Devin Brown – Final Presentation
Tuesday, April 23, 2019
Work of B. Hartwig

The Tetrahedron Algebra and its fin. dim'l irreducibles

Motivation: Let $\mathbb{F} = \overline{\mathbb{F}}$, $V = d+1$ dim'l v.s. / $\mathbb{F}$
char($\mathbb{F}$) = 0

Let $\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ be the Leonard system associated to the hypercube $H(d,2)$

I.e. for $0 \le i \le d$, Φ has parameters

$\theta_i = d - 2i$ $\quad \theta_i^* = d - 2i$

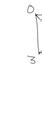
$\phi_i = -2i(d-i+1)$ $\quad \phi_i^* = 2i(d-i+1)$

Recall tetrahedron diagram for Φ



the diagram shows 6 (or 12) decomps of V

For notational convenience label



$\mathrm{II} = \{0,1,2,3\}$

$[i,j]$ = corresponding decomposition frame

Ex: the decomp $\{E_i V\}_{i=0}^d$ is $[3,2]$

$\{E_i^* V\}_{i=0}^d$ is $[0,1]$

For distinct $i, j \in \mathrm{II}$ define $X_{ij} \in \mathrm{End}(V)$ as follows

Let $\{W_k\}_{k=0}^d$ be the decomp $[i,j]$

For $0 \le l \le d$ We is eigenspace for X_{ij} with eigenvalue $2l-d$

e.g. $A = X_{23}$ $\quad A^* = X_{10}$

We saw that the X_{ij} satisfied the relations of the tetrahedron Lie algebra $\boxtimes$

Definition! Let $\boxtimes$ denote the Lie algebra over $\mathbb{F}$ defined by generators $\{X_{ij} \mid i,j \in \mathrm{II},\ i \ne j\}$ and the following relations

(i) For all distinct $r,s \in \mathrm{II}$

$$X_{rs} + X_{sr} = 0$$

(ii) For all mutually distinct $r,s,t \in \mathrm{II}$

$$[X_{rs}, X_{st}] = 2X_{rs} + 2X_{st}$$

(iii) For all mutually distinct $r,s,t,u \in \mathrm{II}$

$$[X_{rs}, [X_{rs}, [X_{rs}, X_{tu}]]] = 4[X_{rs}, X_{tu}]$$

Then V is an irreducible $\boxtimes$-module.

Problem. Classify the finite dimensional irreducible $\boxtimes$-modules.

Strategy consider the subalgebras generated by X_{rs} and X_{tu} for mutually distinct $r,s,t,u \in \mathrm{II}$.

↳ These generate a copy of the so-called Onsager Lie algebra

Remk. In TD system with $\Theta_i = q^{2i-d}$, $\Theta_i^* = q^{d-2i}$ makes V an irreducible $\mathcal{U}_q$ module.

Definition 1.2: Let $\mathcal{O}$ denote the Lie algebra over $\mathbb{F}$ with generators X, Y and relations

$$[X,[X,[X,Y]]] = 4[X,Y]$$

and $\quad [Y,[Y,[Y,X]]] = 4[Y,X]$

Remk. This Lie algebra first appeared in statistical mechanics in the study of the 2-D Ising model.

For r,s,t,u as above, we have injective Lie algebra homomorphisms

$$\mathcal{O} \longrightarrow \boxtimes$$
$$X \mapsto X_{rs}$$
$$Y \mapsto X_{tu}$$

Thus an $\mathcal{O}$-module V is a sum of 3 Onsager subalgebras

Thm (Hartwig–Terwilliger)

$\boxtimes$ is a direct sum of its 3 Onsager subalgebras

(2) Prove this by giving an isomorphism $\boxtimes \xrightarrow{\sim} \mathfrak{sl}_2 \otimes \mathbb{F}[t,t^{-1},(t-1)^{-1}]$. Now we discuss the classification of irreducible $\mathcal{O}$-modules.

Facts (due to Davies)

Let V be a finite dim'l irreducible $\mathcal{O}$-module. Then

- The actions of X, Y are diagonalizable
- There exists $d \ge 0$ and $0 \ne \alpha \in \mathbb{F}$ so that
  $\{d-2i+d \mid 0 \le i \le d\} = \{\text{eigenvalues for } X\}$
  $\{d-2i+\alpha(d) \mid 0 \le i \le d\} = \{\text{eigenvalues for } Y\}$

↳ we say V is of type $(d, \alpha(d))$

- Replacing X, Y by $X - \alpha I$ and $Y - \alpha^2 I$ it suffices to classify irrep of type $(d, 0)$.

To define the modules relevant to the classification we map $\mathcal{O}$ into the loop algebra $L(\mathfrak{sl}_2) := \mathfrak{sl}_2 \otimes_{\mathbb{F}} \mathbb{F}[t,t^{-1}]$

Roan Thm is an injection

$$\mathcal{O} \longrightarrow L(\mathfrak{sl}_2)$$
$$X \mapsto e_0 t + f_0 t^{-1}$$
$$Y \mapsto e_1 t + f_1 t^{-1}$$

For any $a \in \mathbb{F}^\times$ we can define

$$\mathrm{ev}_a : L(\mathfrak{sl}_2) \to \mathfrak{sl}_2$$
by $\quad \mathrm{ev}_a(u \otimes g(t)) = g(a)u$

for all $u \in \mathfrak{sl}_2$ and $g(t) \in \mathbb{F}[t,t^{-1}]$.

Given an irrep V of $\mathfrak{sl}_2$ with $\dim \ge 2$ the pullback along ev_a gives a module $V(a)$ of $\mathcal{O}$

Then

- $V(a)$ is irreducible iff $a \ne \pm 1$
- $V(a) \cong V(b) \iff a = b^{\pm 1}$

↳ Depends on choice of V!

Theorem (Date–Roan '00)

Every fin dim'l irreducible $\mathcal{O}$-module of type $(0,0)$ is isomorphic to a tensor product of evaluation modules

$$V_1(a_1) \otimes \cdots \otimes V_1(a_n).$$

where $a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}$ are mutually distinct.

Further, any two such tensor products are isomorphic iff one can be obtained from the other by permuting factors and replacing some a_i's with a_i^{-1}

↳ Now we can state the classification of irreducible $\boxtimes$-modules

Theorem 1.7 (Hartwig)

Let V be a f.d. irreducible $\boxtimes$-module. Then there exists a unique $\mathcal{O}$-module structure on V so that X, Y act as X_{01}, X_{23} resp. Thm V is type $(0,0)$ and irreducible as an $\mathcal{O}$ module.

Theorem 1.8 (Hartwig)

Let V be a f.d. irreducible $\mathcal{O}$ module of type $(0,0)$. Then there is a unique $\boxtimes$-module structure on V such that X, Y act as X_{01}, X_{23} and V is an irreducible $\boxtimes$-module.

Rmk. We have a bijection

$$\left\{ \begin{array}{c} \text{F.d. Irreducible} \\ \mathcal{O}\text{-modules of type } (0,0) \end{array} \right\} / \cong \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{F.d. Irreducible} \\ \boxtimes\text{-modules} \end{array} \right\} / \cong$$