s^* q^3-1)(s^* q^{2d+2}-1)(s^{*2} q^{2i+2}-1)}{(q-1)(s^* q^{2i}-1)(s^* q^{2i+2}-1)(s^{*2} q^{2d+4}-1)}, \quad 1 \leq i \leq d-1 \quad (3.15)$$

$$a_d = -\frac{(q^{2d}-1)(s^* q^3-1)(s^{*2} q^{2d+2}-1)}{(q-1)(s^* q^{2d}-1)(s^{*2} q^{2d+4}-1)}, \quad (3.16)$$

$$b_i = -\frac{q(q^{2d}-q^{2i})(s^* q^2-1)(s^* q^3-1)(s^{*2} q^{2i+2}-1)}{(q^2-1)(s^* q^{2i+1}-1)(s^* q^{2i+2}-1)(s^{*2} q^{2d+4}-1)}, \quad 1 \leq i \leq d-1 \quad (3.17)$$

$$b_0 = -\frac{q(q^{2d}-1)(s^* q^3-1)(s^* q+1)}{(q^2-1)(s^{*2} q^{2d+4}-1)}, \quad (3.18)$$

$$c_i = \frac{(q^{2i}-1)(s^* q^2-1)(s^* q^3-1)(s^{*2} q^{2d+2i+2}-1)}{(q^2-1)(s^* q^{2i}-1)(s^* q^{2i+1}-1)(s^{*2} q^{2d+4}-1)}, \quad 1 \leq i \leq d-1 \quad (3.19)$$

$$c_d = \frac{(q^{2d}-1)(s^* q^2-1)(s^* q^3-1)(s^{*2} q^{2d+1}+1)}{(q^2-1)(s^* q^{2d}-1)(s^{*2} q^{2d+4}-1)}, \quad (3.20)$$

$$\theta_i = -\frac{(q^i-1)(q^{2d+1-i}-1)(s^* q^2-1)(s^* q^3-1)}{(q^2-1)(s^{*2} q^{2d+4}-1)} + b_0, \quad 0 \leq i \leq d \quad (3.21)$$

$$a_d^* = -\frac{(q^{2d}-q)(q^{d+1}-1)(s^* q^{d+1}+1)}{q^d(q-1)^2(s^* q^{2d+1}+1)}, \quad (3.22)$$

$$b_i^* = -\frac{(q^{2d}-q)(q^{2d+1}-q^i)(s^*q^{i+1}+1)}{(q-1)(q^{2d+1}-q^{2i})(s^*q^{2d+1}+1)}, \quad 1 \leq i \leq d-1 \quad (3.23)$$

$$b_0^* = -\frac{(q^{2d}-q)(s^*q+1)}{(q-1)(s^*q^{2d+1}+1)}, \quad (3.24)$$

$$c_i^* = \frac{(q^{2d}-q)(q^i-1)(s^*q^{2d-i+2}+1)}{q^i(q-1)(q^{2d-2i+1}-1)(s^*q^{2d+1}+1)}, \quad 1 \leq i \leq d \quad (3.25)$$

$$\theta_i^* = -\frac{(q^{2d}-q)(s^*q^{2i+1}+1)}{q^i(q-1)(s^*q^{2d+1}+1)}, \quad 0 \leq i \leq d \quad (3.26)$$

Proof. By the proof of [19, Theorem 3] and in the notation of [1, Theorem 3.5.1], Γ is of type (I) with the restrictions $r_1 = -s^*$, $r_2 = -q^{-d-1}$, $r_3 = q^{-d-1}$, and $s = q^{-2d-2}$. This gives complex numbers $q \neq 0$ and s^* such that (3.15)–(3.26) hold. The inequalities (3.13), (3.14) are from [19, Theorem 3(d)], and (3.12) follows from the inequality

$$sq^i \neq 1, \quad 2 \leq i \leq 2d \quad (3.27)$$

of [20, Note 2.2(iii)]. It remains to be shown that we can assume (3.11). If $s^* = 0$, then $a_1 = -1 - q$, so that q is an integer and (3.11) holds. If $s^* \neq 0$ and $|q| < 1$, then we may replace q by q^{-1} and s^* by s^{*-1} . Observe that the equations (3.12)–(3.26) are unchanged under this substitution, and that (3.11) now holds as desired. $\square$

Lemma 3.2.2 *Let Γ be as in Lemma 3.2.1. Then:*

$$(i) \quad \theta_i^* \neq \theta_j^*; \quad 0 \leq i, j \leq d, \quad i \neq j \quad (3.28)$$

$$(ii) \quad \theta_i^* \neq -\theta_j^*. \quad 0 \leq i, j \leq d \quad (3.29)$$

Proof of (i). The dual eigenvalues of any Q-polynomial structure are distinct [20, Lemma 3.11(iii)]. $\square$

Proof of (ii). By (3.26),

$$\theta_i^* + \theta_j^* = -\frac{(q^{2d}-q)(s^*q^{i+j+1}+1)(q^i+q^j)}{q^{i+j}(q-1)(s^*q^{2d+1}+1)}, \quad 0 \leq i, j \leq d \quad (3.30)$$

No factor in the numerator of (3.30) is zero by (3.12) and (3.14). $\square$

Let $\Gamma = (X, R)$ denote a distance-regular graph. Recall from page 9 the standard module V , with orthonormal basis $\{\hat{x} \mid x \in X\}$. Recall from Definition 2.2.3 the set of vertices $\Delta_{ij}(x, y)$ and the vector $\delta_{ij}(x, y) \in V$.

Lemma 3.2.3 (Terwilliger [19]) *Let $\Gamma = (X, R)$ be as in Lemma 3.2.1. Fix integers i, j with $0 \leq i, j \leq d$. Fix distinct vertices $x, y \in X$, and set $h = \partial(x, y)$. Then:*

$$(i) \quad 0 = \langle EV, \delta_{ij}(x, y) - p_{ij}^h \frac{\theta_j^* \theta_h^* - \theta_i^* \theta_0^*}{\theta_h^{*2} - \theta_0^{*2}} \hat{x} - p_{ij}^h \frac{\theta_j^* \theta_h^* - \theta_i^* \theta_0^*}{\theta_h^{*2} - \theta_0^{*2}} \hat{y} \rangle; \quad (3.31)$$

$$(ii) \quad 0 = \langle EV, \delta_{ii}(x, y) - p_{ii}^h \frac{\theta_i^*}{\theta_h^* + \theta_0^*} (\hat{x} + \hat{y}) \rangle. \quad (3.32)$$

Proof. Similar to the proof of Lemma 2.2.4. $\square$

3.3 The Parameters q and s^* are in $\mathbb{R}$

Lemma 3.3.1 *Let Γ be as in Lemma 3.2.1. Then the following are equivalent:*

- (i) $a_1 = 0$;
- (ii) $s^* q^{2d+2} = 1$;
- (iii) $a_i = 0$ for $1 \leq i \leq d-1$ and $a_d \neq 0$.

Proof. Immediate from (3.15), (3.16), and (3.12)–(3.14). $\square$

Lemma 3.3.2 *Let Γ be as in Lemma 3.2.1, and suppose $a_1 \neq 0$. Then*

$$p_{2,i-1}^i \neq 0. \quad 2 \leq i \leq d \quad (3.33)$$

Proof. By a counting argument,

$$p_{2,i-1}^i = \frac{c_i}{c_2} (a_i + a_{i-1} - a_1), \quad 2 \leq i \leq d \quad (3.34)$$

[2, Lemma 4.1.7]. Evaluating (3.34) using (3.15), (3.16), for $2 \leq i \leq d-1$ we have

$$p_{2,i-1}^i = - \frac{c_i(q^2+1)(q^{2i-2}-1)(s^*q^2-1)(s^*q^3-1)(s^*q^{2d+2}-1)(s^{*2}q^{2i+2}-1)}{c_2(q-1)(s^*q^4-1)(s^*q^{2i-2}-1)(s^*q^{2i+2}-1)(s^{*2}q^{2d+4}-1)}, \quad (3.35)$$

and in the case $i = d$ we obtain

$$p_{2,d-1}^d = - \frac{c_d(q^2+1)(q^{2d-2}-1)(s^*q^2-1)(s^*q^3-1)(s^{*2}q^{2d+2}-1)}{c_2(q-1)(s^*q^4-1)(s^*q^{2d-2}-1)(s^{*2}q^{2d+4}-1)}. \quad (3.36)$$

No factor in the numerator of (3.35), (3.36) is zero by (3.12)–(3.14) and Lemma 3.3.1, so $p_{2,i-1}^i \neq 0$ for $2 \leq i \leq d$ as desired. $\square$

Lemma 3.3.3 *Let $\Gamma = (X, R)$ be as in Lemma 3.2.1, and suppose $a_1 \neq 0$. Fix an integer i with $2 \leq i \leq d$ and vertices $x, y, z \in X$ with $\partial(x, y) = i$, $\partial(x, z) = i-1$, and $\partial(y, z) = 2$. (Such x, y, z exist by Lemma 3.3.2.) Define L_i, H_i by*

$$L_i = |\{w \in \Delta_{11}(y, z) \mid \partial(x, w) = i-1\}|, \quad (3.37)$$

$$H_i = |\{w \in \Delta_{11}(y, z) \mid \partial(x, w) = i\}|. \quad (3.38)$$

Then:

(i)

$$c_2 = L_i + H_i; \quad (3.39)$$

(ii)

$$L_i = \frac{(s^* q^2 - 1)(s^* q^3 - 1)(s^* q^{2i+2} - 1)(s^{*2} q^{2d+6} - 1)}{(s^* q^4 - 1)(s^* q^5 - 1)(s^* q^{2i} - 1)(s^{*2} q^{2d+4} - 1)}, \quad (3.40)$$

$$H_i = \frac{q^2(s^* q^2 - 1)(s^* q^3 - 1)(s^* q^{2i-2} - 1)(s^{*2} q^{2d+6} - 1)}{(s^* q^4 - 1)(s^* q^5 - 1)(s^* q^{2i} - 1)(s^{*2} q^{2d+4} - 1)}. \quad (3.41)$$

In particular, L_i and H_i are constants independent of x, y, z .

Proof of (i). Observe

$$L_i + H_i = |\Delta_{11}(y, z)| \quad (3.42)$$

$$= c_2. \quad (3.43)$$

□

Proof of (ii). By Lemma 3.2.3(ii) and (2.14),

$$0 = \langle E\hat{x}, \delta_{11}(y, z) - c_2 \frac{\theta_1^*}{\theta_2^* + \theta_0^*}(\hat{y} + \hat{z}) \rangle \quad (3.44)$$

$$= L_i \theta_{i-1}^* + H_i \theta_i^* - c_2 \frac{\theta_1^*(\theta_i^* + \theta_{i-1}^*)}{\theta_2^* + \theta_0^*}. \quad (3.45)$$

Eliminating H_i in (3.45) using (3.39) and solving for L_i , we obtain

$$L_i = c_2 \frac{\theta_1^*(\theta_i^* + \theta_{i-1}^*) - \theta_i^*(\theta_2^* + \theta_0^*)}{(\theta_{i-1}^* - \theta_i^*)(\theta_2^* + \theta_0^*)}. \quad (3.46)$$

Evaluating (3.46) using (3.19) and (3.26), we obtain (3.40). A similar argument gives (3.41). □

Lemma 3.3.4 *Let Γ be as in Lemma 3.2.1, and suppose $a_1 \neq 0$. Let L_i, H_i be as in Lemma 3.3.3. Then:*

(i) $L_i \neq 0$ and $H_i \neq 0$ for $2 \leq i \leq d$;

(ii) $L_2 \leq L_3$.

Proof of (i). Each term in the numerator of (3.40), (3.41) is non-zero by (3.13), (3.14), and Lemma 3.3.1. □

Proof of (ii). By Lemma 3.3.2 there exist vertices $x, y, z \in X$ with $\partial(x, y) = 3$, $\partial(x, z) = 2$, and $\partial(y, z) = 2$. Since $L_3 \neq 0$, there exists a vertex $u \in \Delta_{11}(y, z)$

with $\partial(x, u) = 2$. Observe that any vertex w adjacent to both x and u has $\partial(y, w) = 2$, since $\partial(x, y) = 3$ and $\partial(u, y) = 1$. In particular,

$$L_2 = |\{w \in \Delta_{11}(x, z) \mid \partial(u, w) = 1\}| \quad (3.47)$$

$$\leq |\{w \in \Delta_{11}(x, z) \mid \partial(y, w) = 2\}| \quad (3.48)$$

$$= L_3, \quad (3.49)$$

□

as desired.

Lemma 3.3.5 *Let Γ be as in Lemma 3.2.1, and suppose $a_1 \neq 0$. Let $\beta = q + q^{-1}$.*

Then:

(i) $\beta^2 L_2 H_3 = c_2^2$, where L_2, H_3 are as in Lemma 3.3.3;

(ii) $\beta^2 \in \mathbb{Q}$;

(iii) $\beta^2 \geq 4$;

(iv) $q \in \mathbb{R}$.

Proof of (i). Compute $\beta^2 L_2 H_3$ using (3.40) and (3.41), and c_2^2 using (3.19).
The expressions obtained are equal. □

Proof of (ii). Immediate from (i) and Lemma 3.3.4(i), since c_2, L_2 , and L_3 are integers. □

Proof of (iii). By (i), Lemma 3.3.3(i), and Lemma 3.3.4(ii),

$$\beta^2 = \frac{c_2^2}{L_2 H_3} \quad (3.50)$$

$$= \frac{c_2^2}{L_2(c_2 - L_3)} \quad (3.51)$$

$$\geq \frac{c_2^2}{L_2(c_2 - L_2)} \quad (3.52)$$

$$\geq 4. \quad (3.53)$$

□

Proof of (iv). Solving $\beta = q + q^{-1}$ for q , we find $q \in \mathbb{R}$ by (iii). □

Lemma 3.3.6 *Let Γ be as in Lemma 3.2.1, and suppose $a_1 \neq 0$. Then $s^* \in \mathbb{R}$.*

Proof. Let t denote the constant

$$t = q \frac{\theta_0^* - \theta_1^*}{\theta_1^* - \theta_2^*} - 1. \quad (3.54)$$

Since $\theta_0^*, \dots, \theta_d^* \in \mathbb{R}$, and $q \in \mathbb{R}$ by Lemma 3.3.5(iv), we have $t \in \mathbb{R}$. Evaluating (3.54) using (3.26), we obtain

$$t = -\frac{q^2 - 1}{s^* q^4 - 1}. \quad (3.55)$$

In particular, $t \neq 0$ by (3.12). Solving (3.55) for s^* ,

$$s^* = \frac{t - q^2 + 1}{q^4 t}, \quad (3.56)$$

where the denominator is non-zero by (3.11). Now $s^* \in \mathbb{R}$ as desired. $\square$

3.4 Some Constants

Lemma 3.4.1 *Let $\Gamma = (X, R)$ be as in Lemma 3.2.1. Fix an integer i with $1 \leq i \leq d$ and vertices $x, y, z \in X$ with $\partial(x, y) = i$, $\partial(x, z) = i - 1$, and $\partial(y, z) = 1$. Define z_i by*

$$z_i = |\{w \in \Delta_{11}(y, z) \mid \partial(x, w) = i - 1\}|. \quad (3.57)$$

Then

$$z_i = a_1 \frac{s^* q^2 (q^{2i-2} - 1)}{(s^* q^2 + 1)(s^* q^{2i} - 1)}. \quad (3.58)$$

In particular, z_i is a constant independent of x, y, z .

Proof. By Lemma 3.2.3(ii) and (2.14),

$$0 = \langle E\hat{x}, \delta_{11}(y, z) - a_1 \frac{\theta_1^*}{\theta_1^* + \theta_0^*} (\hat{y} + \hat{z}) \rangle \quad (3.59)$$

$$= z_i \theta_{i-1}^* + (a_1 - z_i) \theta_i^* - a_1 \frac{\theta_1^* (\theta_i^* + \theta_{i-1}^*)}{\theta_1^* + \theta_0^*}. \quad (3.60)$$

Solving for z_i , we obtain

$$z_i = a_1 \frac{\theta_1^* \theta_{i-1}^* - \theta_0^* \theta_i^*}{(\theta_{i-1}^* - \theta_i^*)(\theta_1^* + \theta_0^*)}. \quad (3.61)$$

Evaluating (3.61) using (3.26), we obtain (3.58) as desired. $\square$

Lemma 3.4.2 *Let Γ be as in Lemma 3.2.1. Then:*

(i)

$$p_{2i}^i = \frac{1}{c_2} (a_i^2 + b_i c_{i+1} + b_{i-1} c_i - a_i a_1 - k) \quad 1 \leq i \leq d \quad (3.62)$$

where c_{d+1} is an indeterminant;

(ii)

$$p_{2d}^d = \frac{q(q^{2d}-1)(q^{2d-2}-1)(s^* q^2-1)(s^* q^5-1)(s^{*2} q^{2d+2}-1)(s^{*2} q^{2d+3}-1)}{(q+1)(q-1)^2(s^* q^{2d-1}-1)(s^* q^{2d}-1)(s^{*2} q^{2d+4}-1)(s^{*2} q^{2d+6}-1)}. \quad (3.63)$$

Proof of (i). Immediate from the recurrence [2, Lemma 4.1.7]. $\square$

Proof of (ii). Evaluate (3.62) using (3.15)–(3.20), recalling $b_d = 0$. $\square$

Lemma 3.4.3 *Let $\Gamma = (X, R)$ be as in Lemma 3.2.1. Fix an integer i with $1 \leq i \leq d$ and vertices $x, y \in X$ with $\partial(x, y) = i$. For each vertex $z \in \Delta_{2i}(y, x)$, define $D_i(z)$ and $U_i(z)$ by*

$$D_i(z) = |\{w \in \Delta_{11}(y, z) \mid \partial(x, w) = i - 1\}|, \quad (3.64)$$

$$U_i(z) = |\{w \in \Delta_{11}(y, z) \mid \partial(x, w) = i + 1\}|. \quad (3.65)$$

Then

$$D_i(z) = \sigma_i U_i(z) - \epsilon_i, \quad (3.66)$$

where

$$\sigma_i = \frac{\theta_i^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}, \quad (3.67)$$

$$\epsilon_i = c_2 \theta_i^* \frac{\theta_0^* - 2\theta_1^* + \theta_2^*}{(\theta_{i-1}^* - \theta_i^*)(\theta_2^* + \theta_0^*)}. \quad (3.68)$$

Proof. Let z be given. By Lemma 3.2.3(ii) and (2.14),

$$0 = \langle E\hat{x}, \delta_{11}(y, z) - c_2 \frac{\theta_1^*}{\theta_2^* + \theta_0^*} (\hat{y} + \hat{z}) \rangle \quad (3.69)$$

$$= D_i(z) \theta_{i-1}^* + U_i(z) \theta_{i+1}^* + (c_2 - D_i(z) - U_i(z)) \theta_i^* - c_2 \frac{2\theta_1^* \theta_i^*}{\theta_2^* + \theta_0^*} \quad (3.70)$$

$$= D_i(z) (\theta_{i-1}^* - \theta_i^*) - U_i(z) (\theta_i^* - \theta_{i+1}^*) + c_2 \theta_i^* \frac{\theta_0^* - 2\theta_1^* + \theta_2^*}{\theta_2^* + \theta_0^*}. \quad (3.71)$$

Solving for $D_i(z)$, we obtain (3.66), as desired. $\square$

Lemma 3.4.4 (Curtin [4]) *Let $\Gamma = (X, R)$ be as in Lemma 3.2.1, Fix an integer i with $1 \leq i \leq d$ and vertices $x, y \in X$ with $\partial(x, y) = i$, and let D_i, U_i be as in Lemma 3.4.3. Then:*

(i) *if $i = d$ then for each $z \in \Delta_{2d}(y, x)$,*

$$D_d(z) = -c_2 \theta_d^* \frac{\theta_0^* - 2\theta_1^* + \theta_2^*}{(\theta_{d-1}^* - \theta_d^*)(\theta_2^* + \theta_0^*)}, \quad (3.72)$$

(ii) *if $a_1 = 0$ and $i < d$ then for each $z \in \Delta_{2i}(y, x)$,*

$$D_i(z) = -c_2 \frac{\theta_0^* \theta_{i+1}^* - 2\theta_1^* \theta_i^* + \theta_2^* \theta_{i+1}^*}{(\theta_{i-1}^* - \theta_{i+1}^*)(\theta_2^* + \theta_0^*)}. \quad (3.73)$$

In particular, if $a_1 = 0$ then $D_i(z)$ is a constant independent of x, y, z .

Proof of (i). Immediate from (3.66), since U_d is identically zero. $\square$

Proof of (ii). Let z be given. By Lemma 3.3.1 we have $a_i = 0$, so each $w \in \Delta_{11}(y, z)$ has either $\partial(x, w) = i - 1$ or $\partial(x, w) = i + 1$. In particular,

$$D_i(z) + U_i(z) = |\Delta_{11}(y, z)| \quad (3.74)$$

$$= c_2. \quad (3.75)$$

Eliminating $U_i(z)$ in (3.66) using (3.75) and solving for $D_i(z)$, we obtain (3.73) as desired. $\square$

Lemma 3.4.5 Let $\Gamma = (X, R)$ be as in Lemma 3.2.1. Fix an integer i with $1 \leq i \leq d - 1$ and vertices $x, y \in X$ with $\partial(x, y) = i$. Let D_i and U_i be as in Lemma 3.4.3. Then:

(i)

$$\sum_{z \in \Delta_{2i}(y, x)} D_i(z) = c_i(b_{i-1} - a_1 + z_i - 1); \quad (3.76)$$

(ii)

$$\sum_{z \in \Delta_{2i}(y, x)} D_i(z)U_i(z) = c_i b_i(c_2 - z_2 - 1). \quad (3.77)$$

Proof of (i). The sum (3.76) counts pairs $w, z \in X$ with $\partial(x, w) = i - 1$, $\partial(x, z) = i$, $\partial(y, w) = 1$, $\partial(y, z) = 2$, and $\partial(w, z) = 1$. There are c_i ways to choose w with $\partial(x, w) = i - 1$, $\partial(y, w) = 1$. Fix such a vertex w . We wish to count the vertices z with $\partial(x, z) = i$, $\partial(y, z) = 2$, and $\partial(w, z) = 1$. There are b_{i-1} vertices z with $\partial(x, z) = i$ and $\partial(w, z) = 1$. Of these, one is y , and $a_1 - z_i$ have $\partial(y, z) = 1$. This leaves $b_{i-1} - a_1 + z_i - 1$ with $\partial(y, z) = 2$. The number of pairs w, z is $c_i(b_{i-1} - a_1 + z_i - 1)$, and (3.76) holds as desired. $\square$

Proof of (ii). The sum (3.77) counts triples $w_1, w_2, z \in X$ with $\partial(x, w_1) = i - 1$, $\partial(x, w_2) = i + 1$, $\partial(y, w_1) = \partial(y, w_2) = \partial(z, w_1) = \partial(z, w_2) = 1$, and $\partial(y, z) = 2$. There are c_i ways to choose w_1 with $\partial(x, w_1) = i - 1$ and $\partial(y, w_1) = 1$, and b_i ways to choose w_2 with $\partial(x, w_2) = i + 1$ and $\partial(y, w_2) = 1$. Fix such vertices w_1, w_2 . Of the c_2 common neighbors of w_1 and w_2 , one is just y , and z_2 are adjacent to y . This leaves $c_2 - z_2 - 1$ common neighbors z of w_1 and w_2 with $\partial(y, z) = 2$. The number of triples w_1, w_2, z is $c_i b_i(c_2 - z_2 - 1)$, and (3.77) holds as desired. $\square$

Lemma 3.4.6 Let $\Gamma = (X, R)$ be as in Lemma 3.2.1, Fix an integer i with $1 \leq i \leq d - 1$ and vertices $x, y \in X$ with $\partial(x, y) = i$. Let D_i be as in Lemma 3.4.3, and define Φ_i by

$$\Phi_i = p_{2i}^i \left(\sum_{z \in \Delta_{2i}(y, x)} D_i(z)^2 \right) - \left(\sum_{z \in \Delta_{2i}(y, x)} D_i(z) \right)^2. \quad (3.78)$$

Then:

$$(i) \quad \Phi_i \geq 0; \quad (3.79)$$

(ii)

$$\begin{aligned} \Phi_i = & -c_i \frac{q^2(s^*q^2-1)^3(s^*q^3-1)^2(s^*q^{2d+1}-1)(s^*q^{2d+2}-1)(s^{*2}q^{2d+3}-1)}{(q-1)^3(q+1)^2(s^*q^4-1)(s^{*2}q^{2d+4}-1)^3(s^{*2}q^{2d+6}-1)} \\ & \times \frac{(q^{2i}-1)(q^{2i}-q^2)(q^{2d}-q^{2i})(s^{*2}q^{2i+2}-1)^2(s^{*2}q^{2i+4}-1)}{(s^*q^{2i-1}-1)(s^*q^{2i}-1)^2(s^*q^{2i+1}-1)(s^*q^{2i+2}-1)(s^*q^{2i+3}-1)} \end{aligned} \quad (3.80)$$

where we delete the terms $s^*q^{2d+1}-1$, $s^*q^{2i+3}-1$ in the case $i = d-1$.

Proof of (i). Let $\bar{D}_i$ denote the average value of D_i , given by

$$\bar{D}_i = \frac{1}{p_{2i}^i} \left(\sum_{z \in \Delta_{2i}(y,x)} D_i(z) \right). \quad (3.81)$$

Then clearly

$$0 \leq \sum_{z \in \Delta_{2i}(y,x)} (D_i(z) - \bar{D}_i)^2. \quad (3.82)$$

Evaluating (3.82) using (3.81), the result follows. $\square$

Proof of (ii). Evaluating (3.78) using (3.66) and (3.76), (3.77),

$$\Phi_i = p_{2i}^i \left(\sum_{z \in \Delta_{2i}(y,x)} \sigma_i D_i(z) U_i(z) - \epsilon_i D_i(z) \right) - \left(\sum_{z \in \Delta_{2i}(y,x)} D_i(z) \right)^2 \quad (3.83)$$

$$= p_{2i}^i c_i (\sigma_i b_i (c_2 - z_2 - 1) - \epsilon_i (b_{i-1} - a_1 + z_i - 1)) - c_i^2 (b_{i-1} - a_1 + z_i - 1)^2. \quad (3.84)$$

Evaluating (3.84) using (3.58), (3.62), (3.67), (3.68), and (3.15)–(3.26), we factor and obtain (3.80). Note that for $i = d-1$, we use (3.20) instead of (3.19) in the expansion of p_{2i}^i . $\square$

3.5 The Parameter s^* is Zero.

Lemma 3.5.1 *Let Γ be as in Lemma 3.2.1. Then the following are equivalent:*

- (i) $p_{2d}^d = 0$;
- (ii) $p_{id}^d = 0$ for $2 \leq i \leq d$ and $p_{1d}^d \neq 0$;
- (iii) $s^{*2}q^{2d+3} = 1$.

Proof of (i) $\Rightarrow$ (ii). Observe that $a_d \neq 0$ by (3.16) and (3.12)–(3.14). Suppose for a contradiction that $p_{id}^d \neq 0$ for some integer i with $3 \leq i \leq d$, and fix $i \geq 3$ minimal with $p_{id}^d \neq 0$.

Fix vertices $x, y, z \in X$ with $\partial(x, y) = d$, $\partial(x, z) = d$, and $\partial(y, z) = i$. Observe that each of the c_i vertices $w \in \Delta_{i-1,1}(y, z)$ has either $\partial(x, w) = d-1$ or $\partial(x, w) = d$, and that the latter case is impossible since $p_{i-1,d}^d = 0$. By Lemma 3.2.3(i), (2.14), and (3.26),

$$0 = \langle E\hat{x}, \delta_{i-1,1}(y, z) - c_i \frac{\theta_1^* \theta_i^* - \theta_{i-1}^* \theta_0^*}{\theta_i^{*2} - \theta_0^{*2}} \hat{y} - c_i \frac{\theta_{i-1}^* \theta_i^* - \theta_1^* \theta_0^*}{\theta_i^{*2} - \theta_0^{*2}} \hat{z} \rangle \quad (3.85)$$

$$= c_i \theta_{d-1}^* - c_i \frac{\theta_1^* \theta_i^* - \theta_{i-1}^* \theta_0^*}{\theta_i^{*2} - \theta_0^{*2}} \theta_d^* - c_i \frac{\theta_{i-1}^* \theta_i^* - \theta_1^* \theta_0^*}{\theta_i^{*2} - \theta_0^{*2}} \theta_d^* \quad (3.86)$$

$$= c_i \frac{\theta_{d-1}^* (\theta_i^* + \theta_0^*) - \theta_d^* (\theta_{i-1}^* + \theta_1^*)}{\theta_i^* + \theta_0^*} \quad (3.87)$$

$$= c_i \frac{(q+1)(q^{2d-1}-1)(s^* q^{2d-i+1}-1)}{q^{d-i}(q^i+1)(s^* q^{2d+1}+1)}. \quad (3.88)$$

No term in the numerator of (3.88) is zero by (3.11)–(3.13), and we have a contradiction. Thus $p_{id}^d = 0$ for $2 \leq i \leq d$, as desired. $\square$

Proof of (ii) $\Rightarrow$ (i). Immediate. $\square$

Proof of (i) $\Leftrightarrow$ (iii). Immediate from (3.63) and (3.11)–(3.14). $\square$

Lemma 3.5.2 *Let Γ be as in Lemma 3.2.1, and suppose $s^* \neq 0$ and $a_1 \neq 0$. Then:*

$$(i) \quad z_i \neq 0; \quad 2 \leq i \leq d \quad (3.89)$$

$$(ii) \quad s^* < 0. \quad (3.90)$$

Proof of (i). Immediate from (3.11), (3.12), and (3.58). $\square$

Proof of (ii). By Lemma 3.3.4(ii) and (3.15), (3.19), (3.40), (3.58),

$$0 \leq \frac{a_1^2(L_3 - L_2)}{c_2 z_2 z_3} \quad (3.91)$$

$$= -\frac{1}{s^*} \left(\frac{s^* q^2 + 1}{q^2 + 1} \right)^2. \quad (3.92)$$

The inequality is strict by (3.12), and the result follows immediately. $\square$

Lemma 3.5.3 *Let Γ be as in Lemma 3.2.1, and suppose $s^* \neq 0$, $a_1 \neq 0$, and $p_{2d}^d \neq 0$. Then*

$$q < -1. \quad (3.93)$$

Proof. By Lemma 3.4.6(i) and (3.19), (3.63), (3.80),

$$0 \leq \frac{\Phi_{d-1}}{c_{d-1}p_{2d}^d} \quad (3.94)$$

$$= -\frac{q^{2d+1}(q^{2d-4}-1)(s^*q^{2d+2}-1)}{(q^{2d}-1)(s^*q^4-1)(s^*q^5-1)(s^*q^{2d-3}-1)} \left(\frac{(s^*q^2-1)(s^*q^3-1)(s^{*2}q^{2d}-1)}{(s^*q^{2d-2}-1)(s^{*2}q^{2d+4}-1)} \right)^2. \quad (3.95)$$

The inequality is strict by Lemma 3.3.1 and (3.11)–(3.14).

If $q > 1$, then $q^{2d-4}-1$ and $q^{2d}-1$ are positive, and $s^*q^{2d+2}-1$, s^*q^4-1 , s^*q^5-1 , and $s^*q^{2d-3}-1$ are negative by (3.90), contradicting (3.95). Thus $q \neq 1$, and $q < -1$ by (3.11) as desired. $\square$

Lemma 3.5.4 Let Γ be as in Lemma 3.2.1, and suppose $s^* \neq 0$, $a_1 \neq 0$, and $p_{2d}^d \neq 0$. Then

$$|s^*| < \frac{1}{q^{2d}}. \quad (3.96)$$

Proof. By the Krein conditions (1.14), (1.25), and (3.25),

$$0 < c_2^* = \frac{(q^{2d-1}-1)(q+1)(s^*q^{2d}+1)}{q(q^{2d-3}-1)(s^*q^{2d+1}+1)}. \quad (3.97)$$

The terms $q^{2d-1}-1$, $q+1$, q , $q^{2d-3}-1$ are negative by (3.93), and the term $s^*q^{2d+1}+1$ is positive by (3.90), (3.93). We are left with

$$0 < s^*q^{2d}+1, \quad (3.99)$$

and the result is immediate. $\square$

Lemma 3.5.5 Let Γ be as in Lemma 3.2.1, and suppose $d \geq 4$, $a_1 \neq 0$, and $p_{2d}^d \neq 0$. Then

$$s^* = 0. \quad (3.100)$$

Proof. Suppose $s^* \neq 0$ for a contradiction. By Lemma 3.5.2(i) and (3.15), (3.58),

$$1 \leq z_2 = -\frac{s^*q^2(q^2-1)(q+1)(s^*q^3-1)(s^*q^{2d+2}-1)}{(s^*q^4-1)^2(s^{*2}q^{2d+4}-1)}. \quad (3.101)$$

Observe $|s^*| < |q^{-7}|$, $|q+1| < |q|$, $|s^*q^3-1| < 1$, $|s^*q^{2d+2}-1| < |q^2+1|$, and $|s^*q^4-1| > 1$ by (3.90), (3.93), and (3.96). Applying these inequalities to (3.102),

$$1 < -\frac{q^4-1}{q^4(s^{*2}q^{2d+4}-1)}. \quad (3.103)$$

Subtracting 1 and simplifying,

$$0 < -\frac{s^{*2}q^{2d+8}-1}{q^4(s^{*2}q^{2d+4}-1)}. \quad (3.104)$$

The terms $s^{*2}q^{2d+8}-1$, $s^{*2}q^{2d+4}-1$ are negative by (3.96), for a contradiction. Thus $s^* = 0$ as desired. $\square$

3.6 Proof of Theorem 3.1.4

Definition 3.6.1 Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter d , vertex set X , and edge set R . Fix an integer i with $0 \leq i \leq d$. By the *distance- i graph* of Γ we mean the graph $\Delta = (X, S)$ with $xy \in S$ if and only if $\partial(x, y) = i$ in Γ .

Proof of Theorem 3.1.4. Let Γ denote a distance-regular graph with diameter $d \geq 4$ and valency $k \geq 3$, and suppose Γ is almost dual imprimitive. Further suppose Γ is neither the graph (i) nor the graph (ii), so that we may apply Lemma 3.2.1.

Suppose $a_1 = 0$. By Lemma 3.3.1, we have $a_i = 0$ for $1 \leq i \leq d-1$ and $a_d \neq 0$. By Lemma 3.4.4, Γ is 2-homogeneous in the sense of Nomura. In particular, Nomura's classification of bipartite and almost bipartite 2-homogeneous distance-regular graphs applies [13]. The only possibility with valency $k \geq 3$ and $a_d \neq 0$ is the graph (ii), which we have excluded. Thus $a_1 \neq 0$.

Suppose $p_{2d}^d = 0$, and let Δ denote the distance- d graph of Γ . Then by Lemma 3.5.1 and [2, Proposition 4.2.10], Δ is a distance-regular graph with diameter d and $a_1 = 0$, Γ is the distance-2 graph of Δ , and Γ and Δ have the same Q-polynomial structures. Observe that Δ is not the graph (i) since the graph (i) has $a_1 \neq 0$. Observe that Δ is not the graph (ii) since then Γ , the distance-2 graph of Δ , is the graph (i) [1, p. 306]. Applying the previous paragraph to the graph Δ , we have a contradiction. Thus $p_{2d}^d \neq 0$.

Now $s^* = 0$ by Lemma 3.5.5. Setting $s^* = 0$ in (3.15)–(3.20), we obtain

$$a_i = \frac{r^{2i} - 1}{r + 1}, \quad 1 \leq i \leq d \quad (3.105)$$

$$b_i = r \frac{r^{2d} - r^{2i}}{r^2 - 1}, \quad 0 \leq i \leq d-1 \quad (3.106)$$

$$c_i = \frac{r^{2i} - 1}{r^2 - 1}, \quad 1 \leq i \leq d \quad (3.107)$$

where $r = -q$. Observe that $r = a_1 + 1$, so that r is an integer with $r \geq 2$. In particular, Γ has classical parameters $(d, r^2, 0, r)$ with $r^2 \neq 1$, and Γ is not a Hamming graph. Furthermore, the constant z_2 is zero by (3.58), so that any edge of Γ is contained in a unique maximal clique and the maximal cliques of Γ are singular lines. Now Γ is a dual polar graph [2, Theorem 9.4.4], and r is a prime power. The graph (iii) is the unique dual polar graph with intersection numbers as in (3.105)–(3.107).

The Q-polynomial structures on (i)–(iii) are well known; see Chapter 9. $\square$

Chapter 4

A Note on the Krein Parameters

4.1 Introduction

Let Y denote a d -class association scheme, with $d \geq 3$. It is well-known that if Y has a P-polynomial structure with intersection numbers p_{ij}^h , then

$$p_{11}^1 \neq 0 \Rightarrow p_{1i}^1 \neq 0, \quad 1 \leq i \leq d-1 \quad (4.1)$$

We show that if Y has a Q-polynomial structure with Krein parameters q_{ij}^h , then

$$q_{11}^1 \neq 0 \Rightarrow q_{1i}^1 \neq 0, \quad 1 \leq i \leq d-1 \quad (4.2)$$

The result (4.1) follows from easy combinatorial arguments [2, Theorem 5.5.1]. However, these arguments do not apply to the Krein parameters of a Q-polynomial structure. Instead, we prove (4.2) using an algebraic approach. Theorem 4.1.1 is our main result:

Theorem 4.1.1 *Let Y denote a d -class association scheme, with $d \geq 3$. Suppose $E_0, \dots, E_d$ is a Q-polynomial structure for Y , with Krein parameters q_{ij}^h . Then*

$$q_{11}^1 \neq 0 \Rightarrow q_{1i}^1 \neq 0, \quad 1 \leq i \leq d-1 \quad (4.3)$$

4.2 Proof of Theorem 4.1.1

We quote Lemma 4.2.1 without proof:

Lemma 4.2.1 (Cameron, Goethals, Seidel [3]) *Let Y denote a d -class association scheme, with vertex set X , primitive idempotents $E_0, \dots, E_d$, and Krein parameters q_{ij}^h . Then*

$$q_{ij}^h = 0 \Leftrightarrow \forall x, y, z \in X, \sum_{u \in X} (E_h)_{ux} (E_i)_{uy} (E_j)_{uz} = 0 \quad (4.4)$$

for $0 \leq h, i, j \leq d$.

Lemma 4.2.2 *Let Y denote a d -class association scheme, with vertex set X . Suppose $E_0, \dots, E_d$ is a Q-polynomial structure for Y , with Krein parameters q_{ij}^h . Then for each integer i with $1 \leq i \leq d-1$ and all vertices $a, a', b, b' \in X$:*

$$\begin{aligned} (i) \quad & \sum_{e \in X} (E_1)_{ea} (E_{i+1})_{ea'} (E_1)_{eb} (E_{i-1})_{eb} \\ &= \frac{q_{1,i-1}^1}{|X|} \sum_{e \in X} (E_1)_{ea} (E_{i+1})_{ea'} (E_i)_{ebi} \end{aligned} \quad (4.5)$$

(ii)

$$\begin{aligned} \sum_{e \in X} (E_1)_{ea} (E_1)_{ea'} (E_{i-1})_{eb} (E_{i+1})_{eb'} \\ = \sum_{e, e' \in X} (E_1)_{ea} (E_1)_{e'a'} (E_{i-1})_{eb} (E_{i+1})_{e'b'} (E_i)_{ee'}; \end{aligned} \quad (4.6)$$

(iii) if $q_{1i}^i = 0$, then

$$\begin{aligned} \sum_{e \in X} (E_1)_{ea} (E_1)_{ea'} (E_i)_{eb} (E_{i+1})_{eb'} \\ = \sum_{e, e' \in X} (E_1)_{ea} (E_1)_{e'a'} (E_i)_{eb} (E_{i+1})_{e'b'} (E_{i+1})_{ee'}. \end{aligned} \quad (4.7)$$

Proof of (i). By (1.12),

$$\sum_{e \in X} (E_1)_{ea} (E_{i+1})_{ea'} (E_1)_{eb} (E_{i-1})_{eb} \quad (4.8)$$

$$= \sum_{e \in X} (E_1)_{ea} (E_{i+1})_{ea'} (E_1 \circ E_{i-1})_{eb} \quad (4.9)$$

$$= \sum_{e \in X} (E_1)_{ea} (E_{i+1})_{ea'} \frac{1}{|X|} \sum_{h=0}^d q_{1,i-1}^h (E_h)_{eb}. \quad (4.10)$$

The terms with $h > i$ are zero by (1.24). The terms with $h < i$ are zero by (1.24) and (4.4). We are left with (4.5), as desired. $\square$

Proof of (ii). By (1.8),

$$\sum_{e \in X} (E_1)_{ea} (E_1)_{ea'} (E_{i-1})_{eb} (E_{i+1})_{eb'} \quad (4.11)$$

$$= \sum_{e, e' \in X} (E_1)_{ea} (E_1)_{e'a'} (E_{i-1})_{eb} (E_{i+1})_{e'b'} I_{ee'} \quad (4.12)$$

$$= \sum_{e, e' \in X} (E_1)_{ea} (E_1)_{e'a'} (E_{i-1})_{eb} (E_{i+1})_{e'b'} \sum_{h=0}^d (E_h)_{ee'}. \quad (4.13)$$

The terms with $h > i$ are zero by (1.24) and (4.4) (with respect to the sum over e). The terms with $h < i$ are zero by (1.24) and (4.4) (with respect to the sum over e'). We are left with (4.6), as desired. $\square$

Proof of (iii). As in the proof of (ii),

$$\sum_{e \in X} (E_1)_{ea} (E_1)_{ea'} (E_i)_{eb} (E_{i+1})_{eb'} \quad (4.14)$$

$$= \sum_{e, e' \in X} (E_1)_{ea} (E_1)_{e'a'} (E_i)_{eb} (E_{i+1})_{e'b'} \sum_{h=0}^d (E_h)_{ee'}. \quad (4.15)$$

The terms with $h > i + 1$ are zero by (1.24) and (4.4) (with respect to the sum over e). The terms with $h < i$ are zero by (1.24) and (4.4) (with respect to the sum over e'). The term with $h = i$ is zero by the hypothesis $q_{1i}^i = 0$ and (4.4) (with respect to the sum over e). We are left with (4.7), as desired. $\square$

Proof of Theorem 4.1.1. Let X denote the vertex set of Y . Observe

$$(E_i \circ E_j)E_h = \frac{q_{ij}^h}{|X|} E_h \quad (4.16)$$

by (1.12), (1.9).

Suppose $q_{1i}^i = 0$ for some integer i , $2 \leq i \leq d - 1$. Then by (4.4) and (4.16), for all $x, y, z \in X$ we have

$$0 = \frac{q_{1,i+1}^i}{|X|} \sum_{u \in X} (E_i)_{ux} (E_i)_{uy} (E_1)_{uz} \quad (4.17)$$

$$= \sum_{u \in X} (E_i)_{ux} ((E_1 \circ E_{i+1})E_i)_{uy} (E_1)_{uz} \quad (4.18)$$

$$= \sum_{u, v \in X} (E_i)_{ux} (E_1)_{uv} (E_{i+1})_{uv} (E_i)_{vy} (E_1)_{uz}. \quad (4.19)$$

Rearranging terms and applying (4.7),

$$= \sum_{v \in X} (E_i)_{vy} \sum_{u \in X} (E_1)_{uv} (E_1)_{uz} (E_i)_{ux} (E_{i+1})_{uv} \quad (4.20)$$

$$= \sum_{v \in X} (E_i)_{vy} \sum_{u, w \in X} (E_1)_{uv} (E_1)_{wz} (E_i)_{ux} (E_{i+1})_{wv} (E_{i+1})_{uw}. \quad (4.21)$$

Since (4.21) holds with respect to arbitrary $x, y, z \in X$, we have

$$0 = \sum_{x, y, z \in X} (E_{i-1})_{xy} (E_1)_{xz} (E_1)_{yz} \sum_{u, v, w \in X} (E_i)_{vy} (E_1)_{uv} (E_1)_{wz} (E_i)_{ux} (E_{i+1})_{wv} (E_{i+1})_{uw}. \quad (4.22)$$

Rearranging using (1.11) and applying (4.6),

$$= \sum_{y, z, v, w \in X} (E_1)_{yz} (E_i)_{vy} (E_1)_{wz} (E_{i+1})_{wv} \sum_{x, u \in X} (E_1)_{xz} (E_1)_{uv} (E_{i-1})_{xy} (E_{i+1})_{uw} (E_i)_{xu} \quad (4.23)$$

$$= \sum_{y, z, v, w \in X} (E_1)_{yz} (E_i)_{vy} (E_1)_{wz} (E_{i+1})_{wv} \sum_{x \in X} (E_1)_{xz} (E_1)_{xv} (E_{i-1})_{xy} (E_{i+1})_{xw}. \quad (4.24)$$

Rearranging using (1.11) and applying (4.6), (4.5),

$$= \sum_{x,z,w \in X} (E_1)_{wz} (E_1)_{xz} (E_{i+1})_{xw} \sum_{y,v \in X} (E_1)_{yz} (E_1)_{vx} (E_{i-1})_{yx} (E_{i+1})_{vw} (E_i)_{yv} \quad (4.25)$$

$$= \sum_{x,z,w \in X} (E_1)_{wz} (E_1)_{xz} (E_{i+1})_{xw} \sum_{y \in X} (E_1)_{yz} (E_1)_{yx} (E_{i-1})_{yx} (E_{i+1})_{yw} \quad (4.26)$$

$$= \frac{q_{1,i-1}^i}{|X|} \sum_{x,z,w \in X} (E_1)_{wz} (E_1)_{xz} (E_{i+1})_{xw} \sum_{y \in X} (E_1)_{yz} (E_{i+1})_{yw} (E_i)_{yx}. \quad (4.27)$$

Rearranging using (1.11) and applying (4.7),

$$= \frac{q_{1,i-1}^i}{|X|} \sum_{y,z \in X} (E_1)_{yz} \sum_{x,w \in X} (E_1)_{xz} (E_1)_{wz} (E_i)_{xy} (E_{i+1})_{wy} (E_{i+1})_{xw} \quad (4.28)$$

$$= \frac{q_{1,i-1}^i}{|X|} \sum_{y,z \in X} (E_1)_{yz} \sum_{x \in X} (E_1)_{xz} (E_1)_{xz} (E_i)_{xy} (E_{i+1})_{xy}. \quad (4.29)$$

Rearranging and applying (4.16),

$$= \frac{q_{1,i-1}^i}{|X|} \sum_{x,y,z \in X} (E_1 \circ E_1)_{xz} (E_i \circ E_{i+1})_{xy} (E_1)_{yz} \quad (4.30)$$

$$= \frac{q_{1,i-1}^i}{|X|} \sum_{x,z \in X} (E_1 \circ E_1)_{xz} ((E_i \circ E_{i+1}) E_1)_{xz} \quad (4.31)$$

$$= \frac{q_{1,i-1}^i q_{i,i+1}^1}{|X|^2} \sum_{x,z \in X} (E_1 \circ E_1)_{xz} (E_1)_{xz}. \quad (4.32)$$

Now by (1.11) and (4.16),

$$= \frac{q_{1,i-1}^i q_{i,i+1}^1}{|X|^2} \sum_{x \in X} ((E_1 \circ E_1) E_1)_{xx} \quad (4.33)$$

$$= \frac{q_{1,i-1}^i q_{i,i+1}^1 q_{11}^1}{|X|^3} \sum_{x \in X} (E_1)_{xx} \quad (4.34)$$

$$= \frac{q_{1,i-1}^i q_{i,i+1}^1 q_{11}^1}{|X|^3} \text{tr}(E_1). \quad (4.35)$$

Since E_1 is a primitive idempotent, we have $\text{tr}(E_1) = \text{rk}(E_1) \neq 0$. By (1.25), all terms of (4.35) other than q_{11}^1 are non-zero, and $q_{11}^1 = 0$. This contradicts our hypothesis, so $q_{ii}^i \neq 0$ for $2 \leq i \leq d-1$ as desired. $\square$

Chapter 5

The Thin and Dual Thin Conditions

5.1 Introduction

In this chapter we show that under an additional hypothesis, the converse to Theorem 4.1.1 holds. We first introduce the *subconstituent algebra* and the *thin* and *dual thin* conditions of Terwilliger. Our main results are Theorem 5.1.1 and Theorem 5.1.2. This chapter presents joint work with P. Terwilliger.

Let Y denote a d -class association scheme with vertex set X , associate matrices $A_0, \dots, A_d$, primitive idempotents $E_0, \dots, E_d$, and Bose-Mesner algebra M . Fix a vertex $x \in X$. By the *dual associate matrices* for Y with respect to x we mean the diagonal matrices $A_i^* = A_i^*(x) \in \text{Mat}_X(\mathbb{R})$ defined for $0 \leq i \leq d$ by

$$(A_i^*)_{yy} = |X|(E_i)_{xy}, \quad y \in X \quad (5.1)$$

By the *dual Bose-Mesner algebra* for Y with respect to x we mean the subalgebra $M^* = M^*(x)$ of $\text{Mat}_X(\mathbb{R})$ generated by the dual associate matrices $A_0^*, \dots, A_d^*$. From (1.12) we obtain

$$A_i^* A_j^* = \sum_{h=0}^d q_{ij}^h A_h^*, \quad 0 \leq i, j \leq d \quad (5.2)$$

In particular, the dual associate matrices form a basis for M^* . If $E_0, \dots, E_d$ is a $\mathbb{Q}$ -polynomial structure for Y , then A_1^* generates M^* [20, Lemma 3.11].

By the *dual idempotents* for Y with respect to x we mean the diagonal matrices $E_i^* = E_i^*(x) \in \text{Mat}_X(\mathbb{R})$ defined for $0 \leq i \leq d$ by

$$(E_i^*)_{yy} = (A_i)_{xy}, \quad y \in X \quad (5.3)$$

Note that the dual idempotents form a second basis for M^* . From (1.3), (1.4) we obtain

$$E_0^* + \dots + E_d^* = I, \quad (5.4)$$

$$E_i^* E_j^* = \delta_{ij} E_i^*, \quad 0 \leq i, j \leq d \quad (5.5)$$

The *subconstituent algebra* for Y with respect to x is the subalgebra $T = T(x)$ of $\text{Mat}_X(\mathbb{R})$ generated by M and M^* . Recall the standard module $V = \mathbb{R}^X$. By a T -module we mean a subspace of V which is closed under multiplication from T . A T -module is said to be *irreducible* if it properly contains no T -modules other than 0. Recall that T is semi-simple, so that V may be decomposed as a direct sum of irreducible T -modules [20, Lemma 3.4].

An irreducible T -module W is said to be *thin* if

$$\dim E_i^* W \leq 1, \quad 0 \leq i \leq d \quad (5.6)$$

and *dual thin* if

$$\dim E_i W \leq 1, \quad 0 \leq i \leq d \quad (5.7)$$

We say Y is *i-thin* with respect to x if every irreducible T -module W with $E_i^* W \neq 0$ is thin, and *i-thin* if for every $x \in X$, Y is *i-thin* with respect to x . We say Y is *dual i-thin* with respect to x if every irreducible T -module W with $E_i W \neq 0$ is dual thin, and *dual i-thin* if for every $x \in X$, Y is dual *i-thin* with respect to x .

Our main results are the following:

Theorem 5.1.1 *Let Y denote a d -class association scheme, with $d \geq 3$. Suppose $A_0, \dots, A_d$ is a P -polynomial structure for Y , with intersection numbers p_{ij}^h , and suppose Y is 1-thin with respect to at least one vertex. Then*

$$p_{11}^1 = 0 \quad \Rightarrow \quad p_{1i}^i = 0, \quad 1 \leq i \leq d-1 \quad (5.8)$$

We prove Theorem 5.1.1 in Section 5.2.

Theorem 5.1.2 *Let Y denote a d -class association scheme, with $d \geq 3$. Suppose $E_0, \dots, E_d$ is a Q -polynomial structure for Y , with Krein parameters q_{ij}^h , and suppose Y is dual 1-thin with respect to at least one vertex. Then*

$$q_{11}^1 = 0 \quad \Rightarrow \quad q_{1i}^i = 0, \quad 1 \leq i \leq d-1 \quad (5.9)$$

We prove Theorem 5.1.2 in Section 5.3.

5.2 Proof of Theorem 5.1.1

Define a symmetric bilinear form on $\text{Mat}_X(\mathbb{R})$ by

$$\langle B, C \rangle = \text{tr}(B^t C), \quad B, C \in \text{Mat}_X(\mathbb{R}) \quad (5.10)$$

Observe that $\langle B, C \rangle$ is just the sum of the entries of $B \circ C$. In particular, the form is positive definite.

Lemma 5.2.1 (Terwilliger [20]) *Let $Y = (X, \{R_i\}_{0 \leq i \leq d})$ denote a d -class association scheme, with associate matrices $A_0, \dots, A_d$ and intersection numbers p_{ij}^h . Fix a vertex $x \in X$, and write $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$. Then:*

(i) for $0 \leq h, h', i, i', j, j' \leq d$,

$$\langle E_i^* A_h E_j^*, E_{i'}^* A_{h'} E_{j'}^* \rangle = \delta_{hh'} \delta_{ii'} \delta_{jj'} k_h p_{ij}^h; \quad (5.11)$$

(ii) for $0 \leq h, i, j \leq d$,

$$E_h^* A_i E_j^* = 0 \quad \Leftrightarrow \quad p_{ij}^h = 0. \quad (5.12)$$

Proof of (i). Observe

$$(E_i^* A_h E_j^*)_{yz} = (E_i^*)_{yy} (A_h)_{yz} (E_j^*)_{zz} \quad (5.13)$$

$$= (A_i)_{xy} (A_h)_{yz} (A_j)_{xz}, \quad (5.14)$$

so that $(E_i^* A_h E_j^*)_{yz} \neq 0$ if and only if $xy \in R_i$, $yz \in R_h$, and $xz \in R_j$. Since the relations $R_0, \dots, R_d$ are disjoint, the matrices $E_i^* A_h E_j^*$ and $E_{i'}^* A_{h'} E_{j'}^*$ have no non-zero entries in common unless $h = h'$, $i = i'$, $j = j'$. In this case there are precisely $k_h p_{ij}^h$ non-zero entries, each equal to 1. The result follows. $\square$

Proof of (ii). Immediate from (i). $\square$

Let Y denote a d -class association scheme, with vertex set X . Suppose $A_0, \dots, A_d$ is a P -polynomial structure for Y , with intersection numbers p_{ij}^h . Fix a vertex $x \in X$, and write $T = T(x)$, $M^* = M^*(x)$, and $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$.

There are three matrices in T which are of particular interest to us (their duals will be used in Section 5.3). These are the *lowering* matrix $L = L(x)$, the *flat* matrix $F = F(x)$, and the *raising* matrix $R = R(x)$, defined by

$$L = \sum_{i=1}^d E_{i-1}^* A_1 E_i^*, \quad (5.15)$$

$$F = \sum_{i=0}^d E_i^* A_1 E_i^*, \quad (5.16)$$

$$R = \sum_{i=0}^{d-1} E_{i+1}^* A_1 E_i^*. \quad (5.17)$$

It is easily shown using (1.17), (5.4), and (5.12) that

$$A_1 = L + F + R. \quad (5.18)$$

Recall that A_1 generates the Bose-Mesner algebra M , so that A_1 and $E_0^*, \dots, E_d^*$ generate T . In particular, L, F, R , and $E_0^*, \dots, E_d^*$ generate T by (5.18).

Lemma 5.2.2 *Let Y denote a d -class association scheme, with vertex set X . Suppose $A_0, \dots, A_d$ is a P -polynomial structure for Y , with intersection numbers p_{ij}^h . Fix a vertex $x \in X$, and write $L = L(x)$ and $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$. If Y is 1-thin with respect to x , then:*

(i) *for W any irreducible T -module with $E_1^* W \neq 0$,*

$$L E_i^* W = 0 \quad \Rightarrow \quad E_i^* W = 0; \quad 2 \leq i \leq d \quad (5.19)$$

(ii) *for $w \in T E_1^* V$,*

$$L E_i^* w = 0 \quad \Rightarrow \quad E_i^* w = 0; \quad 2 \leq i \leq d \quad (5.20)$$

(iii) for $B \in TE_1^*$,

$$LE_i^* B = 0 \Rightarrow E_i^* B = 0, \quad 2 \leq i \leq d \quad (5.21)$$

Proof of (i). Let W be given. Fix an integer i with $2 \leq i \leq d$, and suppose $LE_i^* W = 0$. Let W' denote the subspace of W defined by

$$W' = E_i^* W + \cdots + E_d^* W. \quad (5.22)$$

Observe by (5.15)–(5.17) and (1.9) that W' is closed under multiplication by L , F , R , and $E_0^*, \dots, E_d^*$. Since T is generated by these matrices, W' is a T -module. Since $E_1^* W' = 0$ and $E_1^* W \neq 0$, W' is a proper submodule of W . Since W is irreducible, we now have $W' = 0$, and $E_i^* W \subseteq W'$ is zero as desired. $\square$

Proof of (ii). Since V may be decomposed into a direct sum of irreducible T -modules, it suffices to show that the result holds for $w \in TE_1^* W$ where W is an irreducible T -module. Fix an integer i with $2 \leq i \leq d$ and an irreducible T -module W , and suppose $w \in TE_1^* W$ has $LE_i^* w = 0$.

Suppose $E_i^* w \neq 0$. Observe $E_1^* W \neq 0$, since $0 \neq E_i^* w \in E_i^* TE_1^* W$. Since Y is 1-thin with respect to x , W is thin and $\dim E_i^* W \leq 1$. In particular, $E_i^* w \in E_i^* W$ spans $E_i^* W$, and $LE_i^* W = 0$. By (i) we have $E_i^* W = 0$, and $E_i^* w = 0$ for a contradiction. Thus $E_i^* w = 0$ as desired. $\square$

Proof of (iii). Immediate from (ii). $\square$

Lemma 5.2.3 *Let Y denote a d -class association scheme, with vertex set X . Suppose $A_0, \dots, A_d$ is a P -polynomial structure for Y , with intersection numbers p_{ij}^h . Fix a vertex $x \in X$, and write $L = L(x)$ and $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$. Then:*

(i) for $1 \leq i \leq d-1$,

$$LE_i^* A_{i+1} E_1^* = b_i E_{i-1}^* A_i E_1^*; \quad (5.23)$$

(ii) for $1 \leq i \leq d$, if $a_{i-1} = 0$ then

$$LE_i^* A_i E_1^* = a_i E_{i-1}^* A_i E_1^*. \quad (5.24)$$

Proof of (i). Let i be given. Observe by (5.5), (1.17), (5.12), (5.4), and (1.7) that

$$LE_i^* A_{i+1} E_1^* = E_{i-1}^* A E_i^* A_{i+1} E_1^* \quad (5.25)$$

$$= E_{i-1}^* A \left(\sum_{h=0}^d E_h^* \right) A_{i+1} E_1^* \quad (5.26)$$

$$= E_{i-1}^* A A_{i+1} E_1^* \quad (5.27)$$

$$= E_{i-1}^* \left(\sum_{h=0}^d p_{1,i+1}^h A_h \right) E_1^* \quad (5.28)$$

$$= b_i E_{i-1}^* A_i E_1^*, \quad (5.29)$$

as desired. $\square$

Proof of (ii). Let i be given, with $a_{i-1} = 0$. Observe as in (i) that

$$LE_i^* A_i E_1^* = E_{i-1}^* A E_i^* A_i E_1^* \quad (5.30)$$

$$= E_{i-1}^* A \left(\sum_{h=0}^d E_h^* \right) A_i E_1^* \quad (5.31)$$

$$= E_{i-1}^* A A_i E_1^* \quad (5.32)$$

$$= E_{i-1}^* \left(\sum_{h=0}^d p_{1i}^h A_h \right) E_1^* \quad (5.33)$$

$$= a_i E_{i-1}^* A_i E_1^*, \quad (5.34)$$

as desired. $\square$

Proof of Theorem 5.1.1. Suppose Y is 1-thin with respect to x , and write $L = L(x)$ and $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$. Suppose $a_1 = 0$, and suppose for a contradiction that $a_i = 0$ for some i with $2 \leq i \leq d-1$. Fix $i \geq 2$ minimal with $a_i \neq 0$. Then by Lemma 5.2.3,

$$0 = L(a_i E_i^* A_{i+1} E_1^* - b_i E_i^* A_i E_1^*), \quad (5.35)$$

and by Lemma 5.2.2(iii),

$$0 = a_i E_i^* A_{i+1} E_1^* - b_i E_i^* A_i E_1^*. \quad (5.36)$$

The summands in (5.36) are nonzero by (5.12) and orthogonal by (5.11), for a contradiction. Thus $a_i = 0$ for $2 \leq i \leq d-1$, as desired. $\square$

5.3 Proof of Theorem 5.1.2

Lemma 5.3.1 (Cameron, Goethals, Seidel [3]) *Let Y denote a d -class association scheme, with vertex set X , primitive idempotents $E_0, \dots, E_d$, and Krein parameters q_{ij}^h . Fix a vertex $x \in X$, and write $A_i^* = A_i^*(x)$ for $0 \leq i \leq d$. Then:*

(i) *for $0 \leq h, h', i, i', j, j' \leq d$,*

$$\langle E_i A_h^* E_j, E_{i'} A_{h'}^* E_{j'} \rangle = \delta_{hh'} \delta_{ii'} \delta_{jj'} k_h^* q_{ij}^h; \quad (5.37)$$

(ii) *for $0 \leq h, i, j \leq d$,*

$$E_h A_i^* E_j = 0 \Leftrightarrow q_{ij}^h = 0. \quad (5.38)$$

Proof of (i). Recall $\text{tr}(BC) = \text{tr}(CB)$, and observe by (1.11), (1.9), (5.1), (1.12), and (1.16) that

$$\langle E_i A_h^* E_j, E_i A_h^* E_j \rangle = \text{tr}(E_j A_h^* E_i E_i A_h^* E_j) \quad (5.39)$$

$$= \text{tr}(E_j E_j A_h^* E_i E_i A_h^*) \quad (5.40)$$

$$= \delta_{ii'} \delta_{jj'} \text{tr}(E_j A_h^* E_i A_h^*) \quad (5.41)$$

$$= \delta_{ii'} \delta_{jj'} \sum_{y,z \in X} (E_j)_{yz} (A_h^*)_{zz} (E_i)_{zy} (A_h^*)_{yy} \quad (5.42)$$

$$= \delta_{ii'} \delta_{jj'} |X|^2 \sum_{y,z \in X} (E_j)_{yz} (E_h)_{xz} (E_i)_{zy} (E_h)_{xy} \quad (5.43)$$

$$= \delta_{ii'} \delta_{jj'} |X|^2 \sum_{y \in X} ((E_i \circ E_j) E_h)_{yx} (E_h)_{xy} \quad (5.44)$$

$$= \delta_{ii'} \delta_{jj'} |X| q_{ij}^h \sum_{y \in X} (E_h)_{yx} (E_h)_{xy} \quad (5.45)$$

$$= \delta_{ii'} \delta_{jj'} |X| q_{ij}^h (E_h)_{xx} \quad (5.46)$$

$$= \delta_{hh'} \delta_{ii'} \delta_{jj'} |X| q_{ij}^h (E_h)_{xx} \quad (5.47)$$

$$= \delta_{hh'} \delta_{ii'} \delta_{jj'} k_h^* q_{ij}^h, \quad (5.48)$$

as desired. $\square$

Proof of (ii). Immediate from (i). $\square$

Let Y denote a d -class association scheme, with vertex set X . Suppose $E_0, \dots, E_d$ is a Q -polynomial structure for Y , with Krein parameters q_{ij}^h . Fix a vertex $x \in X$, and write $T = T(x)$, $M^* = M^*(x)$, and $A_i^* = A_i^*(x)$ for $0 \leq i \leq d$.

The *dual lowering matrix* $L^* = L^*(x)$, the *dual flat matrix* $F^* = F^*(x)$, and the *dual raising matrix* $R^* = R^*(x)$ are defined by

$$L^* = \sum_{i=1}^d E_{i-1} A_1^* E_i, \quad (5.49)$$

$$F^* = \sum_{i=0}^d E_i A_1^* E_i, \quad (5.50)$$

$$R^* = \sum_{i=0}^{d-1} E_{i+1} A_1^* E_i. \quad (5.51)$$

It is easily shown using (1.22), (1.8), and (5.38) that

$$A_1^* = L^* + F^* + R^*. \quad (5.52)$$

Recall that A_1^* generates M^* , so that A_1^* and $E_0, \dots, E_d$ generate T . In particular, L, F, R , and $E_0^*, \dots, E_d^*$ generate T by (5.52).

Lemma 5.3.2 *Let Y denote a d -class association scheme, with vertex set X . Suppose $E_0, \dots, E_d$ is a P -polynomial structure for Y , with Krein parameters q_{ij}^h . Fix a vertex $x \in X$, and write $L^* = L^*(x)$ and $A_i^* = A_i^*(x)$ for $0 \leq i \leq d$. If Y is dual 1-thin with respect to x , then:*

(i) for W any irreducible T -module with $E_1 W \neq 0$,

$$L^* E_i W = 0 \Rightarrow E_i W = 0; \quad 2 \leq i \leq d \quad (5.53)$$

(ii) for $w \in TE_1 V$,

$$L^* E_i w = 0 \Rightarrow E_i w = 0; \quad 2 \leq i \leq d \quad (5.54)$$

(iii) for $B \in TE_1$,

$$L^* E_i B = 0 \Rightarrow E_i B = 0. \quad 2 \leq i \leq d \quad (5.55)$$

Proof. Similar to the proof of Lemma 5.2.2. $\square$

Lemma 5.3.3 Let Y denote a d -class association scheme, with vertex set X . Suppose $E_0, \dots, E_d$ is a P -polynomial structure for Y , with Krein parameters q_{ij}^h . Fix a vertex $x \in X$, and write $L^* = L^*(X)$ and $A_i^* = A_i^*(x)$ for $0 \leq i \leq d$. Then:

$$(i) \text{ for } 1 \leq i \leq d, \quad L^* E_i A_{i+1}^* E_1 = b_i^* E_{i-1} A_i^* E_1; \quad (5.56)$$

(ii) for $1 \leq i \leq d$, if $a_{i-1}^* = 0$ then

$$L^* E_i A_i^* E_1 = a_i^* E_{i-1} A_i^* E_1. \quad (5.57)$$

Proof. Similar to the proof of Lemma 5.2.3. $\square$

Proof of Theorem 5.1.2. Similar to the proof of Theorem 5.1.1. $\square$

Chapter 6

2Q Implies Locally Strongly Regular

6.1 Introduction.

Let Γ denote a distance-regular graph. If Γ admits a Q-polynomial structure and the eigenvalues of Γ are not all integers, then Γ admits a second Q-polynomial structure [1, p. 360]. Since the existence of a single Q-polynomial structure gives us regularity conditions on Γ [20], [21], [22], we expect that the interaction of two Q-polynomial structures will give us extra regularity conditions. In this chapter and in chapter 7 we find two such conditions.

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 2$. By a kite of length i and width j we mean a 4-tuple $x, y, z, w \in X$ such that:

- (i) the vertices $x, y, z, w \in X$ are distinct;
- (ii) $\partial(x, y) = 1$, $\partial(x, z) = 1$ and $\partial(y, z) = j$;
- (iii) $\partial(x, w) = i$, $\partial(y, w) = i - 1$, and $\partial(z, w) = i - 1$.

We say that the 3-tuple x, y, z is the base of the kite x, y, z, w . By (i), we have $1 \leq j \leq 2$ and $2 \leq i \leq d$.

If $x, y, z \in X$ have $\partial(x, y) = \partial(x, z) = 1$, then for $2 \leq i \leq d$ define the kite numbers $f_i(x, y, z)$ by

$$f_i(x, y, z) = \frac{1}{p_{i,i-1}^1} |\{w \in X \mid x, y, z, w \text{ is a kite of length } i\}| - 1. \quad (6.1)$$

Observe that there are $p_{i,i-1}^1(f_i(x, y, z) + 1)$ kites of length i and base x, y, z .

In Theorem 6.2.3 we show that in a distance-regular graph with diameter $d \geq 3$ which admits more than one Q-polynomial structure, the kite number $f_i(x, y, z)$ depends only on i and $\partial(y, z)$. In Corollary 6.2.4 we use Theorem 6.2.3 to show that such a graph has all vertex neighborhood graphs strongly regular.

6.2 Results

Let $\Gamma = (X, R)$ denote a distance-regular graph, and suppose Γ admits a Q-polynomial structure with primary eigenvalue θ and dual eigenvalues $\theta_0^*, \dots, \theta_d^*$.

Then

$$\theta\theta_i^* = c_i\theta_{i-1}^* + a_i\theta_i^* + b_i\theta_{i+1}^*, \quad 0 \leq i \leq d \quad (6.2)$$

where θ_{-1}^* and θ_{d+1}^* are indeterminants [2, p. 128].

A number of useful identities follow from (6.2). If we set $i = 0$ in (6.2) then, recalling $c_0 = 0$, $a_0 = 0$, and $b_0 = k$,

$$\theta\theta_0^* = k\theta_1^*. \quad (6.3)$$

If we set $i = 1$ in (6.2) then, recalling $c_1 = 1$ and $a_i + b_i + c_i = k$,

$$\theta\theta_1^* = \theta_0^* + (k - b_1 - 1)\theta_1^* + b_1\theta_2^*. \quad (6.4)$$

Since the dual eigenvalues are distinct [20, Lemma 3.11(iii)], we may combine (6.3) and (6.4) to obtain

$$\theta = b_1 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_1^*} - 1. \quad (6.5)$$

This fact will be useful in the proof of Theorem 6.2.3 and of Theorem 7.2.3.

We quote the following Lemma without proof:

Lemma 6.2.1 (Terwilliger [15]) *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Suppose Γ admits a Q -polynomial structure, with dual eigenvalues $\theta_0^*, \dots, \theta_d^*$. Fix $x, y, z \in X$ with $\partial(x, y) = \partial(x, z) = 1$, and let $j = \partial(y, z)$. Then the kite numbers $f_i(x, y, z)$ satisfy the relation*

$$f_i(x, y, z) = \alpha_i(\theta)f_2(x, y, z) + \gamma_{ij}(\theta), \quad 2 \leq i \leq d \quad (6.6)$$

where

$$\alpha_i(\theta) = \frac{(\theta_1^* - \theta_2^*)(\theta_0^* + \theta_1^* - \theta_{i-1}^* - \theta_i^*)}{(\theta_0^* - \theta_2^*)(\theta_{i-1}^* - \theta_i^*)}, \quad (6.7)$$

$$\gamma_{i1}(\theta) = \frac{\theta_1^* - \theta_{i-1}^*}{\theta_{i-1}^* - \theta_i^*}, \quad (6.8)$$

$$\gamma_{i2}(\theta) = \frac{b_i(\theta_i^* - \theta_{i+1}^*)}{b_1(\theta_{i-1}^* - \theta_i^*)} - \frac{b_2(\theta_2^* - \theta_3^*)(\theta_0^* + \theta_1^* - \theta_{i-1}^* - \theta_i^*)}{b_1(\theta_0^* - \theta_2^*)(\theta_{i-1}^* - \theta_i^*)} + \frac{(\theta_0^* - \theta_2^*)(\theta_1^* - \theta_{i-1}^*)}{(\theta_0^* - \theta_1^*)(\theta_{i-1}^* - \theta_i^*)}. \quad (6.9)$$

Note 6.2.2 In Terwilliger's notation, f_i is $e_i - 1$ and $\gamma_{i1}(\theta)$ is $\beta_i(\theta) + \alpha_i(\theta) - 1$. We have extended the result to the kites of width 2, calculating $\gamma_{i2}(\theta)$ in a manner completely analogous to Terwilliger's calculation of $\beta_i(\theta)$.

Theorem 6.2.3 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Suppose Γ admits distinct Q -polynomial structures with primary eigenvalues θ and ψ , and suppose $a_1 \neq 0$. Fix vertices $x, y, z \in X$ with $\partial(x, y) = \partial(x, z) = 1$, and let $j = \partial(y, z)$. Then referring to (6.7)–(6.9):*

(i)

$$\alpha_3(\theta)f_2(x, y, z) + \gamma_{3j}(\theta) = \alpha_3(\psi)f_2(x, y, z) + \gamma_{3j}(\psi); \quad (6.10)$$

(ii)

$$\alpha_3(\theta) \neq \alpha_3(\psi); \quad (6.11)$$

(iii)

$$f_2(x, y, z) = \frac{\gamma_{3j}(\psi) - \gamma_{3j}(\theta)}{\alpha_3(\theta) - \alpha_3(\psi)}. \quad (6.12)$$

In particular, the kite numbers $f_i(x, y, z)$ depend only on i and $\partial(y, z)$. We will write f_{ij} for $f_i(x, y, z)$, where $j = \partial(y, z)$.

Proof of (i). Theorem 6.2.1 holds both with respect both Q -polynomial structures on Γ . Equating the expressions for $f_3(x, y, z)$ with respect to the two structures, we obtain (6.10) as desired. $\square$

Proof of (ii). By $a_1 \neq 0$, we may choose x, y, z so that $j = 1$. Then by (i),

$$\alpha_3(\theta)f_2(x, y, z) + \gamma_{31}(\theta) = \alpha_3(\psi)f_2(x, y, z) + \gamma_{31}(\psi). \quad (6.13)$$

Suppose $\alpha_3(\theta) = \alpha_3(\psi)$. Then by (6.13), we have $\gamma_{31}(\theta) = \gamma_{31}(\psi)$. We will show that θ is determined by $\alpha_3(\theta)$ and $\gamma_{31}(\theta)$; since ψ is determined in the same way by $\alpha_3(\psi)$ and $\gamma_{31}(\psi)$, we will have $\theta = \psi$ for a contradiction.

By expanding the definitions (6.7), (6.8) of $\alpha_3(\theta)$ and $\gamma_{31}(\theta)$ we have

$$\alpha_3(\theta) - \gamma_{31}(\theta) = \frac{(\theta_1^* - \theta_2^*)(\theta_1^* - \theta_3^*)}{(\theta_0^* - \theta_2^*)(\theta_2^* - \theta_3^*)}, \quad (6.14)$$

$$2\gamma_{31}(\theta) - \alpha_3(\theta) + 1 = \frac{(\theta_0^* - \theta_1^*)(\theta_1^* - \theta_3^*)}{(\theta_0^* - \theta_2^*)(\theta_2^* - \theta_3^*)}. \quad (6.15)$$

Since the dual eigenvalues $\theta_0^*, \dots, \theta_d^*$ are distinct, the expression on the left in (6.15) is not zero. This allows us to write

$$\frac{\alpha_3(\theta) - \gamma_{31}(\theta)}{2\gamma_{31}(\theta) - \alpha_3(\theta) + 1} = \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_1^*}. \quad (6.16)$$

Combining (6.16) with the identity (6.5), we obtain

$$\theta = b_1 \frac{\alpha_3(\theta) - \gamma_{31}(\theta)}{2\gamma_{31}(\theta) - \alpha_3(\theta) + 1} - 1. \quad (6.17)$$

Now θ is determined by $\alpha_3(\theta)$ and $\gamma_{31}(\theta)$ as desired, and $\alpha_3(\theta) \neq \alpha_3(\psi)$. $\square$

Proof of (iii). Solving (6.10) for $f_2(x, y, z)$, we obtain (6.12) as desired. The denominator is non-zero by (ii). $\square$

Recall from Definition 2.3.2 that a graph Γ is said to be *strongly regular* with parameters (ν, k, λ, μ) if Γ has ν vertices and is regular of degree k , adjacent vertices of Γ have λ common neighbors, and non-adjacent vertices of Γ have μ common neighbors.

Corollary 6.2.4 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, and suppose Γ admits more than one Q -polynomial structure. Fix a vertex $x \in X$, and let $\Gamma_1 = (Y, S)$ denote the induced graph on the vertices adjacent to x in Γ . Then Γ_1 is strongly regular, with no edges if a_1 is zero, and with parameters*

$$(k, a_1, a_1 - b_1(f_{21} + 1) - 1, c_2 - b_1(f_{22} + 1) - 1) \quad (6.18)$$

if a_1 is not zero.

Proof. If the intersection number a_1 of Γ is zero, then the graph Γ_1 has no edges and is trivially strongly regular. So we suppose $a_1 \neq 0$, and Theorem 6.2.3 applies.

Observe that Γ_1 has k vertices and is regular of degree a_1 , as desired. Fix distinct vertices $y, z \in Y$, and let $j = \partial(y, z)$. Then y and z have p_{11}^j common neighbors w in X . Of these, one is x . Those w with $\partial(x, w) = 2$ form kites of length 2, width j , and base x, y, z ; there are $p_{21}^1(f_{2j} + 1)$ such vertices. This leaves $p_{11}^j - p_{21}^1(f_{2j} + 1) - 1$ common neighbors of y and z which are in Y .

If y and z are adjacent in Γ_1 , then $j = 1$ and y and z have $a_1 - b_1(f_{21} + 1) - 1$ common neighbors in Y , as desired. If y and z are not adjacent in Γ_1 , then $j = 2$ and y and z have $c_2 - b_1(f_{22} + 1) - 1$ common neighbors in Y , again as desired. $\square$

Chapter 7

2Q Implies Thin

7.1 Introduction.

In this chapter we show that a distance-regular graph which admits more than one Q -polynomial structure is *thin*. Our main result is Theorem 7.1.1.

Let $\Gamma = (X, R)$ denote a distance-regular graph, and recall the subconstituent algebra $T = T(x)$ from page 37. We say that Γ is *thin* if for every vertex $x \in X$, every $T(x)$ -module is thin, and that Γ is *dual thin* if for every vertex $x \in X$, every $T(x)$ -module is dual thin. Observe that thin implies i -thin for $0 \leq i \leq d$, and dual thin implies dual i -thin for $0 \leq i \leq d$.

Theorem 7.1.1 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, and suppose Γ admits more than one Q -polynomial structure. Then Γ is thin and dual thin.*

7.2 Proof of Theorem 7.1.1

We break the proof of Theorem 7.1.1 into cases according to the value of the intersection number a_2 . Lemma 7.2.2 shows that Γ is thin in the case $a_2 = 0$, and Lemma 7.2.4 shows that Γ is thin in the case $a_2 \neq 0$. Lemma 7.2.1 is used to prove both results. Any distance-regular graph which is thin is also dual thin [22, Theorem 5.1(iii)].

Recall the matrices $L(x)$, $F(x)$, $R(x)$ from page 39. Note that R is being used both for the edge set of Γ and for the matrix $R(x)$; the intended meaning will be clear from context. Recall the matrix commutator $[B, C] = BC - CB$.

Lemma 7.2.1 (Terwilliger) *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, and suppose Γ has a Q -polynomial structure with primary eigenvalue θ and dual eigenvalues $\theta_0^*, \dots, \theta_d^*$. Fix a vertex $x \in X$, and write $L = L(x)$, $F = F(x)$, $R = R(x)$, and $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$. Then:*

(i) [22, Lemma 5.5(i)]

$$[F, LR]E_i^* = (\mu_i(\theta) - 1)[F, RL]E_i^*, \quad 1 \leq i \leq d-1 \quad (7.1)$$

where

$$\mu_i(\theta) = \frac{\theta_{i-1}^* - \theta_{i+1}^*}{\theta_i^* - \theta_{i+1}^*}; \quad 1 \leq i \leq d-1 \quad (7.2)$$

(ii) [22, Theorem 5.1(v)] Γ is thin if and only if

$$[F, RL]E_i^* = 0, \quad 2 \leq i \leq d-1 \quad (7.3)$$

Lemma 7.2.2 Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Suppose Γ has a Q -polynomial structure with primary eigenvalue θ , and suppose the intersection number a_2 is zero. Then:

(i) there exist real numbers $\gamma(\theta)$, $g_i^-(\theta)$, and $g_i^+(\theta)$ such that

$$g_i^-(\theta)a_{i-2} + a_{i-1} + g_i^+(\theta)a_i = \gamma(\theta), \quad 2 \leq i \leq d \quad (7.4)$$

$$g_i^+(\theta) \neq 0; \quad 2 \leq i \leq d-1 \quad (7.5)$$

(ii) $a_i = 0$ for $1 \leq i \leq d-1$;

(iii) Γ is thin.

Proof of (i). Fix a vertex $x \in X$, and write $L = L(x)$, $F = F(x)$, $R = R(x)$, and $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$. Then by [22, Lemma 5.5(iii), Lemma 5.6] there exist real numbers $\gamma(\theta)$, $g_i^-(\theta)$, and $g_i^+(\theta)$ satisfying (7.5) so that

$$(g_i^-(\theta)FL^2 + LFL + g_i^+(\theta)L^2F - \gamma(\theta)L^2)E_i^* = 0. \quad 2 \leq i \leq d \quad (7.6)$$

Let δ_i denote the vector

$$\delta_i = \sum_{\substack{y \in X \\ \theta(x,y)=i}} \hat{y}, \quad 0 \leq i \leq d \quad (7.7)$$

and observe by (5.15) and (5.16) that

$$L\delta_i = b_{i-1}\delta_{i-1}, \quad 1 \leq i \leq d \quad (7.8)$$

$$F\delta_i = a_i\delta_i. \quad 0 \leq i \leq d \quad (7.9)$$

Applying (7.6) to the vector δ_i and using (7.8) and (7.9), we obtain

$$b_{i-2}b_{i-1}(g_i^-(\theta)a_{i-2} + a_{i-1} + g_i^+(\theta)a_i - \gamma(\theta))\delta_{i-2} = 0, \quad 2 \leq i \leq d \quad (7.10)$$

and (7.4) follows immediately. $\square$

Proof of (ii). If $a_1 \neq 0$ then $a_2, \dots, a_{d-1} \neq 0$ [2, Proposition 5.5.1(i)]. Since $a_2 = 0$, we have $a_1 = 0$; trivially $a_0 = 0$. Setting $i = 2$ in (7.4), we find that $\gamma(\theta) = 0$, and (7.4) becomes

$$g_i^-(\theta)a_{i-2} + a_{i-1} + g_i^+(\theta)a_i = 0. \quad 2 \leq i \leq d \quad (7.11)$$

Recalling from (7.5) that $g_i^+(\theta)$ is not zero for $2 \leq i \leq d-1$, a simple induction shows that $a_1, \dots, a_{d-1}$ are zero, as desired. $\square$

Proof of (iii). Observe by (ii) and (5.16) that $FE_i^* = 0$ for $1 \leq i \leq d-1$. Also, observe by (5.15) and (5.17) that the matrices RL and E_i^* commute. Now

$$[F, RL]E_i^* = FRLE_i^* - RLFE_i^* \quad (7.12)$$

$$= FE_i^*RL - RLFE_i^* \quad (7.13)$$

$$= 0 \quad (7.14)$$

for $1 \leq i \leq d-1$. Applying Lemma 7.2.1(ii) we see that Γ is thin, as desired. $\square$

Lemma 7.2.3 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Suppose Γ has a Q -polynomial structure with primary eigenvalue θ , and suppose the intersection number a_2 is not zero.*

Fix an integer i with $2 \leq i \leq d-1$, and vertices $x, y, z \in X$ with $\partial(x, y) = 2$, $\partial(y, z) = i-1$, and $\partial(x, z) = i+1$. Let σ and σ' denote the numbers

$$\sigma = |\{u \in X \mid \partial(x, u) = 2, \partial(y, u) = 1, \partial(z, u) = i-1\}|, \quad (7.15)$$

$$\sigma' = |\{v \in X \mid \partial(x, v) = 1, \partial(y, v) = 2, \partial(z, v) = i+1\}|. \quad (7.16)$$

Then

$$b_1 = (\theta + b_1 + 1) \left(1 - \frac{\sigma' + \sigma(\mu_i(\theta) - 1)}{a_2 \mu_i(\theta)} \right), \quad (7.17)$$

where $\mu_i(\theta)$ is as in (7.2). In particular, the eigenvalue θ may be calculated from $\mu_i(\theta)$, the integers σ and σ' , and the intersection numbers b_1 and a_2 .

Proof. Let θ have corresponding primitive idempotent E and dual eigenvalues $\theta_0^*, \dots, \theta_d^*$. Recall from Definition 2.2.3 the vector $\delta_{ij} = \delta_{ij}(x, y)$. By [23, Theorem 3.3], [17, Theorem 1.1] we have

$$0 = \langle E\hat{z}, \delta_{21} - \delta_{12} - a_2 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_2^*} (\hat{y} - \hat{x}) \rangle. \quad (7.18)$$

There are exactly a_2 vertices $u \in X$ with $\partial(x, u) = 2$ and $\partial(y, u) = 1$. Of these, σ have $\partial(z, u) = i-1$, leaving $a_2 - \sigma$ with $\partial(z, u) = i$. Applying (2.14),

$$|X| \langle E\hat{z}, \delta_{21} \rangle = \sigma \theta_{i-1}^* + (a_2 - \sigma) \theta_i^*. \quad (7.19)$$

Similarly, of the a_2 vertices $v \in X$ with $\partial(x, v) = 1$ and $\partial(y, v) = 2$, exactly σ' have $\partial(z, v) = i+1$, leaving $a_2 - \sigma'$ with $\partial(z, v) = i$; we have

$$|X| \langle E\hat{z}, \delta_{12} \rangle = \sigma' \theta_{i+1}^* + (a_2 - \sigma') \theta_i^*. \quad (7.20)$$

Evaluating (7.18) using (7.19), (7.20), and (2.14), we obtain

$$0 = (\sigma \theta_{i-1}^* + (a_2 - \sigma) \theta_i^*) - (\sigma' \theta_{i+1}^* + (a_2 - \sigma') \theta_i^*) - a_2 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_2^*} (\theta_{i-1}^* - \theta_{i+1}^*) \quad (7.21)$$

$$= \sigma(\theta_{i-1}^* - \theta_i^*) + \sigma'(\theta_i^* - \theta_{i+1}^*) - a_2 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_2^*} (\theta_{i-1}^* - \theta_{i+1}^*). \quad (7.22)$$

Rearranging (7.22) and applying (7.2), we have

$$\frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_2^*} = \frac{\sigma'(\theta_i^* - \theta_{i+1}^*) + \sigma(\theta_{i-1}^* - \theta_i^*)}{a_2(\theta_{i-1}^* - \theta_{i+1}^*)} \quad (7.23)$$

$$= \frac{\sigma' + \sigma(\mu_i(\theta) - 1)}{a_2 \mu_i(\theta)}. \quad (7.24)$$

Now by (6.5) and (7.24),

$$b_1 = (\theta + b_1 + 1) \left(1 - \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_2^*} \right) \quad (7.25)$$

$$= (\theta + b_1 + 1) \left(1 - \frac{\sigma' + \sigma(\mu_i(\theta) - 1)}{a_2 \mu_i(\theta)} \right), \quad (7.26)$$

as desired. $\square$

Lemma 7.2.4 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Suppose Γ admits distinct Q -polynomial structures with primary eigenvalues θ and ψ , and suppose the intersection number a_2 is not zero. Then Γ is thin.*

Proof. Suppose for a contradiction that Γ is not thin. Fix a vertex $x \in X$, and write $L = L(x)$, $F = F(x)$, $R = R(x)$, and $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$. Then by Lemma 7.2.1(ii) there exists an integer i with $2 \leq i \leq d-1$ so that $[F, RL]E_i^*$ is not zero. Fix such an integer i .

Since $[F, RL]E_i^*$ is not zero, we can solve (7.1) for $\mu_i(\theta)$, obtaining a constant which does not depend on θ . In particular, $\mu_i(\theta) = \mu_i(\psi)$. Now (7.17) applies to both θ and ψ , forcing $\theta = \psi$. This is a contradiction, and Γ is thin as desired. $\square$

Chapter 8

Twice Q-polynomial Distance-Regular Graphs

8.1 Introduction

In this chapter we classify the distance-regular graphs with diameter $d \geq 5$ and valency $k \geq 3$ which admit more than one Q-polynomial structure. We use this classification and an argument of Bannai and Ito to show that the eigenvalues of a Q-polynomial distance-regular graph with diameter $d \geq 5$ and valency $k \geq 3$ are all integers. The bounds on d and k in this result are tight: the Hadamard graphs and Taylor graphs may have eigenvalues which are not integers, and have diameter $d = 4$ and diameter $d = 3$, respectively; the distance-regular graphs with valency $k = 2$ are precisely the ordinary n -gons, which have non-integer eigenvalues for $n = 5$ and $n \geq 7$.

Our main results are Theorem 8.1.2 and Corollary 8.1.3. In the proof of Theorem 8.1.2 we apply the results of Chapters 2, 3, 4, 5, and 7.

Bannai and Ito study distance-regular graphs with diameter $d \geq 34$ and valency $k \geq 3$ which admit more than one Q-polynomial structure [1, Section 3.7]. They find relations between the two Q-polynomial structures, and use these relations to show that the eigenvalues of a Q-polynomial distance-regular graph with diameter $d \geq 34$ and valency $k \geq 3$ are all integers. Brouwer, Cohen, and Neumaier conjecture that our result Corollary 8.1.3 holds [2, p. 247].

Example 8.1.1 The following examples of distance-regular graphs have precisely two Q-polynomial structures. In each case, $E_0, \dots, E_d$ denotes the natural ordering of the primitive idempotents of the graph. See Lemmas 2.1.1, 3.1.1, and 3.1.2 for the definitions of dual bipartite, almost dual bipartite, and almost dual antipodal Q-polynomial structures, respectively.

(i) The cube $H(d, 2)$, d even:

$E_0, \dots, E_d$	dual bipartite;
$E_0, E_{d-1}, E_2, \dots, E_{d-2}, E_1, E_d$	dual bipartite.

(ii) The half-cube $\frac{1}{2}H(2d+1, 2)$:

$E_0, \dots, E_d$	almost dual bipartite;
$E_0, E_2, E_4, \dots, E_5, E_3, E_1$	almost dual antipodal.

(iii) The antipodal quotient of the cube $\tilde{H}(2d+1, 2)$:

$E_0, \dots, E_d$	almost dual antipodal;
$E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, \dots$	almost dual bipartite.

(iv) The dual polar graph on $[{}^2A_{2d-1}(r)]$, $r \geq 2$ a prime power:

$$\begin{array}{ll} E_0, \dots, E_d & \text{almost dual antipodal;} \\ E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, \dots & \text{almost dual bipartite.} \end{array}$$

For more information on these graphs, see Chapter 9.

Theorem 8.1.2 *Let Γ denote a distance-regular graph with diameter $d \geq 5$ and valency $k \geq 3$. Then Γ has more than one Q -polynomial structure if and only if Γ is one of:*

- (i) the cube $H(d, 2)$, d even;
- (ii) the half-cube $\frac{1}{2}H(2d+1, 2)$;
- (iii) the antipodal quotient of the cube $\tilde{H}(2d+1, 2)$;
- (iv) the dual polar graph on $[{}^2A_{2d-1}(r)]$, $r \geq 2$ a prime power.

In particular, Γ has at most two Q -polynomial structures.

We prove Theorem 8.1.2 in Section 8.2. We mention the following corollary to Theorem 8.1.2, which is of independent interest:

Corollary 8.1.3 *Let Γ denote a distance-regular graph with diameter $d \geq 5$ and valency $k \geq 3$, and suppose Γ admits a Q -polynomial structure. Then the eigenvalues of Γ are all integers.*

Proof. Suppose Γ has an eigenvalue which is not an integer. Then Γ has a second Q -polynomial structure [1, p. 360], and must be one of Theorem 8.1.2(i)–(iv). The eigenvalues of these graphs are all integers, for a contradiction. $\square$

8.2 Proof of Theorem 8.1.2

Lemma 8.2.1 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 4$. Suppose $E_0, \dots, E_d$ and $\tilde{E}_0, \dots, \tilde{E}_d$ are Q -polynomial structures for Γ , with Krein parameters q_{ij}^h and $\tilde{q}_{ij}^h$, respectively. If $q_{11}^1 \neq 0$ and $\tilde{q}_{11}^1 \neq 0$, then:*

- (i) $E_1 = \tilde{E}_d$;
- (ii) $E_d = \tilde{E}_1$;
- (iii) $d = 4$.

Proof of (i). Let i, j denote the integers with $E_i = \tilde{E}_1$ and $E_1 = \tilde{E}_j$. Recall $E_0 = \tilde{E}_0$ and $E_1 \neq \tilde{E}_1$, so that $i, j \geq 2$. Suppose $j \neq d$ for a contradiction. Then by Theorem 4.1.1,

$$0 \neq \tilde{q}_{1j}^j \tag{8.1}$$

$$= q_{i1}^1, \tag{8.2}$$

and $i = 2$ by (1.24). In particular $i \neq d$; switching i, j in the above argument, we obtain $j = 2$.

Let l denote the integer with $E_l = \tilde{E}_4$. Since $i = j = 2$, we have $E_2 = \tilde{E}_1$ and $E_1 = \tilde{E}_2$. Thus $l \geq 3$, and by (1.24),

$$0 = q_{l1}^1 \quad (8.3)$$

$$= \tilde{q}_{42}^2, \quad (8.4)$$

contradicting (1.25). Thus $j = d$, as desired. $\square$

Proof of (ii). Similar to (i). $\square$

Proof of (iii). Let k denote the integer with $E_2 = \tilde{E}_k$. Observe $k \neq 0$ by $E_0 = \tilde{E}_0$ and $k \neq d$ by (i). By Theorem 4.1.1 and (ii),

$$0 \neq \tilde{q}_{1k}^k \quad (8.5)$$

$$= q_{d2}^2. \quad (8.6)$$

By (1.24) we have $d \leq 4$, and $d = 4$ as desired. $\square$

Proof of Theorem 8.1.2. By Lemma 8.2.1 and our hypothesis $d \geq 5$, some $\mathbb{Q}$ -polynomial structure for Γ has $q_{11}^1 = 0$. Fix such a structure. By Theorem 7.1.1 and Theorem 5.1.2 this structure has $q_{1i}^i = 0$ for $1 \leq i \leq d-1$. In particular, this structure is either dual bipartite or almost dual bipartite. In the first case, Γ must be one of the graphs listed in Theorem 2.1.2. The only such graph with $d \geq 5$, $k \geq 3$, and a second $\mathbb{Q}$ -polynomial structure is (i). In the second case, Γ is one of the graphs (ii), (iii), (iv) by Theorem 3.1.4. $\square$

Chapter 9

Examples

9.1 Introduction

In this chapter we define the graphs which are mentioned elsewhere in the thesis. For each graph, we give the intersection array $\{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$, the eigenvalues $\theta_0, \dots, \theta_d$, and the Q-polynomial structures. We quote from [1], [2], and [22].

In each case, the graph is known to be distance-regular with diameter d . We denote the natural ordering of the primitive idempotents by $E_0, \dots, E_d$.

9.2 Johnson Graphs

The *Johnson Graph* $J(d, n) = (X, R)$ is defined for integers $n \geq 2d \geq 0$, and has

$$\begin{aligned} X &= \text{all subsets of } \{1, \dots, n\} \text{ of order } d, \\ R &= \{xy \in X \times X \mid |x \cap y| = d - 1\}, \end{aligned}$$

$$\begin{aligned} b_i &= (d - i)(n - d - i), & 0 \leq i \leq d \\ c_i &= i^2, & 0 \leq i \leq d \\ \theta_i &= (d - i)(n - d - i) - i. & 0 \leq i \leq d \end{aligned}$$

The natural ordering of the primitive idempotents is the unique Q-polynomial structure on $J(d, n)$:

$$E_0, \dots, E_d \quad \text{dual bipartite if } n = 2d.$$

9.3 Hamming Graphs and Cubes

The *Hamming Graph* $H(d, n) = (X, R)$ is defined for integers $d \geq 0$, $n \geq 1$, and has

$$\begin{aligned} X &= \text{all } d\text{-tuples from the set } \{1, \dots, n\}, \\ R &= \{xy \in X \times X \mid x, y \text{ differ in exactly 1 coordinate}\}, \end{aligned}$$

$$\begin{aligned} b_i &= (d - i)(n - 1), & 0 \leq i \leq d - 1 \\ c_i &= i, & 1 \leq i \leq d \\ \theta_i &= n(d - i) - d. & 0 \leq i \leq d \end{aligned}$$

Note that $H(d, 2)$ is just the vertex graph of the d -cube. The natural ordering of the primitive idempotents is a Q-polynomial structure on $H(d, n)$:

$$E_0, \dots, E_d \quad \text{dual bipartite if } n = 2.$$

The cube $H(d, 2)$ has a second Q-polynomial structure if d is even:

$$E_0, E_{d-1}, E_2, \dots, E_{d-2}, E_1, E_d \quad \text{dual bipartite.}$$

9.4 Half Cubes

The cube $H(d, 2)$ is bipartite, with a unique bipartite half. The half-cube $\frac{1}{2}H(2d, 2)$ has

$$\begin{aligned} b_i &= (d-i)(2d-2i-1), & 0 \leq i \leq d-1 \\ c_i &= i(2i-1), & 1 \leq i \leq d \\ \theta_i &= 2(d-i)^2 - d. & 0 \leq i \leq d \end{aligned}$$

The natural ordering of the primitive idempotents is the unique Q-polynomial structure on $\frac{1}{2}H(2d, 2)$:

$$E_0, \dots, E_d \quad \text{dual bipartite.}$$

The half-cube $\frac{1}{2}H(2d+1, 2)$ has

$$\begin{aligned} b_i &= (d-i)(2d-2i+1), & 0 \leq i \leq d-1 \\ c_i &= i(2i-1), & 1 \leq i \leq d \\ \theta_i &= 2(d-i)^2 + d - 2i. & 0 \leq i \leq d \end{aligned}$$

There are two Q-polynomial structures on $\frac{1}{2}H(2d+1, 2)$:

$$\begin{aligned} E_0, \dots, E_d & \quad \text{almost dual bipartite;} \\ E_0, E_2, E_4, \dots, E_5, E_3, E_1 & \quad \text{almost dual antipodal.} \end{aligned}$$

9.5 Antipodal Quotients of Cubes

The cube $H(d, 2)$ is antipodal. For $n = 2d$ and $n = 2d+1$, The antipodal quotient of the cube $\tilde{H}(n, 2)$ has diameter d and

$$\begin{aligned} b_i &= n-i, & 0 \leq i \leq d-1 \\ c_i &= i, & 1 \leq i \leq d \\ \theta_i &= n-4i. & 0 \leq i \leq d \end{aligned}$$

The natural ordering of the primitive idempotents is the unique Q-polynomial structure on $\tilde{H}(2d, 2)$:

$$E_0, \dots, E_d.$$

There are two Q-polynomial structures on $\tilde{H}(2d+1, 2)$:

$$\begin{aligned} E_0, \dots, E_d & \quad \text{almost dual antipodal;} \\ E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, \dots & \quad \text{almost dual bipartite.} \end{aligned}$$

9.6 Dual Polar Graphs

The *Hermitian Dual Polar Graph* $[{}^2A_{2d-1}(r)]$ is defined for $r \geq 2$ a prime power. Let U denote a $2d$ -dimensional vector space over $\text{GF}(r^2)$, together with a nondegenerate hermitian form. A subspace of U on which the form vanishes completely is said to be *isotropic*. The graph $[{}^2A_{2d-1}(r)]$ has

$$\begin{aligned} X &= \text{all maximal isotropic subspaces of } U, \\ R &= \{xy \in X \times X \mid \dim(x \cap y) = d-1\}, \end{aligned}$$

$$b_i = r \frac{r^{2d} - r^{2i}}{r^2 - 1}, \quad 0 \leq i \leq d$$

$$c_i = \frac{r^{2i} - 1}{r^2 - 1}, \quad 0 \leq i \leq d$$

$$\theta_i = r \frac{r^{2d-2i} - 1}{r^2 - 1} - \frac{r^{2j} - 1}{r^2 - 1}, \quad 0 \leq i \leq d$$

There are two Q -polynomial structures on $[{}^2A_{2d-1}(r)]$:

$$\begin{aligned} E_0, \dots, E_d & \quad \text{almost dual antipodal;} \\ E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, \dots & \quad \text{almost dual bipartite.} \end{aligned}$$

9.7 Hadamard Graphs

The *Hadamard Graphs* of order 4γ are the distance-regular graphs with diameter $d = 4$ and intersection array

$$\{4\gamma, 4\gamma - 1, 2\gamma, 1; 1, 2\gamma, 4\gamma - 1, 4\gamma\},$$

where $\gamma \geq 1$ is an integer. These graphs have eigenvalues

$$4\gamma, 2\sqrt{\gamma}, 0, -2\sqrt{\gamma}, -4\gamma,$$

and two Q -polynomial structures:

$$\begin{aligned} E_0, E_1, E_2, E_3, E_4 & \quad \text{dual bipartite;} \\ E_0, E_3, E_2, E_1, E_4 & \quad \text{dual bipartite.} \end{aligned}$$

Infinitely many examples are known to exist, including examples whenever $4\gamma - 1$ is a prime power [12].

9.8 Taylor Graphs

The *Taylor Graphs* are the distance-regular graphs with diameter $d = 3$ and intersection array

$$\{k, k - a_1 - 1, 1; 1, k - a_1 - 1, k\},$$

where $a_1 \geq 1$ and $k \geq a_1 + 2$ are integers. These graphs have eigenvalues

$$k, \quad n, \quad -1, \quad -m,$$

where n and $-m$ are the roots of $\theta^2 + (k - 2a_1 - 1)\theta - k = 0$, and two Q-polynomial structures:

E_0, E_1, E_2, E_3	dual bipartite;
E_0, E_2, E_1, E_3	dual bipartite.

Infinitely many examples are known to exist, including graphs with $k = 2a_1 + 1$ whenever $2a_1 + 1$ is a prime power. (Observe that these have $n = m = \sqrt{k}$.)

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