

\mathbb{F} alg closed $0 \neq q \in \mathbb{F}$ $q^4 \neq 1$

Given any AW polys $\{p_n\}_{n=0}^{\infty}$

$$p_n = p_n(x; a, b, c, d/q)$$

In th 54 we displayed a module structure on $\mathbb{F}[x]$ for $\Delta = \Delta_{q^{1/2}}$ such that the gen A of Δ acts on $\mathbb{F}[x]$ as mult by x

We now obtain a similar result using $\hat{H}_{q^{1/2}}$

Let y denote an indet. Consider

$\mathbb{F}[y, y^{-1}]$ the \mathbb{F} -alg of Laurent polys in y

Def
$$x = y + y^{-1}$$

Subalg of $\mathbb{F}[y, y^{-1}]$ gen by x has basis

$$1, y + y^{-1}, y^2 + y^{-2}, y^3 + y^{-3}, \dots$$

This subalg is iso $\mathbb{F}[x]$

I identify $\mathbb{F}[x]$ with subalg of $\mathbb{F}[y, y^{-1}]$ gen by x

Consider generator $A = y + y^{-1}$ for $\hat{H}_{q^{1/2}}$ ($\gamma = 606$)

We display an $\hat{H}_{q^{1/2}}$ -module str on $\mathbb{F}[y, y^{-1}]$

such that Y acts as mult by y . This $\hat{H}_{q^{1/2}}$ -module

is iso to the one from th 40 (with q replaced by $q^{1/2}$)

Fix square roots as in #54:

$$q^{1/2}, a^{1/2}, b^{1/2}, c^{1/2}, d^{1/2}$$

Define

$$k_0 = (ab)^{1/2}$$

$$k_1 = (ab^{-1})^{1/2}$$

$$k_2 = (cd)^{1/2}$$

$$k_3 = (q^{-1}cd)^{1/2}$$

so that

$$a = k_0 k_1$$

$$b = k_0 k_1^{-1}$$

$$c = q^{1/2} k_2 k_3$$

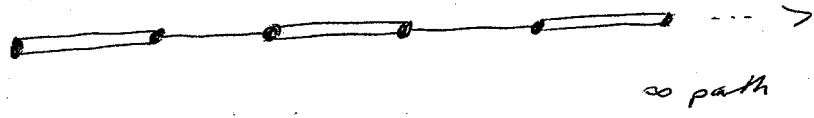
$$d = q^{1/2} k_2^{-1} k_3$$

[this matches #42 with q replaced by $q^{1/2}$]

Recall $\hat{H}_{q^{1/2}}$ - module

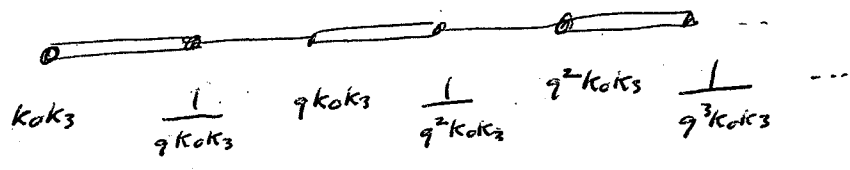
$$V = V(k_0, k_1, k_2, k_3)$$

from th 40. Diagrams

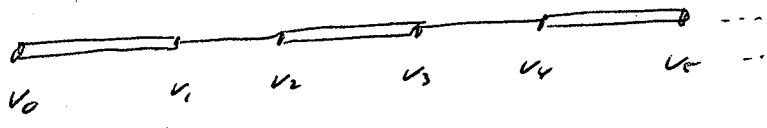


Recall for above diagram the nodes are eigenvalues

of $X = t_3 t_0$ on V_i equals are



Eigenbasis for X :



$$G_2 v_i = v_{i+2} \quad i \text{ even} \quad i = 0, 1, 2, \dots$$

$$G_0 v_i = v_{i-2} \quad i \text{ odd}$$

Recall by const prior to th 40,

$$t_0 v_0 = k_0 v_0,$$

$$t_3 v_0 = k_3 v_0.$$

Going to show the following is a basis for V :

$$Y^n v_0 \quad n \in \mathbb{Z}$$

Recall

$$V = \frac{t_0 - k_0^{\rightarrow}}{k_0 - k_0^{\rightarrow}} V + \frac{t_0 - k_0}{k_0^{\rightarrow} - k_0} V \quad ds$$

↑
eigspace for t_0 equal k_0

↑
eigspace for t_0 equal k_0^{\rightarrow}

Recall

$$\frac{t_0 - k_0^{\rightarrow}}{k_0 - k_0^{\rightarrow}} V \quad \text{has basis}$$

$$\frac{t_0 - k_0^{\rightarrow}}{k_0^{\rightarrow} - k_0} v_{2i} \quad i = 0, 1, 2, \dots$$

and

$$\frac{t_0 - k_0}{k_0^{\rightarrow} - k_0} V \quad \text{has basis}$$

$$\frac{t_0 - k_0}{k_0^{\rightarrow} - k_0} v_{2i} \quad i = 1, 2, \dots$$

Note $t_0 v_0 = k_0 v_0$ so

$$\frac{t_0 - k_0^{\rightarrow}}{k_0 - k_0^{\rightarrow}} v_0 = v_0$$

By M 42

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i} \in \mathbb{F} \quad p_i(A; a, b, c, d / q) v_0$$

$$i = 0, 1, 2, \dots$$

$$A = Y + Y^{-1} \quad Y = t_0 b_1$$

Recall p_i has degree i for $i = 0, 1, 2, \dots$

The following is a basis for $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V$:

$$(Y + Y^{-1})^n v_0$$

$$n = 0, 1, 2, \dots$$

(*)

By M 44

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_{2i} \in \mathbb{F} \quad p_i(A; q a, q b, c, d / q) \frac{t_0 - k_0}{k_0^{-1} - k_0} v_2$$

$$i = 1, 2, 3, \dots$$

So the following is a basis for $\frac{t_0 - k_0}{k_0^{-1} - k_0} V$:

$$(Y + Y^{-1})^n \frac{t_0 - k_0}{k_0^{-1} - k_0} v_2$$

$$n = 0, 1, 2, \dots \quad (**)$$

LEM 55 With above notation

the following agree up to a nonzero scalar factor:

(c) $\frac{t_0 - k_0}{k_0^{-1} - k_0} v_2$

(cc) $(Y \mp abY^{-1} - a - b) v_0$

pf Recall $Y = t_0 t_1$

By the const of the $\hat{H}_{q^{1/2}}$ -module V in $M \neq 0$

we are given the action of t_0, t_1 on the basis $\{v_i\}_{i=0}^{\infty}$

Recall

	v_0	v_1	v_2	v_3
t_0	$\gamma_0(q^{-1/2} k_0^{-1} k_3^{-1})$		\circ	
t_1				
v_2	\circ		$\gamma_1(q^{-3/2} k_0^{-1} k_3^{-1})$	
v_3				

$$T_1(\theta) = \begin{pmatrix} \alpha(\theta) & \beta(\theta) \\ \gamma(\theta) & \delta(\theta) \end{pmatrix}$$

$$\alpha(\theta) = \frac{\theta(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{\theta - \theta^{-1}}$$

$$\gamma(\theta) = \frac{1}{\theta^{-1} - \theta}$$

t_0 :

	v_0	v_1	v_2	...
v_0	k_0	\circ		\circ
v_1	\circ	$\tau_0(q^{-1}k_0^{-1}k_3^{-1})$		\circ
v_2				
	\circ	\circ		

$$T_0(\theta) = \begin{pmatrix} \theta d(\theta) & -\frac{b(\theta)}{\theta} \\ -\theta c(\theta) & \frac{a(\theta)}{\theta} \end{pmatrix}$$

$$d(\theta) = \frac{\theta^{-1}(k_3 + k_3^{-1}) - k_0 - k_0^{-1}}{\theta^{-1} - \theta}$$

$$b(\theta) = \frac{G(\theta, k_0, k_3)}{\theta - \theta^{-1}}$$

$$c(\theta) = \frac{1}{\theta^{-1} - \theta}$$

$$a(\theta) = \frac{\theta(k_3 + k_3^{-1}) - k_0 - k_0^{-1}}{\theta - \theta^{-1}}$$

Σ_0

$$\Upsilon v_0 = t_0 t_1 v_0$$

$$= t_0 \left(\alpha \left(q^{-1/2} k_0^{-1} k_3^{-1} \right) v_0 + \gamma \left(q^{-1/2} k_0^{-1} k_3^{-1} \right) v_1 \right)$$

$$= k_0 \alpha \left(q^{-1/2} k_0^{-1} k_3^{-1} \right) v_0 + \gamma \left(q^{-1/2} k_0^{-1} k_3^{-1} \right) t_0 v_1$$

$$= k_0 \alpha \left(q^{-1/2} k_0^{-1} k_3^{-1} \right) v_0$$

$$+ \gamma \left(q^{-1/2} k_0^{-1} k_3^{-1} \right) q^{-1} k_0^{-1} k_3^{-1} d \left(q^{-1} k_0^{-1} k_3^{-1} \right) v_1$$

$$- \gamma \left(q^{-1/2} k_0^{-1} k_3^{-1} \right) q^{-1} k_0^{-1} k_3^{-1} c \left(q^{-1} k_0^{-1} k_3^{-1} \right) v_2$$

$$\begin{aligned}
 Y^{-1} k_0 &= t_1^{-1} t_0^{-1} v_0 \\
 &= k_0^{-1} t_1^{-1} v_0 \\
 &= k_0^{-1} (T_1 - t_1) v_0 \\
 &= k_0^{-1} (k_1 + k_1^{-1} - t_1) v_0 \\
 &= k_0^{-1} \left(k_1 + k_1^{-1} - \alpha \left(q^{-1/2} k_0^{-1} k_3^{-1} \right) \right) v_0 \\
 &\quad - k_0^{-1} \gamma \left(q^{-1/2} k_0^{-1} k_3^{-1} \right) v_0
 \end{aligned}$$

Also

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_2 = \frac{-q k_0 k_3 b \left(q^{-1/2} k_0^{-1} k_3^{-1} \right) v_1 + \left(q k_0 k_3 a \left(q^{-1/2} k_0^{-1} k_3^{-1} \right) - k_0 \right) v_2}{k_0^{-1} - k_0}$$

Evaluating $Y v_0, Y^{-1} v_0, \frac{t_0 - k_0}{k_0^{-1} - k_0} v_0$ by these connects we get the result. \square

now

Find a vector $\psi \in \text{Span}(Y, Y^{-1}, 1)$ such that

$$t_0 \psi v_0 = k_0^{-1} \psi v_0$$

Write

$$\psi = Y + rY^{-1} + s$$

Find r, s

Require

$$0 = (t_0 - k_0^{-1}) \psi v_0$$

$$= (t_0 - k_0^{-1}) Y v_0 + r (t_0 - k_0^{-1}) Y^{-1} v_0 + s (k_0 - k_0^{-1}) v_0 \quad *$$

Recall

$$t_0 Y = Y^{-1} t_0 + Y T_0 - T_1$$

$$t_0 Y^{-1} = Y t_0 - Y T_0 + T_1$$

So

$$(t_0 - k_0^{-1}) Y v_0 = k_0 Y^{-1} v_0 + k_0 Y v_0 - (k_1 + k_1^{-1}) v_0$$

$$(t_0 - k_0^{-1}) Y^{-1} v_0 = -k_0^{-1} Y v_0 - k_0^{-1} Y^{-1} v_0 + (k_1 + k_1^{-1}) v_0$$

* becomes

0 =

term	coef
$Y v_0$	$k_0 - r k_0^{-1}$
$Y^{-1} v_0$	$k_0 - r k_0^{-1}$
v_0	$-k_1 - k_1^{-1} + r(k_1 + k_1^{-1}) + s(k_0 - k_0^{-1})$

Require each coef is 0

$$r = k_0^2 = ab$$

$$0 = (k_1 k_1^{-1})(r-1) + \cancel{1}(k_0 - k_0^{-1})$$

" "

$$(k_0^2 - 1) k_0^{-1}$$

" "

$$(r-1) k_0^{-1}$$

$$0 = k_1 + k_1^{-1} + k_0^{-1} r$$

$$r = -k_0(k_1 + k_1^{-1})$$
$$= -k_0 k_1^{-1}(k_1^2 + 1)$$
$$= -b(a/b + 1)$$
$$= -a - b$$

$$\psi = \gamma + r\gamma^{-1} + 1$$
$$= \gamma + ab\gamma^{-1} - a - b$$

Cor 56 With the above notation
the following is a basis for V :

$$Y^n v_0 \quad n \in \mathbb{Z}$$

pf By ~~*, **~~, LEM 55 and since $ab = k_0^2 \neq 1$ \square

Cor 57 With above notation
 \exists iso of \mathbb{F} -vector spaces

$$\begin{array}{ccc} \mathbb{F}[y, y^{-1}] & \longrightarrow & V \\ y^n & \longrightarrow & Y^n v_0 \quad n \in \mathbb{Z} \end{array}$$

pf By Cor 56 \square

Ref to Cor 57,

Via ζ we pull back the $\hat{H}_{q^{1/2}}$ -module str
from V to $\mathbb{F}[y, y^{-1}]$.

Now $\mathbb{F}[y, y^{-1}]$ becomes a $\hat{H}_{q^{1/2}}$ -module.

LEM 58

For the above $\hat{H}_{9/2}$ module $\mathbb{F}[y, y^{-1}]$

$$t_0 \cdot 1 = k_0 \cdot 1$$

$$t_3 \cdot 1 = k_3 \cdot 1$$

pf t_0 sends $1 \rightarrow v_0$

and

$$t_0 \cdot v_0 = k_0 v_0$$

$$t_3 \cdot v_0 = k_3 v_0$$

□

LEM 59

For the above $\hat{H}_{9/2}$ module $\mathbb{F}[y, y^{-1}]$

$$Y \cdot f = y f$$

$\forall f \in \mathbb{F}[y, y^{-1}]$

pf wlog

$$f = y^n \quad n \in \mathbb{Z}$$

$$Y \cdot y^n \underset{\text{in } \mathbb{F}[y, y^{-1}]}{=} Y \cdot y^n \underset{\text{in } V}{v_0}$$

$$= Y^{n+1} \underset{\text{in } V}{v_0}$$

$$= y^{n+1} \underset{\text{in } \mathbb{F}[y, y^{-1}]}{v_0}$$

□
464

For the above $\hat{H}_{q^{1/2}}$ -module $\mathbb{F}[y, y^{-1}]$ we now

describe the action of each t_i

By LS9 and

Since $t_0 t_1 = Y,$

$$t_2 t_3 = q^{-1/2} Y^{-1}$$

so to describe the actions of

t_0, t_3

Lecture 39 Mon Dec 5

\mathbb{F} alg dned $0 \neq q \in \mathbb{F}$ $q^4 \neq 1$

Recall our situation:

Given AN polys $\{p_n\}_{n=0}^{\infty}$

$$p_n = p_n(x; a, b, c, d/q)$$

$\{p_n\}_{n=0}^{\infty}$ is basis for $\mathbb{F}[x]$

View $x = y + y^{-1}$ y indet

Identity

$$\mathbb{F}[x] = \mathbb{F}[y + y^{-1}]$$

$$= \text{Span} \{ (y + y^{-1})^n \mid n = 0, 1, 2, \dots \}$$

$$= \text{Span} \{ 1, y + y^{-1}, y^2 + y^{-2}, \dots \}$$

In Cor 57 we constructed an \hat{H}_{q^2} module structure on $\mathbb{F}[y, y^{-1}]$ s.t.

• $t_0 \cdot 1 = k_0 \cdot 1$

• $t_3 \cdot 1 = k_3 \cdot 1$

• $Y = t_0 t_3$ acts on $\mathbb{F}[y, y^{-1}]$ as mult by q

Define

$$\psi \in F[y, y^{-1}]$$

by

$$\begin{aligned}\psi &= y + aby^{-1} - a - b \\ &= (y-a)(y-b)y^{-1}\end{aligned}$$

By LSS

$$t_0 \circ \psi = k_0^{-1} \psi$$

LEMMA For the above $\hat{H}_{g^{1/2}}$ -module $\mathbb{F}[y, y^{-1}]$,

3

$$(i) \quad t_0 \circ fg = f t_0 \circ g \quad \forall f \in \mathbb{F}[y, y^{-1}] \\ \forall g \in \mathbb{F}[y, y^{-1}]$$

$$(ii) \quad \mathbb{F}[y, y^{-1}] = \mathbb{F}[y, y^{-1}] + \mathbb{F}[y, y^{-1}] \psi \\ (\text{ds } \psi)$$

(iii) $\mathbb{F}[y, y^{-1}]$ is the eigenspace for t_0 with eigenvalue k_0

(iv) $\mathbb{F}[y, y^{-1}] \psi$ is the eigenspace for t_0 with eigenvalue k_0^{-1}

pf (i) In $\hat{H}_{g^{1/2}}$
 $Y + Y^{-1}$ commutes with t_0 ($Y = t_0 t_1$).
 Also Y acts on $\mathbb{F}[y, y^{-1}]$ as mult by y .

(ii) $ab \neq 1$, so the following are bases for the same space:

$$1, y, y^{-1}$$

$$1, y + y^{-1}, \psi$$

468

(iii), (iv) $\forall f(y) \in F[y]$

$$\begin{aligned} t_0 \circ f(y) &= t_0 \circ f(y) 1 \\ &= f(y) \underbrace{t_0 \circ 1}_{\substack{\text{"} \\ k_0 1}} \\ &= k_0 f(y) \end{aligned}$$

$$\begin{aligned} t_0 \circ f(y) \psi &= f(y) \underbrace{t_0 \circ \psi}_{\substack{\text{"} \\ k_0 \psi}} \\ &= k_0 f(y) \psi \end{aligned}$$

Result follows in view of (iii)

□

Prop 61 For the above $\hat{H}_{q^{1/2}}$ -module

$$\mathbb{F}[y, y^{-1}]$$

t_0 acts as follows:

$$\forall f \in \mathbb{F}[y, y^{-1}],$$

$$t_0 \cdot f(y) = (ab)^{1/2} f(y)$$

+

$$\frac{(y-a)(y-b)}{(ab)^{1/2} y} \frac{f(y) - f(y^{-1})}{y - y^{-1}}$$

Note $\frac{f(y) - f(y^{-1})}{y - y^{-1}} \in \mathbb{F}[y, y^{-1}]$ since

$\mathbb{F}[y, y^{-1}]$ has basis $\{y^n\}_{n \in \mathbb{Z}}$ and

$$\frac{y^n - y^{-n}}{y - y^{-1}} = y^{n-1} + y^{n-3} + \dots + y^{3-n} + y^{1-n}$$

$\forall n \in \mathbb{Z}$

pf wlog $f(y) = y^n$ $n \in \mathbb{Z}$

6

show

$$t_0 \cdot y^n = (ab)^{1/2} y^n + \frac{(y-a)(y-b)}{(ab)^{1/2} y} \frac{y^n - y^{-n}}{y - y^{-1}} \quad *$$

By L60 (ii) $\exists h, k \in \mathbb{F}[y, y^{-1}]$ s.t

$$y^n = h(y) + k(y) \psi(y)$$

By L60 (iii), (iv)

$$t_0 \cdot y^n = k_0 h(y) + k_0^{-1} k(y) \psi(y) \quad **$$

Find h, k :

the map $y \rightarrow y^{-1}$ leaves h, k invariant, so

$$y^{-n} = h(y) + k(y) \psi(y^{-1})$$

$$\text{So } y^n - y^{-n} = k(y) (\psi(y) - \psi(y^{-1}))$$

$$\text{obs } \psi(y) - \psi(y^{-1}) = (y - y^{-1})(1 - ab)$$

$$\text{so } k(y) = \frac{1}{1-ab} \frac{y^n - y^{-n}}{y - y^{-1}} \quad **$$

Obs

7

$$h(y) = y^n - k(y) \psi(y)$$

So using **

$$t_0 \cdot y^n = k_0 \left(y^n - k(y) \psi(y) \right) + k_0^{-1} k(y) \psi(y)$$

$$= k_0 y^n + \underbrace{(k_0^{-1} - k_0)}_{k_0^{-1}(1 - k_0^2) = k_0^{-1}(1 - ab)} k(y) \psi(y)$$

$$= k_0 y^n + \frac{\psi(y)}{k_0} \frac{y^n - y^{-n}}{y - y^{-1}} \quad k_0 = (ab)^{1/2}$$

$$= (ab)^{1/2} y^n + \frac{(y-a)(y-b)}{(ab)^{1/2} y} \frac{y^n - y^{-n}}{y - y^{-1}} \quad \checkmark$$

We have shown *.

□

Recall the projections

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}$$

$$\frac{t_0 - k_0}{k_0^{-1} - k_0}$$

We now show how these act on $\mathbb{F}[y, y^{-1}]$

472

LEM 62 For the above $\hat{H}_{q^{1/2}}$ -module
 $F[y, y^{-1}]$,

$\forall f \in F[y, y^{-1}]$

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} f(y) = \frac{\psi(y)}{1-ab} \frac{f(y) - f(y^{-1})}{y - y^{-1}}$$

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} f(y) = \frac{1}{1-ab} \frac{\psi(y) f(y^{-1}) - \psi(y^{-1}) f(y)}{y - y^{-1}}$$

where we recall

$$\psi(y) = (y-a)(y-b)y^{-1}$$

PF

Use Prop 61.

□

We now find the action of t_3 on the $\hat{H}_{q^{1/2}}$ module $\mathbb{F}[y, y^{-1}]$.

Recall $t_3 \circ 1 = k_3$

We will need an element

$$\phi \in \text{Span}\{y, y^{-1}, 1\}$$

such that

$$t_3 \cdot \phi = k_3^{-1} \phi$$

Write

$$\phi = y + r y^{-1} + a$$

find r, a

Require

$$0 = (t_3 - k_3^{-1}) \phi$$

$$= (t_3 - k_3^{-1}) \cdot y + r (t_3 - k_3^{-1}) \cdot y^{-1} + a (k_3 - k_3^{-1}) \quad *$$

Recall in $\hat{H}_{q^{1/2}}$

$$t_3 y = q^{-1} y^{-1} t_3 - q^{-1} y^{-1} T_3 + q^{-1/2} T_2$$

$$t_3 y^{-1} = q y t_3 + y^{-1} T_3 - q^{1/2} T_2$$

So

$$(t_3 - k_3^{-1}) \cdot y =$$

term	coef
y	$-k_3^{-1}$
y^{-1}	$q^{-1} k_3 - q^{-1} k_3 - q^{-1/2} k_3^{-1}$
1	$q^{-1/2} (k_2 + k_2^{-1})$

$$(t_3 - k_3^{-1}) \cdot y^{-1} =$$

term	coef
y	$q k_3$
y^{-1}	$-\cancel{k_3^{-1}} + k_3 + \cancel{k_3^{-1}}$
1	$-q^{1/2} (k_2 + k_2^{-1})$

* becomes

$$0 =$$

term	coef
y	$-k_3^{-1} + r q k_3$
y^{-1}	$-q^{-1} k_3^{-1} + r k_3$
1	$q^{-1/2} (k_2 + k_2^{-1}) - r q^{1/2} (k_2 + k_2^{-1}) + \Delta (k_3 - k_3^{-1})$

Require each coef is 0

$$r = q^{-1} k_3^{-2} = c^{-1} d^{-1}$$

$$\Delta (k_3 - k_3^{-1}) = (k_2 + k_2^{-1}) (q^{1/2} r - q^{-1/2})$$

$$= (q r - 1) q^{-1/2} (k_2 + k_2^{-1})$$

$$= (k_3^{-2} - 1) q^{-1/2} (k_2 + k_2^{-1})$$

$$= -(k_3 - k_3^{-1}) k_3^{-1} q^{-1/2} (k_2 + k_2^{-1})$$

$$\Delta = -\frac{(k_2 + k_2^{-1}) q^{-1/2}}{k_3} = -\frac{(k_2^2 + 1) q^{-1/2}}{k_2 k_3} = -c^{-1} d^{-1}$$

475

Def 63 For the above $\hat{H}_{q^{1/2}}$ -module $\mathbb{F}[y, y^{-1}]$ //

define

$$\begin{aligned} \phi &= y + c^{-1}d^{-1}y^{-1} - c^{-1} - d^{-1} \\ &= (y - c^{-1})(y - d^{-1})y^{-1} \end{aligned}$$

By ans

$$t_3 \circ \phi = k_3^{-1} \phi$$

— o —

Back in $\hat{H}_{q^{1/2}}$ recall

$$t_3 \text{ commutes with } \begin{array}{cc} t_2 t_3 + (t_2 t_3)^{-1} \\ \parallel & \parallel \\ q^{-1/2} y^{-1} & q^{1/2} y \end{array}$$

$$t_2 t_3 + (t_2 t_3)^{-1} = q^{1/2} (y + q^{-1} y^{-1})$$

In our study of the t_0 action on $\mathbb{F}[y, y^{-1}]$
 $y + y^{-1}$ played a key role.

In our study of the t_3 action on $\mathbb{F}[y, y^{-1}]$
 $y + q^{-1} y^{-1}$ will play a similar role.

LEM 64 For the $\hat{H}_{9/2}$ -module $\mathbb{F}[y, y^{\dagger}]$

$$(i) \quad t_3 \cdot fg = f t_3 \cdot g \quad \forall f \in \mathbb{F}[y + q^{-1}y^{\dagger}], \\ \forall g \in \mathbb{F}[y, y^{\dagger}]$$

$$(ii) \quad \mathbb{F}[y, y^{\dagger}] = \mathbb{F}[y + q^{-1}y^{\dagger}] + \mathbb{F}[y + q^{-1}y^{\dagger}] \phi \\ \text{(ds \& us)}$$

(iii) $\mathbb{F}[y + q^{-1}y^{\dagger}]$ is the eigenspace for t_3 with eigenval k_3

(iv) $\mathbb{F}[y + q^{-1}y^{\dagger}] \phi$ is the eigenspace for t_3 with eigenval k_3^{\dagger}

pf (i) We saw earlier

t_3 commutes with $y + q^{-1}y^{\dagger}$

$$(ii) \quad \phi = y + c^{-1}d^{-1}y^{\dagger} - c^{-1} - d^{-1}$$

$$c^{-1}d^{-1} \neq q^{-1}$$

So each of the following $\ast, \ast\ast$ is a basis for the same space:

$$1, y, y^{\dagger} \quad \ast$$

$$1, y + y^{\dagger}, \phi \quad \ast\ast$$

(iii), (iv) By (i), (ii) and since

$$t_3 \cdot 1 = k_3 \cdot 1$$

$$t_3 \cdot \phi = k_3^{\dagger} \cdot \phi$$

477 \square

Prop 65 For the above $\hat{H}_{q^{1/2}}$ module $\mathbb{F}[q, q^{-1}]$, 13

t_3 acts as follows.

$$t_3 \cdot f(y) = (q^a c d)^{1/2} f(y)$$

$$- \frac{(y - c^{-1})(y - d^{-1})}{(q^a c d)^{1/2} y} \cdot \frac{f(y) - f(q^{-2} y^2)}{y - q^{-2} y^2}$$

pt WLOG $f(y) = y^n$

19

show

$$t_3 \cdot y^n = (q^{-c}d)^{1/2} y^n - \frac{(y-c^{\tau})(y-d^{\tau})}{(q^{c^{\tau}}d^{\tau})^{1/2} y} \frac{y^n - (q^{\tau}y^{\tau})^n}{y - q^{\tau}y^{\tau}} \quad *$$

By L64(ii) $\exists h(y), k(y) \in \mathbb{F}[y + q^{\tau}y^{\tau}]$ s.t.

$$y^n = h(y) + k(y) \phi(y)$$

So by L64(iii), (iv)

$$t_3 \cdot y^n = k_3 h(y) + k_3^{\tau} k(y) \phi(y) \quad **$$

Find h, k

more $y \rightarrow q^{\tau}y^{\tau}$ leaves $y + q^{\tau}y^{\tau}$ inv
 $h(y), k(y)$ inv

$$\text{so } (q^{\tau}y^{\tau})^n = h(y) + k(y) \phi(q^{\tau}y^{\tau})$$

$$\text{so } \frac{y^n - (q^{\tau}y^{\tau})^n}{y - q^{\tau}y^{\tau}} = k(y) \frac{\phi(y) - \phi(q^{\tau}y^{\tau})}{y - q^{\tau}y^{\tau}}$$

$$= k(y) (1 - c^{\tau}d^{\tau})$$

$$k(y) = \frac{1}{1 - q^c d^{\tau}} \frac{y^n - (q^{\tau}y^{\tau})^n}{y - q^{\tau}y^{\tau}}$$

$$h(k) = y^n - k(y) \phi(y)$$

So using **

$$t_3 \cdot y^n = k_3 \left(y^n - k(y) \phi(y) \right) + k_3^{-1} k(y) \phi(y)$$

$$= k_3 y^n + \underbrace{(k_3^{-1} - k_3)}_{\text{"}} k(y) \phi(y)$$

$$k_3^{-1} (1 - k_3^2) = k_3^{-1} (1 - q^2 c d)$$

$$= (q^2 c d)^{1/2} y^n +$$

$$- \frac{\phi(y)}{(q c^2 d^2)^{1/2}}$$

$$\frac{y^n - (q^2 y^2)^n}{y - q^2 y^2}$$

We have shown *

□

LEM 66

For the above $H_{q^{1/2}}$ module $\mathbb{F}[y, y^{-1}]$

16

$$\forall f \in \mathbb{F}[y, y^{-1}]$$

$$\frac{t_3 - k_3}{k_3^{-1} - k_3} f(y) = \frac{\phi(y)}{1 - qc^{2d}} \frac{f(y) - f(q^2 y^{-1})}{y - q^2 y^{-1}}$$

$$\frac{t_3 - k_3^{-1}}{k_3 - k_3^{-1}} f(y) = \frac{1}{1 - qc^{2d}} \frac{\phi(y)f(q^2 y^{-1}) - \phi(q^2 y^{-1})f(y)}{y - q^2 y^{-1}}$$

pf Use Prop 65

□

Another view of $\mathbb{F}[y, y^{-1}]$

We write everything in terms of $\{k_i\}_{i \in \mathbb{Z}}$
instead of a, b, c, d. Work with \hat{H}_g instead of $\hat{H}_g^{1/2}$.

Until further notice let

y, k_0, k_1, k_2, k_3

denote mutually com units

Consider \mathbb{F} -alg

$$V = \mathbb{F}[y^{\pm 1}, k_0^{\pm 1}, k_1^{\pm 1}, k_2^{\pm 1}, k_3^{\pm 1}]$$

View an element f in V as a Laurent poly in y with

$$\text{coeff in } \mathbb{F}[k_0^{\pm 1}, k_1^{\pm 1}, k_2^{\pm 1}, k_3^{\pm 1}]$$

Thm 67 the above \mathbb{F} -vector space V

has a \hat{H}_q -module str such that $\forall f \in V,$

$$t_0 \cdot f(y) = k_0 f(y) +$$

$$\frac{(y - k_0 k_1)(y - k_0 k_1^{-1})}{k_0 y} \frac{f(y) - f(y^{-1})}{y - y^{-1}},$$

$$t_3 \cdot f(y) = k_3 f(y) +$$

$$\frac{(y - q^{-1} k_2^{-1} k_3^{-1})(y - q^{-1} k_2 k_3)}{y} \frac{f(y) - f(q^{-2} y^{-1})}{y - q^{-2} y^{-1}},$$

$$Y \cdot f(y) = y f(y).$$

The actions of t_1, t_2 are obtained using

$$t_0 t_1 = Y, \quad t_2 t_3 = q^{-1} Y.$$

Moreover the action of $t_i + t_i^{-1}$ on V is

$$(t_i + t_i^{-1}) f = (k_i + k_i^{-1}) f \quad (\text{see II})$$

pt this is reformulation of Prop 61, 65

F alg closed $0 \neq q \in F$

Given AW polys $\{p_n\}_{n=0}^{\infty}$

$$p_n = p_n(x; a, b, c, d | q)$$

Recall the $\hat{H}_{q^{1/2}}$ module $F[y, y^{-1}]$ from Props 6.1, 6.5
 ($x = y + y^{-1}$)

Recall:

$$F[y, y^{-1}] = F[y + y^{-1}] \uparrow + F[y + y^{-1}] \Psi \quad (ds \text{ vs})$$

\uparrow to eigenspace equal k_0 \uparrow to eigenspace equal k_0

$\Psi = y + ay^{-1} - a - b$

$F[y + y^{-1}]$ has basis the AW polys

$$p_n(y + y^{-1}; a, b, c, d | q) \quad n = 0, 1, 2, \dots$$

\uparrow

eigenvectors for $X + X^{-1}$ with equal $(X = t_3 t_0)$

$$k_0 k_3 q^n + \frac{1}{k_0 k_3 q^n}$$

$F[y + y^{-1}] \Psi$ has basis the AW polys

$$p_n(y + y^{-1}; q^a, q^b, c, d | q) \Psi \quad n = 0, 1, 2, \dots$$

\uparrow

eigenvectors for $X + X^{-1}$ with unequal

$$k_0 k_3 q^{na} + \frac{1}{k_0 k_3 q^{nb}}$$

Recall

$$e_n = aq^n + a^{-1}q^{-n}$$

$$n = 0, 1, 2, \dots$$

$\mathbb{F}[y+y^{-1}]_1$ has basis

$$1, \quad y+y^{-1}-e_0, \quad (y+y^{-1}-e_0)(y+y^{-1}-e_0), \quad \dots$$

Rel this basis

$$y+y^{-1} \quad \text{is} \quad \text{LBD}$$

(by #51)

$$x+x^{-1} \quad \text{is} \quad \text{UBD}$$

$\mathbb{F}[y+y^{-1}]_\Psi$ has basis

$$1_\Psi, \quad (y+y^{-1}-e_0)_\Psi, \quad (y+y^{-1}-e_0)(y+y^{-1}-e_0)_\Psi, \quad \dots$$

Rel this basis

$$y+y^{-1} \quad \text{is} \quad \text{LBD}$$

(by #51)

$$x+x^{-1} \quad \text{is} \quad \text{UBD}$$

Next goal: display a basis for $\mathbb{F}[y+y^{-1}]$
with respect to each

y is Lower triangular

x is upper triangular.

LEM 68

The following is a basis for $\mathbb{F}[q, q^{-1}]$

Name	vector
u_0	1
u_{-1}	$1 - aq^{-1}$
u_1	$(1 - qa) (1 - aq^{-1})$
u_{-2}	$(1 - qa^{-1}) (1 - qa) (1 - aq^{-1})$
u_2	$(1 - q^2a) (1 - qa^{-1}) (1 - qa) (1 - aq^{-1})$
\vdots	\vdots

Moreover

$$(1 - q^n a q^{-1}) u_n = u_{n-1}$$

$n = 0, 1, 2, \dots$

$$(1 - q^n a) u_n = u_n$$

$n = 1, 2, 3, \dots$

pf Routine

486

□

LEM 69

 y, y^{-1} act on basis $\{u_n\}_{n \in \mathbb{Z}}$ as follows.

9

(i) For $n \geq 0$

$$y u_n = q^n a u_n + q^{-n+1} a^{-1} u_{n-1} - q^{-n+1} a^{-1} u_{n+1}$$

$$y^{-1} u_n = q^{-n} a^{-1} u_n - q^{-n} a^{-1} u_{n+1}$$

(ii) For $n \geq 1$

$$y u_{-n} = q^{-n} a^{-1} u_{-n} - q^{-n} a^{-1} u_n$$

$$y^{-1} u_{-n} = q^n a u_{-n} + q^{-n} a^{-1} u_n - q^{-n} a^{-1} u_{-n-1}$$

pf routine verification using L68

□

Note 70: relative the basis

$u_0, u_1, u_2, u_3, \dots$

The matrices representing γ, γ^{-1} look as follows

$\gamma:$

	u_0	u_1	u_2	u_3	\dots
u_0	*				
u_1	*	*			
u_2	*	*	*		
u_3			*	*	
u_4			*	*	*
u_5				*	*
u_6					*
\vdots					\dots

$\gamma^{-1}:$

	u_0	u_1	u_2	u_3	\dots
u_0	*				
u_1	*	*			
u_2		*	*		
u_3		*	*	*	
u_4			*	*	*
u_5				*	*
u_6					*
\vdots					\dots

LEM 71

$$t_0 \cdot u_0 = (ab)^{1/2} u_0$$

$\forall n \geq 1$

u_{-n}, u_n
is a basis for a t_0 -inv subspace of $F[q, q^{-1}]$

Rel the basis

$$t_0: \frac{1}{q^n (ab)^{1/2}}$$

$$\begin{pmatrix} 1 & (1-q^n)(1-abq^n) \\ -1 & abq^n - 1 + q^n \end{pmatrix}$$

Aside

For $n \geq 1$ and rel the basis

u_{n-1}, u_n

to:

$$\frac{1}{q^n (ab)^{1/2}}$$

$$\begin{pmatrix} abq^n - 1 + q^n & -(1-q^n)(1-abq^n) \\ 1 & 1 \end{pmatrix}$$

p f We invoke Prop 61. By construction

$$u_{-n} = \underbrace{(1 - q^{2n} a y^{-1})(1 - q^{2n-2} a y^{-1}) \cdots (1 - q a y^{-1})(1 - a y^{-1})}_{S(y) \text{ symmetric in } y, y^{-1} \in \mathbb{F}[y, y^{-1}]}$$

$$\begin{aligned} u_n &= (1 - q^n a y) u_{-n} \\ &= S(y) (1 - q^n a y)(1 - a y^{-1}) \end{aligned}$$

By Prop 61

$$\begin{aligned} \text{to } u_n &= (ab)^{1/2} u_{-n} = S(y)(1 - a y^{-1}) \\ &+ \frac{(y-a)(y-b)}{(ab)^{1/2} y} \underbrace{\frac{S(y)(1 - a y^{-1}) - S(y)(1 - a y)}{y - y^{-1}}}_{a S(y)} \end{aligned}$$

$$= \underbrace{S(y)(1 - a y^{-1})}_{u_{-n}} \left((ab)^{1/2} + \frac{a(y-b)}{(ab)^{1/2}} \right)$$

$$= \frac{a}{(ab)^{1/2}} \left(q^{-n} a^{-n} u_{-n} - q^{-n} a^{-n} u_n \right)$$

Also

$$to. u_n = (ab)^{1/2} u_n +$$

$$\frac{(y-a)(y-b)}{(ab)^{1/2} y} S(y) \underbrace{\frac{(1-q^na)(1-aq^n) - (1-q^naq^n)(1-aq)}{y-y^2}}_{a(1-q^n)}$$

$$= (ab)^{1/2} S(y) (1-q^na)(1-aq^n)$$

$$+ \frac{(y-b)a(1-q^n)}{(ab)^{1/2}} \underbrace{S(y)(1-aq^n)}_{u_n}$$

$$= (ab)^{1/2} u_n + \frac{a(1-q^n)}{(ab)^{1/2}} \underbrace{y u_n}_{u_n} - \frac{ab(1-q^n)}{(ab)^{1/2}} u_n$$

$q^{-n} a^n (u_n - u_n)$

term	coef
u_{-n}	$-\frac{ab(1-q^n)}{(ab)^{1/2}} + \frac{1-q^n}{(ab)^{1/2} q^n}$
u_n	$(ab)^{1/2} - \frac{1-q^n}{(ab)^{1/2} q^n}$



Note 72: Relative the basis

$u_0, u_1, u_2, u_3, \dots$

The matrices rep to ± 1 look as follows

± 1
to :

	u_0	u_1	u_2	u_3	u_4	u_5
u_0	*					
u_1		*	*			
u_2			*	*		
u_3				*	*	
u_4					*	*
u_5						*
\vdots						\vdots

We now find the action of t_3 on the basis vectors

LEM 73

(i) For $n \geq 0$

$$t_3 \cdot u_n = (q^{\pm cd})^{1/2} u_n$$

(ii) For $n \geq 1$

$$t_3 \cdot u_{-n} =$$

term	coef
u_{-n}	$-\frac{(1-acq^{n+1})(1-adq^{n+1})}{(q^{\pm cd})^{1/2}}$
u_{-n}	$(q^{\pm d})^{1/2}$
u_n	$(q^{\pm cd})^{1/2}$

For $n \geq 0$

$$t_3^{-1} \cdot u_n = (qc^{-1}d^{-1})^{1/2} u_n$$

For $n \geq 1$

$$t_3^{-1} \cdot u_{-n} =$$

term	coef
u_{n-1}	$\frac{(1-acq^{n-1})(1-adq^{n-1})}{(q^{-1}cd)^{1/2}}$
u_n	$(q^{-1}cd)^{1/2}$
u_n	$-(q^{-1}cd)^{1/2}$

pf (i) Obs

11

$$u_n = (1 - q^n a y)(1 - q^{n-1} a y^{-1}) \cdots (1 - q a y)(1 - a y^{-1})$$

this inv under $y \rightarrow q^2 y^{-1}$, so contained in $\mathbb{F}[y + q^2 y^{-1}]$

Done by L69 (iii) and since

$$k_3 = (q^{-1} c d)^{1/2}$$

$$(ii) \quad u_{-n} = (1 - q^{n+1} a y^{-1})(1 - q^n a y) \cdots (1 - q a y)(1 - a y^{-1})$$

$u_{-n} \in \mathbb{F}[y + q^2 y^{-1}]$

$$u_n = (1 - q^n a y)(1 - q^{n-1} a y^{-1}) u_{n-1}$$

By Prop 65.

$$t_3 \cdot u_{-n} = (q^{-1} c d)^{1/2} u_{-n}$$

$$= \frac{(y - c^{-1})(y - d^{-1})}{(q c^{-1} d^{-1})^{1/2} y} u_{n-1} \underbrace{\frac{1 - q^{n+1} a y^{-1} - (1 - q^n a y)}{y - q^2 y^{-1}}}_{q^n a}$$

$$(y - c^T)(y - d^T)y^T = r \cdot 1 + s(1 - q^{n+1}ay^T) + t(1 - q^n ay)(1 - q^{n+1}ay^T)$$

find r, s, t

set $y = q^{n+1}a$:

$$r = \frac{(q^{n+1}a - c^T)(q^{n+1}a - d^T)q^{1-n}a^T}{acd q^{n+1}}$$

$$t = -q^{-n}a^T$$

$$s = q^{-n}a^T(1 - qc^T d^T)$$

$t_3 \cdot u_{-n} =$

term	coef
u_{n+1}	$\frac{-q^n a}{(qc^T d^T)^{1/2}} \cdot \frac{(1 - acq^{n+1})(1 - adq^{n+1})}{acd q^{n+1}}$
u_n	$(q^T cd)^{1/2} + \frac{-q^n a}{(qc^T d^T)^{1/2}} \cdot \frac{1 - qc^T d^T}{q^n a} \quad (= (qc^T d^T)^{1/2})$
u_n	$\frac{-q^n a}{(qc^T d^T)^{1/2}} \cdot \frac{-1}{q^n a}$

Simplify to get result.

Note 74 Relative the bases

$u_0, u_{-1}, u_1, u_{-2}, u_2, \dots$

the matrices rep $t_3^{\pm 1}$ look as follows

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3
u_0	*	*					
u_{-1}		*					
u_1		*	*	*			
u_{-2}				*			
u_2				*	*	*	
u_{-3}						*	
u_3						*	*
\vdots							\dots

Continue to discuss split basis

for the $\hat{H}_{q^{1/2}}$ module $\{u_n\}_{n \in \mathbb{Z}}$ $\mathbb{F}[q, q^{-1}]$

$$u_0 = 1$$

$$u_{-1} = (1 - aq^{-1})^{-1}$$

$$u_1 = (1 - qa^2)(1 - aq^{-1})^{-1}$$

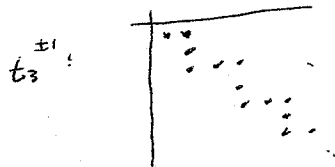
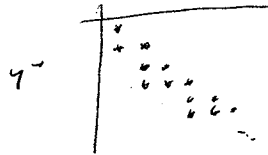
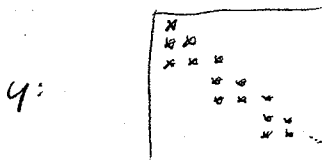
$$(1 - q^{-1}aq^{-1})u_n = u_{n+1}$$

$$n = 0, 1, 2, \dots$$

$$(1 - q^2aq)u_{-n} = u_n$$

$$n = 1, 2, \dots$$

So far: rel $u_0, u_{-1}, u_1, u_{-2}, u_2, \dots$



We now give the action of $X^{\pm 1}$ ($X = t_3 t_0$) on the basis $\{u_n\}_{n \in \mathbb{Z}}$

LEM 75

$$X_+ u_0 = (q^{-1}abcd)^{1/2} u_0$$

For $n \geq 1$

$$X_+ u_n = \frac{1}{(q^{-1}abcd)^{1/2}} \text{ times}$$

term	coef
u_{n+1}	$-q^{-n}(1-q^n)(1-abq^n)(1-acq^{n+1})(1-adq^{n+1})$
u_{-n}	$q^{-n}(1-q^n)(1-abq^n)$
u_n	$abcdq^{n+1}$

$$X, u_{-n} = \frac{1}{(q^{-n}abcd)^{1/2}} \text{ terms}$$

u_{n-1}	$-q^{-n}(1-acq^{2n})(1-adq^{2n})$
u_{-n}	q^{-n}

$$X^{-1}, u_0 = \frac{1}{(q^{-1}abcd)^{1/2}} u_0$$

$$X^{-1}, u_{-1} = \frac{1}{(q^{-1}abcd)^{1/2}} \left((1-ac)(1-ad)u_0 + abcd u_{-1} \right)$$