Lecture 38 Friday Dec 2

$\mathbb{F}$ alg closed $0 \neq q \in \mathbb{F}$ $q^4 \neq 1$

Given any AW polys $\{p_n\}_{n=0}^{\infty}$

$$p_n = p_n(x; a, b, c, d \mid q)$$

In th 54 we displayed a module structure on $\mathbb{F}[x]$ for $\Delta = \Delta_{q^{1/2}}$ such that the gen A of Δ acts on $\mathbb{F}[x]$ as mult by x

We now obtain a similar result using $\hat{H}_{q^{1/2}}$.

Let y denote an indet. Consider $\mathbb{F}[y, y^{-1}]$ the $\mathbb{F}$-alg of Laurent polys in y

Def

$$x = y + y^{-1}$$

Subalg of $\mathbb{F}[y, y^{-1}]$ gen by x has basis

$$1,\ y+y^{-1},\ y^2+y^{-2},\ y^3+y^{-3},\ \ldots$$

This subalg is iso $\mathbb{F}[x]$

Identify $\mathbb{F}[x]$ with subalg of $\mathbb{F}[y, y^{-1}]$ gen by x

Consider generator $A = Y + Y^{-1}$ for $\hat{H}_{q^{1/2}}$ $(Y = t_0 t_1)$

We display an $\hat{H}_{q^{1/2}}$-module str on $\mathbb{F}[y, y^{-1}]$ such that Y acts as mult by y. This $\hat{H}_{q^{1/2}}$-module 452

will be iso to the one from th 40 (with q replaced by $q^{1/2}$).

Fix square roots as in th 54:

$$q^{1/2}, \quad a^{1/2} \quad b^{1/2} \quad c^{1/2} \quad d^{1/2}$$

Define

$$k_0 = (ab)^{1/2}$$

$$k_1 = (ab^{-1})^{1/2}$$

$$k_2 = (cd^{-1})^{1/2}$$

$$k_3 = (q^{-1}cd)^{1/2}$$

so that

$$a = k_0 k_1$$

$$b = k_0 k_1^{-1}$$

$$c = q^{1/2} k_2 k_3$$

$$d = q^{1/2} k_2^{-1} k_3$$

[this matches th 42 with q replaced by $q^{1/2}$]

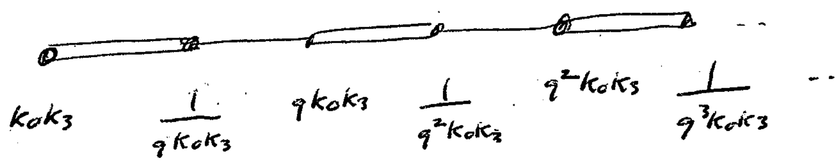
Recall $\hat{H}_{q^{1/2}}$-module

$$V = V(k_0, k_1, k_2, k_3)$$

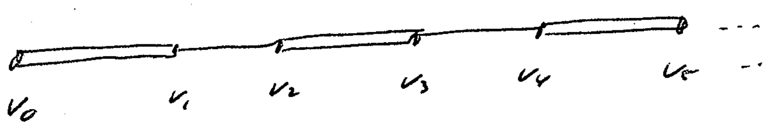
from th 40. Diagram is

∞ path

Recall for above diagram the nodes are eigenspaces of $X = t_3 t_0$ on V. eigvals are



Eigenbasis for X:



$G_2 v_i = v_{i+1}$ i even $i = 0, 1, 2, \ldots$

$G_0 v_i = v_{i+1}$ i odd

Recall by constr prior to th 40,

$$t_0 v_0 = k_0 v_0,$$

$$t_3 v_0 = k_3 v_0.$$

Going to show the following is a basis for V:

$$Y^n v_0 \qquad n \in \mathbb{Z}$$

Recall

$$V = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V + \frac{t_0 - k_0}{k_0^{-1} - k_0} V \qquad \text{ds}$$

↑ eigspace for t_0 equal k_0 (first summand)

↑ eigspace for t_0 equal k_0^{-1} (second summand)

Recall

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V \quad \text{has basis}$$

$$\frac{t_0 - k_0^{-1}}{k_0^{-1} - k_0} v_{2i} \qquad i = 0, 1, 2, \ldots$$

and

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} V \quad \text{has basis}$$

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_{2i} \qquad i = 1, 2, \ldots$$

[Note $t_0 v_0 = k_0 v_0$ so

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_0 = v_0$$]

By Thm 42

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i} \in \mathbb{F}\, p_i(A;\, a, b, c, d \,/\, q)\, v_0$$

$i = 0, 1, 2, \ldots$

$A = Y + Y^{-1} \qquad Y = t_0 t_1$

Recall p_i has degree i for $i = 0, 1, 2, \ldots$ so

the following is a basis for $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V$:

$$(Y + Y^{-1})^n v_0 \qquad n = 0, 1, 2, \ldots \qquad (*)$$

By Thm 44

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_{2i} \in \mathbb{F}\, p_{i-1}(A;\, qa, qb, c, d \,/\, q)\, \frac{t_0 - k_0}{k_0^{-1} - k_0} v_2$$

$i = 1, 2, 3, \ldots$

So the following is a basis for $\frac{t_0 - k_0}{k_0^{-1} - k_0} V$:

$$(Y + Y^{-1})^n \frac{t_0 - k_0}{k_0^{-1} - k_0} v_2 \qquad n = 0, 1, 2, \ldots \qquad (**)$$

LEM 55 With above notation

the following agree up to a nonzero scalar factor:

(i) $$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_2$$

(ii) $$(Y + abY^{-1} - a - b)\, v_0$$

pf Recall $Y = t_0 t_1$

By the constr of the $\hat{H}_{q^{1/2}}$-module V in Th 40

we are given the action of t_0, t_1 on the basis $\{v_i\}_{i=0}^{\infty}$

Recall

t_1:

	v_0 v_1	v_2 v_3	
v_0 v_1	$\Upsilon_1(q^{-1/2}k_0^{-1}k_3^{-1})$	0	...
v_2 v_3	0	$\Upsilon_1(q^{-3/2}k_0^{-1}k_3^{-1})$	
		⋮	

$$T_1(\theta) = \begin{pmatrix} \alpha(\theta) & \beta(\theta) \\ \gamma(\theta) & \delta(\theta) \end{pmatrix}$$

$$\alpha(\theta) = \frac{\theta(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{\theta - \theta^{-1}}$$

$$\gamma(\theta) = \frac{1}{\theta^{-1} - \theta}$$

t_0 :

	v_0	v_1 … v_2	
v_0	k_0	0	0
v_1 … v_2	0	$r_0(q^{-1}k_0^{-1}k_3^{-1})$	0
	0	0	

$$r_0(\theta) = \begin{pmatrix} \theta d(\theta) & \frac{-b(\theta)}{\theta} \\ -\theta c(\theta) & \frac{a(\theta)}{\theta} \end{pmatrix}$$

$$d(\theta) = \frac{\theta^{-1}(k_3 + k_3^{-1}) - k_0 - k_0^{-1}}{\theta^{-1} - \theta} \qquad b(\theta) = \frac{G(\theta, k_0, k_3)}{\theta - \theta^{-1}}$$

$$c(\theta) = \frac{1}{\theta^{-1} - \theta} \qquad a(\theta) = \frac{\theta(k_3 + k_3^{-1}) - k_0 - k_0^{-1}}{\theta - \theta^{-1}}$$

So

$$Y v_0 = t_0 t_1 v_0$$

$$= t_0 \left(\alpha(q^{-1/2} k_0^{-1} k_3^{-1}) v_0 + \gamma(q^{-1/2} k_0^{-1} k_3^{-1}) v_1 \right)$$

$$= k_0 \alpha(q^{-1/2} k_0^{-1} k_3^{-1}) v_0 + \gamma(q^{-1/2} k_0^{-1} k_3^{-1}) t_0 v_1$$

$$= k_0 \alpha(q^{-1/2} k_0^{-1} k_3^{-1}) v_0$$

$$+ \gamma(q^{-1/2} k_0^{-1} k_3^{-1}) q^{-1} k_0^{-1} k_3^{-1} d(q^{-1} k_0^{-1} k_3^{-1}) v_1$$

$$- \gamma(q^{-1/2} k_0^{-1} k_3^{-1}) q^{-1} k_0^{-1} k_3^{-1} c(q^{-1} k_0^{-1} k_3^{-1}) v_2$$

$$
\begin{aligned}
Y^{-1} k_0 &= t_1^{-1} t_0^{-1} v_0 \\
&= k_0^{-1} t_1^{-1} v_0 \\
&= k_0^{-1} (T_1 - t_1) v_0 \\
&= k_0^{-1} (k_1 + k_1^{-1} - t_1) v_0 \\
&= k_0^{-1} \left(k_1 + k_1^{-1} - \alpha (q^{-1/2} k_0^{-1} k_3^{-1}) \right) v_0 . \\
&\quad - k_0^{-1} \gamma (q^{-1/2} k_0^{-1} k_3^{-1}) v_1
\end{aligned}
$$

Also

$$
\frac{t_0 - k_0}{k_0^{-1} - k_0} v_2 = \frac{-q k_0 k_3 \, b(q^{-1} k_0^{-1} k_3^{-1}) v_1 + (q k_0 k_3 \, a(q^{-1} k_0^{-1} k_3^{-1}) - k_0) v_2}{k_0^{-1} - k_0}
$$

Evaluating $Y v_0$, $Y^{-1} v_0$, $\frac{t_0 - k_0}{k_0^{-1} - k_0} v_0$ by these er... we get the result. □

Now

Find a vector $\psi \in \mathrm{Span}(Y, Y^{-1}, 1)$ such that

$$t_0 \psi v_0 = k_0^{-1} \psi v_0$$

Write

$$\psi = Y + rY^{-1} + s$$

Find r, s

Require

$$0 = (t_0 - k_0^{-1}) \psi v_0$$

$$= (t_0 - k_0^{-1}) Y v_0 + r(t_0 - k_0^{-1}) Y^{-1} v_0 + s(k_0 - k_0^{-1}) v_0 \qquad *$$

Recall

$$t_0 Y = Y^{-1} t_0 + Y T_0 - T_1$$

$$t_0 Y^{-1} = Y t_0 - Y T_0 + T_1$$

So

$$(t_0 - k_0^{-1}) Y v_0 = k_0 Y^{-1} v_0 + k_0 Y v_0 - (k_1 + k_1^{-1}) v_0$$

$$(t_0 - k_0^{-1}) Y^{-1} v_0 = -k_0^{-1} Y v_0 - k_0^{-1} Y^{-1} v_0 + (k_1 + k_1^{-1}) v_0$$

$*$ becomes

0 =

term	coef
$Y v_0$	$k_0 - r k_0^{-1}$
$Y^{-1} v_0$	$k_0 - r k_0^{-1}$
v_0	$-k_1 - k_1^{-1} + r(k_1 + k_1^{-1}) + s(k_0 - k_0^{-1})$

Require each coef is 0

$$r = k_0^2 = ab$$

$$0 = (k_1 + k_1^{-1})(r-1) \quad + s(k_0 - k_0^{-1})$$

$$= (k_0^2 - 1)\, k_0^{-1}$$

$$= (r-1)\, k_0^{-1}$$

$$0 = k_1 + k_1^{-1} + k_0^{-1} s$$

$$s = -k_0(k_1 + k_1^{-1})$$

$$= -k_0 k_1^{-1}(k_1^2 + 1)$$

$$= -b(a/b + 1)$$

$$= -a - b$$

$$\psi = Y + rY^{-1} + s$$

$$= Y + ab\, Y^{-1} - a - b$$

Cor 56 With the above notation the following is a basis for V:

$$Y^n v_0 \qquad n \in \mathbb{Z}$$

pf By *, **, Lem 55 and since $ab = k_0^2 \neq 1$ □

Cor 57 With above notation ∃ iso of $\mathbb{F}$-vector spaces

$$\natural : \begin{array}{ccc} \mathbb{F}[y, y^{-1}] & \longrightarrow & V \\ y^n & \longrightarrow & Y^n v_0 \end{array} \qquad n \in \mathbb{Z}$$

□

pf By Cor 56

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Ref to Cor 57,

Via $\natural$ we pull back the $\hat{H}_{q^{1/2}}$ - module str from V to $\mathbb{F}[y, y^{-1}]$.

Now $\mathbb{F}[y, y^{-1}]$ becomes a $\hat{H}_{q^{1/2}}$-module.

LEM 58 For the above $\hat{H}_{q^{1/2}}$ module $\mathbb{F}[y, y^{-1}]$

$$t_0 . 1 = k_0 1$$

$$t_3 . 1 = k_3 1$$

pf φ sends $1 \to v_0$

and

$$t_0 . v_0 = k_0 v_0$$

$$t_3 . v_0 = k_3 v_0$$

$\square$

LEM 59 For the above $\hat{H}_{q^{1/2}}$ -module $\mathbb{F}[y, y^{-1}]$

$$Y . f = y f \qquad \forall f \in \mathbb{F}[y, y^{-1}]$$

pf WLOG $f = y^n$ $n \in \mathbb{Z}$

$$\underset{\text{in } \mathbb{F}[y,y^{-1}]}{Y . y^n} = \underset{\text{in } V}{Y . Y^n v_0}$$

$$= \underset{\text{in } V}{Y^{n+1} v_0}$$

$$= \underset{\text{in } \mathbb{F}[y,y^{-1}]}{y^{n+1}}$$

$\square$

For the above $\hat{H}_{q^{1/2}}$-module $\mathbb{F}[y, y^{-1}]$ we now

describe the action of each t_i

By L5.9 and
Since $$t_0 t_1 = Y,$$

$$t_2 t_3 = q^{-1/2} Y^{-1}$$

suf to describe the actions of

$$t_0, \; t_3$$

Lecture 39 Mon Dec 5

$\mathbb{F}$ alg closed $0 \neq q \in \mathbb{F}$ $q^4 \neq 1$

Recall our situation:

Given AW polys $\{p_n\}_{n=0}^{\infty}$ $p_n = p_n(x; a,b,c,d \mid q)$

$\{p_n\}_{n=0}^{\infty}$ is basis for $\mathbb{F}[x]$

View $x = y + y^{-1}$ y indet

Identify

$$\mathbb{F}[x] = \mathbb{F}[y+y^{-1}]$$
$$= \text{Span}\{(y+y^{-1})^n \mid n = 0,1,2,..\}$$
$$= \text{Span}\{1, y+y^{-1}, y^2+y^{-2}, \dots\}$$

In Cor 57 we constructed an $\hat{H}_{q^{1/2}}$-module structure on $\mathbb{F}[y,y^{-1}]$ s.t.

- $t_0 . 1 = k_0 1$
- $t_3 . 1 = k_3 1$
- $Y = t_0 t_1$ acts on $\mathbb{F}[y,y^{-1}]$ as mult by y

Define

$$\psi \in \mathbb{F}[y, y^{-1}]$$

by

$$\psi = y + aby^{-1} - a - b$$

$$= (y-a)(y-b)\, y^{-1}$$

By LSS

$$t_0 \cdot \psi = k_0^{-1} \psi$$

LEM 60 For the above $\hat{H}_{q^{1/2}}$-module $\mathbb{F}[y, y^{-1}]$,

(i) $t_0 . fg = f\, t_0 . g \qquad \forall f \in \mathbb{F}[y+y^{-1}]$, $\forall g \in \mathbb{F}[y, y^{-1}]$

(ii) $\mathbb{F}[y, y^{-1}] = \mathbb{F}[y+y^{-1}] + \mathbb{F}[y+y^{-1}]\,\psi$ (ds sus)

(iii) $\mathbb{F}[y+y^{-1}]$ is the eigenspace for t_0 with eigval k_0

(iv) $\mathbb{F}[y+y^{-1}]\,\psi$ is the eigenspace for t_0 with eigval k_0^{-1}

pf (i) In $\hat{H}_{q^{1/2}}$
$Y+Y^{-1}$ commutes with t_0 $\quad (Y = t_0 t_1)$.
Also Y acts on $\mathbb{F}[y, y^{-1}]$ as mult by y.

(ii) $ab \neq 1$, so the following are bases for the same space:

$1, y, y^{-1}$ ~~$\otimes$~~

$1, y+y^{-1}, \psi$ ~~$\otimes\otimes$~~

(iii), (iv) $\forall f(y) \in \mathbb{F}[y, y^{-1}]$

$$t_0 \circ f(y) = t_0 \circ f(y) 1 = f(y) \underbrace{t_0 \circ 1}_{= k_0 1} = k_0 f(y)$$

$$t_0 \circ f(y) \varphi = f(y) \underbrace{t_0 \circ \varphi}_{= k_0^{-1} \varphi} = k_0^{-1} f(y) \varphi$$

Result follows in view of (ii) □

Prop 61 For the above $\hat{H}_{q^{1/2}}$ module

$\mathbb{F}[y, y^{-1}]$

t_0 acts as follows:

$\forall f \in \mathbb{F}[y, y^{-1}]$,

$$t_0 \cdot f(y) = (ab)^{1/2} f(y) + \frac{(y-a)(y-b)}{(ab)^{1/2}\, y} \frac{f(y) - f(y^{-1})}{y - y^{-1}}$$

Note $\frac{f(y)-f(y^{-1})}{y-y^{-1}} \in \mathbb{F}[y, y^{-1}]$ since

$\mathbb{F}[y, y^{-1}]$ has basis $\{y^n\}_{n \in \mathbb{Z}}$ and

$$\frac{y^n - y^{-n}}{y - y^{-1}} = y^{n-1} + y^{n-3} + \cdots + y^{3-n} + y^{1-n} \qquad \forall n \in \mathbb{Z}$$

pf WLOG $f(y) = y^n$ $n \in \mathbb{Z}$

show

$$t_0 \cdot y^n = (ab)^{1/2} y^n + \frac{(y-a)(y-b)}{(ab)^{1/2} y} \frac{y^n - y^{-n}}{y - y^{-1}} \quad *$$

By L60 (ii) $\exists$ $h, k \in \mathbb{F}[y+y^{-1}]$ s.t

$$y^n = h(y) + k(y)\,\psi(y)$$

By L60 (iii), (iv)

$$t_0 \cdot y^n = k_0 h(y) + k_0^{-1} k(y)\,\psi(y) \quad **$$

Find h, k:
the move $y \to y^{-1}$ leaves h, k invariant, so

$$y^{-n} = h(y) + k(y)\,\psi(y^{-1})$$

So

$$y^n - y^{-n} = k(y)\left(\psi(y) - \psi(y^{-1})\right)$$

obs

$$\psi(y) - \psi(y^{-1}) = (y - y^{-1})(1 - ab)$$

so

$$k(y) = \frac{1}{1-ab} \frac{y^n - y^{-n}}{y - y^{-1}}$$

Obs

$$h(y) = y^n - k(y)\,\Psi(y)$$

So using **

$$t_0 \cdot y^n = k_0\left(y^n - k(y)\,\Psi(y)\right) + k_0^{-1} k(y)\,\Psi(y)$$

$$= k_0 y^n + \underbrace{(k_0^{-1} - k_0)}_{k_0^{-1}(1-k_0^2) = k_0^{-1}(1-ab)} k(y)\,\Psi(y)$$

$$= k_0 y^n + \frac{\Psi(y)}{k_0}\,\frac{y^n - y^{-n}}{y - y^{-1}} \qquad k_0 = (ab)^{1/2}$$

$$= (ab)^{1/2} y^n + \frac{(y-a)(y-b)}{(ab)^{1/2}\,y}\,\frac{y^n - y^{-n}}{y - y^{-1}} \quad ✓$$

We have shown *. □

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Recall the projections

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} \qquad \frac{t_0 - k_0}{k_0^{-1} - k_0}$$

We now show how these act on $\mathbb{F}[y, y^{-1}]$

LEM 62 For the above $\hat{H}_{q,k}$-module $\mathbb{F}[y, y^{-1}]$,

$\forall f \in \mathbb{F}[y, y^{-1}]$

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} f(y) = \frac{\varphi(y)}{1 - ab} \frac{f(y) - f(y^{-1})}{y - y^{-1}}$$

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} f(y) = \frac{1}{1 - ab} \frac{\varphi(y) f(y^{-1}) - \varphi(y^{-1}) f(y)}{y - y^{-1}}$$

where we recall $\varphi(y) = (y-a)(y-b)y^{-1}$ □

Pf Use Prop 61.

We now find the action of t_3 on the $\hat{H}_{q^{1/2}}$ module $\mathbb{F}[y, y^{-1}]$.

Recall $t_3 . 1 = k_3$

We will need an element

$$\phi \in \mathrm{Span}\{ y, y^{-1}, 1 \}$$

such that

$$t_3 . \phi = k_3^{-1} \phi$$

Write

$$\phi = y + r y^{-1} + s$$

find r, s

Require

$$0 = (t_3 - k_3^{-1}) \phi$$

$$= (t_3 - k_3^{-1}).y + r(t_3 - k_3^{-1}).y^{-1} + s(k_3 - k_3^{-1}) \quad *$$

Recall in $\hat{H}_{q^{1/2}}$

$$t_3 Y = q^{-1} Y^{-1} t_3 - q^{-1} Y^{-1} T_3 + q^{-1/2} T_2$$

$$t_3 Y^{-1} = q Y t_3 + Y^{-1} T_3 - q^{1/2} T_2$$

So

$(t_3 - k_3^{-1}).y =$

term	coef
y	$-k_3^{-1}$
y^{-1}	~~$q^{-1} k_3$~~ ~~$- q^{-1} k_3$~~ $- q^{-1} k_3^{-1}$
1	$q^{-1/2}(k_2 + k_2^{-1})$

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$(t_3 - k_3^{-1}) \cdot y^{-1} =$

term	coef
y	$q k_3$
y^{-1}	$-k_3^{-1} + k_3 + k_3^{-1}$
1	$-q^{1/2}(k_2 + k_2^{-1})$

* becomes

0 =

term	coef
y	$-k_3^{-1} + r q k_3$
y^{-1}	$-q^{-1} k_3^{-1} + r k_3$
1	$q^{-1/2}(k_2 + k_2^{-1}) - r q^{1/2}(k_2 + k_2^{-1}) + s(k_3 - k_3^{-1})$

Require each coef is 0

$$r = q^{-1} k_3^{-2}$$
$$= c^{-1} d^{-1}$$

$$s(k_3 - k_3^{-1}) = (k_2 + k_2^{-1})(q^{1/2} r - q^{-1/2})$$

where

$$q^{1/2} r - q^{-1/2} = (q r - 1)\, q^{-1/2}$$
$$= (k_3^{-2} - 1)\, q^{-1/2}$$
$$= -(k_3 - k_3^{-1})\, k_3^{-1} q^{-1/2}$$

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$$s = -\frac{(k_2 + k_2^{-1})}{k_3} q^{-1/2} = -\frac{(k_2^2 + 1)}{k_2 k_3} q^{-1/2} = -c^{-1} - d^{-1}$$

Def 63 For the above $\hat{H}_{q^{1/2}}$-module $\mathbb{F}[y, y^{-1}]$

define

$$\phi = y + c^{-1}d^{-1}y^{-1} - c^{-1} - d^{-1}$$

$$= (y - c^{-1})(y - d^{-1})y^{-1}$$

By construction

$$t_3 \cdot \phi = k_3^{-1}\phi$$

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Back in $\hat{H}_{q^{1/2}}$ recall

t_3 commutes with $t_2 t_3 + (t_2 t_3)^{-1}$

$$t_2 t_3 = q^{-1/2} y^{-1}, \qquad (t_2 t_3)^{-1} = q^{1/2} y$$

$$t_2 t_3 + (t_2 t_3)^{-1} = q^{1/2}(y + q^{-1}y^{-1})$$

In our study of the t_0 action on $\mathbb{F}[y, y^{-1}]$ $y + y^{-1}$ played a key role.

In our study of the t_3 action on $\mathbb{F}[y, y^{-1}]$ $y + q^{-1}y^{-1}$ will play a similar role.

LEM 64 For the $\hat{H}_{q^{1/2}}$-module $\mathbb{F}[y, y^{-1}]$

(i) $t_3 . fg = f\, t_3 . g \quad \forall f \in \mathbb{F}[y + q^{-1}y^{-1}],$
$\forall g \in \mathbb{F}[y, y^{-1}]$

(ii) $\mathbb{F}[y, y^{-1}] = \mathbb{F}[y + q^{-1}y^{-1}] + \mathbb{F}[y + q^{-1}y^{-1}]\, \phi$
(ds sus)

(iii) $\mathbb{F}[y + q^{-1}y^{-1}]$ is the eigenspace for t_3 with eigval k_3

(iv) $\mathbb{F}[y + q^{-1}y^{-1}]\, \phi$ is the eigenspace for t_3 with eigval k_3^{-1}

pf (i) We saw earlier
t_3 commutes with $y + q^{-1}y^{-1}$

(ii) $\phi = y + c^{-1}d^{-1}y^{-1} - c^{-1} - d^{-1}$
$c^{-1}d^{-1} \neq q^{-1}$
So each of the following $*$, $**$ is a basis for the same space:

$1, \; y, \; y^{-1}$ $\qquad *$

$1, \; y + q^{-1}y^{-1}, \; \phi$ $\qquad **$

(iii), (iv) By (i), (ii) and since
$t_3 . 1 = k_3 1 \qquad t_3 . \phi = k_3^{-1} \phi$

Prop 65 For the above $\hat{H}_{q^{1/2}}$ module $\mathbb{F}[y, y^{-1}]$, 13

t_3 acts as follows.

$$t_3 . f(y) = (q^{-1} c d)^{1/2} f(y)$$

$$- \frac{(y - c^{-1})(y - d^{-1})}{(q c^{-1} d^{-1})^{1/2} y} \frac{f(y) - f(q^{-1} y^{-1})}{y - q^{-1} y^{-1}}$$

pf WLOG $f(y) = y^n$

show

$$t_3 . y^n = (q^{-1} c d)^{1/2} y^n - \frac{(y-c^{-1})(y-d^{-1})}{(q c^{-1} d^{-1})^{1/2} y} \frac{y^n - (q^{-1}y^{-1})^n}{y - q^{-1}y^{-1}} \quad *$$

By L64(ii) $\exists$ $h(y), k(y) \in \mathbb{F}[y + q^{-1}y^{-1}]$ s.t.

$$y^n = h(y) + k(y)\, \phi(y)$$

So by L64(iii), (iv)

$$t_3 . y^n = k_3 h(y) + k_3^{-1} k(y)\, \phi(y) \quad **$$

Find h, k

Move $y \rightarrow q^{-1}y^{-1}$ leaves $y + q^{-1}y^{-1}$ inv

" $h(y), k(y)$ inv

so $(q^{-1}y^{-1})^n = h(y) + k(y)\, \phi(q^{-1}y^{-1})$

so

$$\frac{y^n - (q^{-1}y^{-1})^n}{y - q^{-1}y^{-1}} = k(y) \frac{\phi(y) - \phi(q^{-1}y^{-1})}{y - q^{-1}y^{-1}}$$

$$= k(y) (1 - c^{-1}d^{-1}q)$$

$$k(y) = \frac{1}{1 - q c^{-1} d^{-1}} \frac{y^n - (q^{-1}y^{-1})^n}{y - q^{-1}y^{-1}}$$

Obs

$$h(k) = y^n - k(y)\ \phi(y)$$

So using **

$$t_3 \cdot y^n = k_3\left(y^n - k(y)\ \phi(y) \right) + k_3^{-1}\ k(y)\ \phi(y)$$

$$= k_3 y^n + \underbrace{\left(k_3^{-1} - k_3\right)}_{k_3^{-1}(1-k_3^2) = k_3^{-1}(1-q^{-1}cd)} k(y)\ \phi(y)$$

$$= (q^{-1}cd)^{1/2} y^n +$$

$$- \frac{\phi(y)}{(qc^{-1}d^{-1})^{1/2}} \frac{y^n - (q^{-1}y^{-1})^n}{y - q^{-1}y^{-1}}$$

we have shown * □

LEM 66 For the above $\hat{H}_{q^{1/2}}$ module $\mathbb{F}[y, y^{-1}]$

$\forall f \in \mathbb{F}[y, y^{-1}]$

$$\frac{t_3 - k_3}{k_3^{-1} - k_3} f(y) = \frac{\phi(y)}{1 - q c^{-1} d^{-1}} \; \frac{f(y) - f(q^{-1} y^{-1})}{y - q^{-1} y^{-1}}$$

$$\frac{t_3 - k_3^{-1}}{k_3 - k_3^{-1}} f(y) = \frac{1}{1 - q c^{-1} d^{-1}} \; \frac{\phi(y) f(q^{-1} y^{-1}) - \phi(q^{-1} y^{-1}) f(y)}{y - q^{-1} y^{-1}}$$

pf Use Prop 65 □

Another view of $\mathbb{F}[q, q^{-1}]$

We write everything in terms of $\{k_i\}_{i \in \mathbb{I}}$ instead of a, b, c, d. Work with $\hat{H}_q$ instead of $\hat{H}_{q^{1/2}}$.

Until further notice let

$$q, \; K_0, K_1, K_2, K_3$$

denote mutually com indets

Consider $\mathbb{F}$-alg

$$V = \mathbb{F}[q^{\pm 1}, K_0^{\pm 1}, K_1^{\pm 1}, K_2^{\pm 1}, K_3^{\pm 1}]$$

View an elemt f in V as a Laurent poly in q with coeffs in $\mathbb{F}[K_0^{\pm 1}, K_1^{\pm 1}, K_2^{\pm 1}, K_3^{\pm 1}]$

thm 67 the above $\mathbb{F}$-vector space V has a $\hat{H}_q$-module str such that $\forall f \in V$,

$$t_0 . f(y) = k_0 f(y) + \frac{(y-k_0k_1)(y-k_0k_1^{-1})}{k_0 y} \frac{f(y)-f(y^{-1})}{y-y^{-1}},$$

$$t_3 . f(y) = k_3 f(y) + \frac{(y-q^{-1}k_2^{-1}k_3^{-1})(y-q^{-1}k_2k_3^{-1})k_3}{y} \frac{f(y)-f(q^{-2}y^{-1})}{y-q^{-2}y^{-1}},$$

$$Y . f(y) = y f(y).$$

The actions of t_1, t_2 are obtained using

$$t_0t_1 = Y, \qquad t_2t_3 = q^{-1}Y^{-1}.$$

Moreover the action of $t_i + t_i^{-1}$ on V is

$$(t_i + t_i^{-1}) f = (k_i + k_i^{-1}) f \qquad (i \in \mathbb{I})$$

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pf This is reformulation of Props 61, 65 $\square$

$\mathbb{F}$ alg closed $0 \neq q \in \mathbb{F}$

Given AW polys $\{p_n\}_{n=0}^{\infty}$ $p_n = p_n(x; a, b, c, d \mid q)$

Recall the $\hat{H}_{q^{1/2}}$-module $\mathbb{F}[y, y^{-1}]$ from Props 61, 65 $(x = y + y^{-1})$

Recall:

$$\mathbb{F}[y, y^{-1}] = \mathbb{F}[y+y^{-1}]\,1 + \mathbb{F}[y+y^{-1}]\,\psi \quad (\text{ds vs})$$

$\psi = y + ab y^{-1} - a - b$

$\mathbb{F}[y+y^{-1}]\,1$ ↑ to eigspace eigval k_0; $\mathbb{F}[y+y^{-1}]\,\psi$ ↑ to eigspace eigval k_0^{-1}

$\mathbb{F}[y+y^{-1}]$ has basis the AW polys

$$p_n(y+y^{-1}; a, b, c, d \mid q) \qquad n = 0, 1, 2 \ldots$$

↑

eigenvector for $X + X^{-1}$ $(X = t_3 t_0)$ with eigval

$$k_0 k_3 q^n + \frac{1}{k_0 k_3 q^n}$$

$\mathbb{F}[y+y^{-1}]\,\psi$ has basis the AW polys

$$p_n(y+y^{-1}; qa, qb, c, d \mid q)\,\psi \qquad n = 0, 1, 2 \ldots$$

↑

eigvector for $X + X^{-1}$ with eigval

$$k_0 k_3 q^{n+1} + \frac{1}{k_0 k_3 q^{n+1}}$$

Recall

$\theta_n = aq^n + a^{-1}q^{-n}$ $n = 0, 1, 2 \ldots$

$\mathbb{F}[y+y^{-1}]1$ has basis

$1, \quad y+y^{-1}-\theta_0, \quad (y+y^{-1}-\theta_1)(y+y^{-1}-\theta_0), \ldots$

rel this basis

$y+y^{-1}$ is LBD (by TH 51)

$x+x^{-1}$ is UBD

$\mathbb{F}[y+y^{-1}]\Psi$ has basis

$1\Psi, \quad (y+y^{-1}-\theta_1)\Psi, \quad (y+y^{-1}-\theta_2)(y+y^{-1}-\theta_1)\Psi, \ldots$

rel this basis (by TH 51)

$y+y^{-1}$ is LBD

$x+x^{-1}$ is UBD

Next goal: display a basis for $\mathbb{F}[y,y^{-1}]$
with respect to which

y is lower triangular

x is upper triangular.

LEM 68 The following is a basis for $\mathbb{F}[y, y^{-1}]$

Name	vector
u_0	1
u_{-1}	$1 - ay^{-1}$
u_1	$(1 - qay)(1 - ay^{-1})$
u_{-2}	$(1 - qay^{-1})(1 - qay)(1 - ay^{-1})$
u_2	$(1 - q^2ay)(1 - qay^{-1})(1 - qay)(1 - ay^{-1})$
$\vdots$	$\vdots$

Moreover

$$(1 - q^n a y^{-1})\, u_n = u_{-n-1} \qquad n = 0, 1, 2 \ldots$$

$$(1 - q^n a y)\, u_{-n} = u_n \qquad n = 1, 2, 3, \ldots$$

pf Routine $\square$

LEM 69 y, y^{-1} act on basis $\{u_n\}_{n \in \mathbb{Z}}$ as follows.

(i) For $n \geq 0$

$$y\, u_n = q^n a\, u_n + q^{-n-1} a^{-1} u_{-n-1} - q^{-n-1} a^{-1} u_{n+1}$$

$$y^{-1} u_n = q^{-n} a^{-1} u_n - q^{-n} a^{-1} u_{-n-1}$$

(ii) For $n \geq 1$

$$y\, u_{-n} = q^{-n} a^{-1} u_{-n} - q^{-n} a^{-1} u_n$$

$$y^{-1} u_{-n} = q^n a\, u_{-n} + q^{-n} a^{-1} u_n - q^{-n} a^{-1} u_{-1-n}$$

pf routine verification using L68 □

Note 70: Relative the basis

$u_0, u_{-1}, u_1, u_{-2}, u_2, \dots$

the matrices representing Y, Y^{-1} look as follows

Y:

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3	...
u_0	*							
u_{-1}	*	*				O		
u_1	*	*	*					
u_{-2}			*	*				
u_2			*	*	*			
u_{-3}		O			*	*		
u_3					*	*	*	
⋮							...	...

Y^{-1}:

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3	...
u_0	*							
u_{-1}	*	*				O		
u_1		*	*					
u_2		*	*	*				
u_2				*	*			
u_{-3}		O		*	*	*		
u_3						*	*	
⋮							...	...

LEM 71

$$t_0 . u_0 = (ab)^{1/2} u_0$$

Fa $n \geq 1$

$$u_{-n}, \quad u_n$$

is a basis fn a t_0-inv subspace of $\mathbb{F}[y, y^{-1}]$

Rel this basis

$$t_0 : \frac{1}{q^n (ab)^{1/2}} \begin{pmatrix} 1 & (1-q^n)(1-abq^n) \\ -1 & abq^n - 1 + q^n \end{pmatrix}$$

Aside

For $n \geq 1$ and rel the basis

$u_{-n}, \quad u_n$

$$t_0^{-1}: \quad \frac{1}{q^n(ab)^{1/2}} \begin{pmatrix} abq^n - 1 + q^n & -(1-q^n)(1-abq^n) \\ 1 & 1 \end{pmatrix}$$

pf We invoke Prop 61. By construction

$$u_{-n} = \underbrace{(1-q^{n-1}ay^{-1})(1-q^{n-1}ay) \cdots (1-qay^{-1})(1-qay)}_{S(y) \text{ sym in } y, y^{-1} \in \mathbb{F}[y+y^{-1}]}(1-ay^{-1})$$

$$u_n = (1-q^n ay)\, u_{-n}$$

$$= S(y)\,(1-q^n ay)(1-ay^{-1})$$

By Prop 61

$$t_0 u_{-n} = (ab)^{1/2} \overset{S(y)(1-ay^{-1})}{u_{-n}}$$

$$+ \frac{(y-a)(y-b)}{(ab)^{1/2}\, y} \underbrace{\frac{S(y)(1-ay^{-1}) - \overset{S(y)}{S(y^{-1})}(1-ay)}{y-y^{-1}}}_{a\,S(y)}$$

$$= \underbrace{S(y)(1-ay^{-1})}_{u_{-n}} \underbrace{\left((ab)^{1/2} + \frac{a(y-b)}{(ab)^{1/2}} \right)}_{\frac{ay}{(ab)^{1/2}}}$$

$$= \frac{a}{(ab)^{1/2}} \left(q^{-n} a^{-1} u_{-n} - q^{-n} a^{-1} u_n \right)$$

Also

$$t_0 \cdot u_n = (ab)^{1/2} u_n +$$

$$\frac{(y-a)(y-b)}{(ab)^{1/2} y} S(y) \underbrace{\frac{(1-q^n a y)(1-a y^{-1}) - (1-q^n a y^{-1})(1-a y)}{y-y^{-1}}}_{a(1-q^n)}$$

$$= (ab)^{1/2} S(y)(1-q^n a y)(1-a y^{-1})$$

$$+ \frac{(y-b)\, a(1-q^n)}{(ab)^{1/2}} \underbrace{S(y)(1-a y^{-1})}_{u_{-n}}$$

$$= (ab)^{1/2} u_n + \frac{a(1-q^n)}{(ab)^{1/2}} \underbrace{y\, u_{-n}}_{\substack{\| \\ q^{-n} a^{-1}(u_{-n} - u_n)}} - \frac{ab(1-q^n)}{(ab)^{1/2}} u_{-n}$$

=

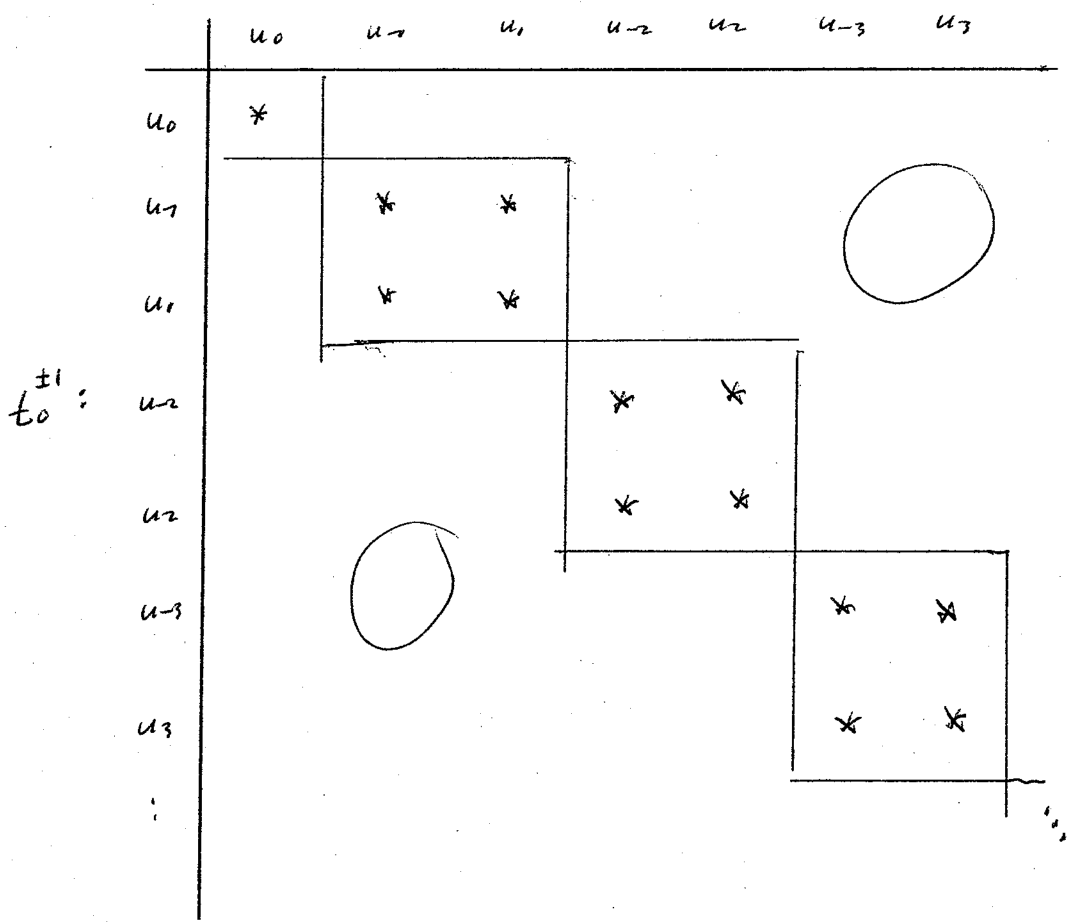
term	coeff
u_{-n}	$-\frac{ab(1-q^n)}{(ab)^{1/2}} + \frac{1-q^n}{(ab)^{1/2} q^n}$
u_n	$(ab)^{1/2} - \frac{1-q^n}{(ab)^{1/2} q^n}$

✓

Note 72: Relative the basis

$u_0, u_{-1}, u_1, u_{-2}, u_2, \ldots$

the matrices rep $t_0^{\pm 1}$ look as follows



We now find the action of t_3 on the basis [illegible]

LEM 73

(i) For $n \geq 0$

$$t_3 . u_n = (q^{-1}cd)^{1/2} u_n$$

(ii) For $n \geq 1$

$$t_3 . u_{-n} =$$

term	coef
u_{n-1}	$-\dfrac{(1-acq^{n-1})(1-adq^{n-1})}{(q^{-1}cd)^{1/2}}$
u_{-n}	$(qc^{-1}d^{-1})^{1/2}$
u_n	$(q^{-1}cd)^{1/2}$

Fn $n \geq 0$

$$t_3^{-1} \cdot u_n = (qc^{-1}d^{-1})^{1/2} u_n$$

Fn $n \geq 1$

$$t_3^{-1} \cdot u_{-n} =$$

term	coef
u_{n-1}	$\dfrac{(1-acq^{n-1})(1-adq^{n-1})}{(q^{-1}cd)^{1/2}}$
u_{-n}	$(q^{-1}cd)^{1/2}$
u_n	$-(q^{-1}cd)^{1/2}$

pf (i) Obs

$$u_n = (1-q^n a y)(1-q^{n-1} a y^{-1}) \quad \cdots \quad (1-qay)(1-ay^{-1})$$

this inv under $y \to q^{-1}y^{-1}$, so contained in $\mathbb{F}[y+q^{-1}y^{-1}]$

Done by L64 (iii) and since

$$k_3 = (q^{-1}cd)^{1/2}$$

(ii) $u_{-n} = (1-q^{n-1} a y^{-1})\underbrace{(1-q^{n-1} ay) \quad \cdots \quad (1-qay)(1-ay^{-1})}_{u_{n-1} \in \mathbb{F}[y+q^{-1}y^{-1}]}$

$$u_n = (1-q^n a y)(1-q^{n-1} a y^{-1})\, u_{n-1}$$

By Prop 65,

$$t_3 . u_{-n} = (q^{-1}cd)^{1/2} u_{-n}$$

$$- \frac{(y-c^{-1})(y-d^{-1})}{(qc^{-1}d^{-1})^{1/2}\, y} \quad u_{n-1} \underbrace{\frac{1-q^{n-1}ay^{-1} - (1-q^{n-1}a\,qy)}{y-q^{-1}y^{-1}}}_{q^n a}$$

$$(y-c^{-1})(y-d^{-1})y^{-1} = r\,1 + s(1-q^{n-1}ay^{-1})$$
$$+\,t(1-q^{n}ay)(1-q^{n-1}ay^{-1})$$

find r, s, t

set $y = q^{n-1}a$:

$$r = (q^{n-1}a-c^{-1})(q^{n-1}a-d^{-1})\,q^{1-n}a^{-1} = \frac{(1-acq^{n-1})(1-adq^{n-1})}{acd\,q^{n-1}}$$

$$t = -q^{-n}a^{-1}$$

$$s = q^{-n}a^{-1}(1-qc^{-1}d^{-1})$$

$t_3 \cdot u_{-n} =$

term	coef
u_{n-1}	$\frac{-q^{n}a}{(qc^{-1}d^{-1})^{1/2}} \cdot \frac{(1-acq^{n-1})(1-adq^{n-1})}{acd\,q^{n-1}}$
u_n	$(q^{-1}cd)^{1/2} + \frac{-q^{n}a}{(qc^{-1}d^{-1})^{1/2}} \cdot \frac{1-qc^{-1}d^{-1}}{q^{n}a}$ $\left(= (qc^{-1}d^{-1})^{1/2}\right)$
u_n	$\frac{-q^{n}a}{(qc^{-1}d^{-1})^{1/2}} \cdot \frac{-1}{q^{n}a}$

Simplify to get result.

Note 74 Relative the bases

$u_0, u_{-1}, u_1, u_{-2}, u_2, \dots$

the matrices rep $t_3^{\pm 1}$ look as follows

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3
u_0	*	*					
u_{-1}		*				O	
u_1		*	*	*			
u_{-2}				*			
u_2				*	*	*	
u_{-3}		O				*	
u_3						*	*
$\vdots$							$\ddots$

Continue to discuss split bases

$\{u_n\}_{n\in\mathbb{Z}}$

for the $\hat{H}_{q^{1/2}}$ module $\mathbb{F}[y, y^{-1}]$

$u_0 = 1$

$u_{-1} = 1 - ay^{-1}$

$u_1 = (1-qay)(1-ay^{-1})$

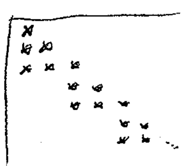
$\vdots$

$(1 - q^n a y^{-1})\, u_n = u_{-n-1}$ $\quad n = 0, 1, 2 \ldots$

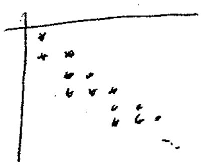
$(1 - q^n a y)\, u_{-n} = u_n$ $\quad n = 1, 2, \ldots$

So far: rel $u_0, u_{-1}, u_1, u_{-2}, u_2 \ldots$

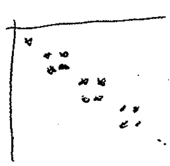
y:



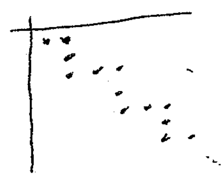
y^{-1}



$t_0^{\pm1}$:



$t_3^{\pm1}$:



We now give the action of $X^{\pm 1}$ $(X = t_3 t_0)$ on the basis $\{u_n\}_{n\in\mathbb{Z}}$

LEM 75

$$X.\, u_0 = (q^{-1}abcd)^{1/2}\, u_0$$

For $n \geq 1$

$$X.\, u_n = \frac{1}{(q^{-1}abcd)^{1/2}} \text{ times}$$

term	coef
u_{n-1}	$-q^{-n}(1-q^n)(1-abq^n)(1-acq^{n-1})(1-adq^{n-1})$
u_{-n}	$q^{-n}(1-q^n)(1-abq^n)$
u_n	$abcdq^{n-1}$

$$X \cdot u_{-n} = \frac{1}{(q^{-1}abcd)^{1/2}} \quad \text{times}$$

u_{n-1}	$-q^{-n}(1-acq^{n-1})(1-adq^{n-1})$
u_{-n}	q^{-n}

$$X^{-1} \cdot u_0 = \frac{1}{(q^{-1}abcd)^{1/2}} \, u_0$$

$$X^{-1} \cdot u_{-1} = \frac{1}{(q^{-1}abcd)^{1/2}} \Big((1-ac)(1-ad)\, u_0 + abcd\, u_{-1} \Big)$$

For $n \geq 1$

$$X^{-1} \cdot u_n = \frac{1}{(q^{-1}abcd)^{1/2}} \text{ times}$$

term	coef
u_{-n}	$-q^{-n}(1-q^n)(1-abq^n)$
u_n	q^{-n}

For $n \geq 2$

$$X^{-1} \cdot u_{-n} = \frac{1}{(q^{-1}abcd)^{1/2}} \text{ times}$$

term	coef
u_{1-n}	$-q^{1-n}(1-q^{n-1})(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})$
u_{n-1}	$q^{1-n}(1-acq^{n-1})(1-adq^{n-1})$
u_{-n}	$abcdq^{n-1}$

Pf: Use $X = t_3 t_0$ and L 71, 73

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Note 76 Relative the basis

$u_0, u_{-1}, u_1, u_{-2}, u_2, \ldots$

the matrices rep X, X^{-1} look as follows

X:

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3
u_0	*	*	*				
u_{-1}		*	*				
u_1			*	*	*		
u_{-2}				*	*		
u_2					*	*	*
u_{-3}						*	*
u_3							*
$\vdots$							

X^{-1}:

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3
u_0	*	*					
u_{-1}		*	*	*			
u_1			*	*			
u_{-2}				*	*	*	
u_2					*	*	
u_{-3}						*	*
u_3							*
$\vdots$							

LEM 77

t_1 acts on the basis $\{u_n\}_{n\in\mathbb{Z}}$ as follows

$t_1 \cdot u_0 = (ab^{-1})^{1/2} u_0 + (a^{-1}b)^{1/2} u_{-1}$

For $n \geq 1$

$t_1 \cdot u_{-n} = (a^{-1}b)^{1/2} u_{-n}$

↑ k_1^{-1}

$t_1 \cdot u_n =$

term	coef.
u_{-n}	$-(ab^{-1})^{1/2}(1-q^n)(1-abq^n)$
u_n	$(ab^{-1})^{1/2}$
u_{-n-1}	$(a^{-1}b)^{1/2}$

$$t_1^{-1} \cdot u_0 = (a^{-1}b)^{1/2}(u_0 - u_{-1})$$

For $n \geq 1$

$$t_1^{-1} \cdot u_{-n} = (ab^{-1})^{1/2} u_{-n}$$

$$t_1^{-1} \cdot u_n =$$

term	coef
u_{-n}	$(ab^{-1})^{1/2}(1-q^n)(1-abq^n)$
u_n	$(a^{-1}b)^{1/2}$
u_{-n-1}	$-(a^{-1}b)^{1/2}$

of Use $y = t_0 t_1$ and L71 □

Note 78 Relative the basis

$u_0, u_{-1}, u_1, u_{-2}, u_2, \ldots$

the matrices rep $t_1^{\pm 1}$ look as follows:

$t_1^{\pm 1}$:

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3	$\cdots$
u_0	*							
u_{-1}	*	*	*			0		
u_1			*					
u_{-2}			*	*	*			
u_2					*			
u_{-3}		0			*	*	*	
u_3							*	
$\vdots$								$\ddots$

LEM 79 t_2 acts on the basis $\{u_n\}_{n\in\mathbb{Z}}$ as follows:

For $n \geq 0$

$u_n, \quad u_{-n-1}$

is basis for a t_2-inv subspace of $\mathbb{F}[y,y^{-1}]$.

Rel to this basis

$$t_2 : \frac{1}{(cd)^{1/2} q^n a} \begin{pmatrix} 1 & (1-acq^n)(1-adq^n) \\ -1 & adq^n - 1 + acq^n \end{pmatrix}$$

$$t_2^{-1} : \frac{1}{(cd)^{1/2} q^n a} \begin{pmatrix} adq^n - 1 + acq^n & -(1-acq^n)(1-adq^n) \\ 1 & 1 \end{pmatrix}$$

Note 80 Relative the basis

$$u_0,\ u_{-1},\ u_1,\ u_{-2},\ u_2,\ \dots$$

the matrices rep $t_2^{\pm 1}$ look as follows

$t_2^{\pm 1}$:

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}
u_0	*	*				0
u_{-1}	*	*				
u_1			*	*		
u_{-2}			*	*		
u_2					*	*
u_{-3}	0				*	*

Summary $\hat{H}_{q^{1/2}}$ module $\mathbb{F}[y, y^{-1}]$

$\hat{H}_{q^{1/2}}$ -gens $\quad t_0^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}$

$X = t_3 t_0 \qquad Y = t_0 t_1$

parameters $\quad a, b, c, d \quad$ or $\quad k_0, k_1, k_2, k_3$

Related by

$k_0 = (ab)^{1/2} \qquad k_1 = (ab^{-1})^{1/2}$

$k_2 = (cd^{-1})^{1/2} \qquad k_3 = (q^{-1}cd)^{1/2}$

$a = k_0 k_1 \qquad b = k_0 k_1^{-1}$

$c = q^{1/2} k_2 k_3 \qquad d = q^{1/2} k_2^{-1} k_3$

Y acts on $\mathbb{F}[y, y^{-1}]$ as mult by y

$t_0 \cdot 1 = k_0 1 \qquad t_3 \cdot 1 = k_3 1$

$\forall f \in \mathbb{F}[y, y^{-1}]$

$$t_0 \cdot f(y) = (ab)^{1/2} f(y) + \frac{(y-a)(y-b)}{(ab)^{1/2} y} \frac{f(y) - f(y^{-1})}{y - y^{-1}}$$

$$t_3 \cdot f(y) = (q^{-1}cd)^{1/2} f(y) - \frac{(y-c^{-1})(y-d^{-1})}{(qc^{-1}d^{-1})^{1/2} y} \frac{f(y) - f(q^{-1}y^{-1})}{y - q^{-1}y^{-1}}$$

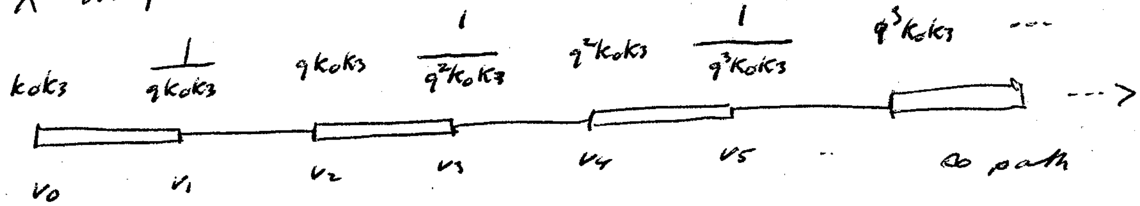
to get the t_1, t_2 actions use

$t_0 t_1 = Y, \qquad t_2 t_3 = q^{-1/2} Y^{-1}$

Summary, cont.

X is diagonalizable on $\mathbb{F}[y, y^{-1}]$

X-diagram is



$v_0 = 1$

$G_2 v_i = v_{i+1}$ for i even $\qquad i = 0, 1, 2, \ldots$

$G_0 v_i = v_{i+1}$ for i odd

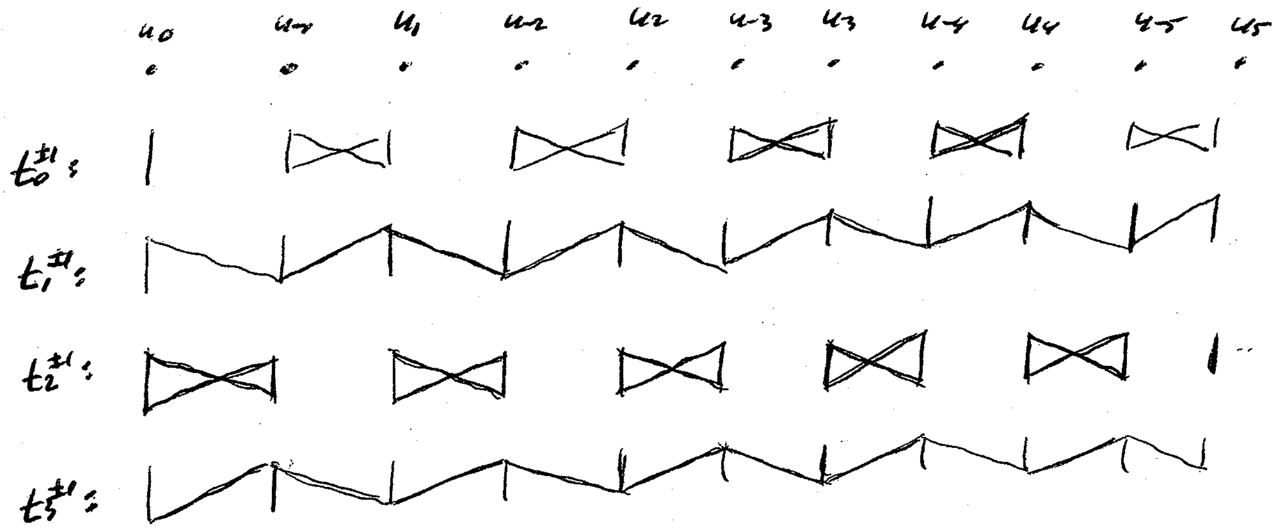
$G_0 = t_0 - t_3 t_0 t_3^{-1} \qquad G_2 = t_2 - t_1 t_2 t_1^{-1}$

Split basis $\{u_n\}_{n \in \mathbb{Z}}$ for $\mathbb{F}[y, y^{-1}]$:

$u_n = (1-q^n a y)(1-q^{n-1} a y^{-1})(1-q^{n-1} a y)(1-q^{n-2} a y^{-1}) \cdots (1-a y^{-1}) \qquad n = 0, 1, 2, \ldots$

$u_{-n} = (1-q^{n-1} a y^{-1})(1-q^{n-1} a y)(1-q^{n-2} a y^{-1}) \cdots (1-a y^{-1}) \qquad n = 1, 2, \ldots$

the $t_i^{\pm 1}$ act on $\{u_n\}_{n \in \mathbb{Z}}$ as follows



y:

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3
u_0	a						
u_{-1}	$q^{-1}a^{-1}$	$q^{-1}a^{-1}$					
u_1	$-q^{-1}a^{-1}$	$-q^{-1}a^{-1}$	qa				
u_{-2}			$q^{-2}a^{-1}$	$q^{-2}a^{-1}$			
u_2			$-q^{-2}a^{-1}$	$-q^{-2}a^{-1}$	q^2a		
u_{-3}					$q^{-3}a^{-1}$	$q^{-3}a^{-1}$	
u_3					$-q^{-3}a^{-1}$	$-q^{-3}a^{-1}$	q^3a
u_{-4}							

SU

y^{-1}:

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3	u_{-4}
u_0	a^{-1}							
u_{-1}	$-a^{-1}$	qa						
u_1		$q^{-1}a^{-1}$	$q^{-1}a^{-1}$					
u_{-2}		$-q^{-1}a^{-1}$	$-q^{-1}a^{-1}$	q^2a				
u_2				$q^{-2}a^{-1}$	$q^{-2}a^{-1}$			
u_{-3}				$-q^{-2}a^{-1}$	$-q^{-2}a^{-1}$	q^3a		
u_3						$q^{-3}a^{-1}$	$q^{-3}a^{-1}$	
u_{-4}						$-q^{-3}a^{-1}$	$-q^{-3}a^{-1}$	q^4a

$\ddots$

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VS

t_0 : $\frac{1}{(ab)^{1/2}}$ times

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3	u_{-4}
u_0	ab							
u_{-1}		q^{-1}	$q^{-1}(1-q)(1-abq)$					
u_1		$-q^{-1}$	$ab-q^{-1}+1$					
u_{-2}				q^{-2}	$q^{-2}(1-q^2)(1-abq^2)$			
u_2				$-q^{-2}$	$ab-q^{-2}+1$			
u_{-3}						q^{-3}	$q^{-3}(1-q^3)(1-abq^3)$	
u_3						$-q^{-3}$	$ab-q^{-3}+1$	
u_{-4}								$\ddots$

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t_0^{-1} : $\frac{1}{(ab)^{1/2}}$ times

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3	u_{-4}
u_0	1							
u_{-1}		$ab-q^{-1}+1$	$-q^{-1}(1-q)(1-abq)$					
u_1		q^{-1}	q^{-1}					
u_{-2}				$ab-q^{-2}+1$	$-q^{-2}(1-q^2)(1-abq^2)$			
u_2				q^{-2}	q^{-2}			
u_{-3}						$ab-q^{-3}+1$	$-q^{-3}(1-q^3)(1-abq^3)$	
u_3						q^{-3}	q^{-3}	
u_{-4}								

t_1: $(ab^{-1})^{1/2}$ times

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3	u_{-4}
u_0	1							
u_{-1}	$a^{-1}b$	$a^{-1}b$	$-(1-q)(1-abq)$					
u_1			1					
u_{-2}			$a^{-1}b$	$a^{-1}b$	$-(1-q^2)(1-abq^2)$			
u_2					1			
u_{-3}					$a^{-1}b$	$a^{-1}b$	$-(1-q^3)(1-abq^3)$	
u_3							1	
u_{-4}							$a^{-1}b$	$a^{-1}b$
u_4								

t_1^{-1}: $(ab^{-1})^{1/2}$ times

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v8

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3	u_{-4}
u_0	$a^{-1}b$							
u_{-1}	$-a^{-1}b$	1	$(1-q)(1-abq)$					
u_1			$a^{-1}b$					
u_{-2}			$-a^{-1}b$	1	$(1-q^2)(1-abq^2)$			
u_2					$a^{-1}b$			
u_{-3}					$-a^{-1}b$	1	$(1-q^3)(1-abq^3)$	
u_3							$a^{-1}b$	
u_{-4}							$-a^{-1}b$	1
u_4								

t_2: $\frac{1}{(cd)^{1/2} a}$

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3	u_{-4}
u_0	1	$(1-ac)(1-ad)$						
u_{-1}	-1	$ad-1+ac$						
u_1			q^{-1}	$q^{-1}(1-acq)(1-adq)$				
u_{-2}			$-q^{-1}$	$ad-q^{-1}+ac$				
u_2					q^{-2}	$q^{-2}(1-acq^2)(1-adq^2)$		
u_{-3}					$-q^{-2}$	$ad-q^{-2}+ac$		
u_3							q^{-3}	$q^{-3}(1-acq^3)(1-adq^3)$
u_{-4}							$-q^{-3}$	$ad-q^{-3}+ac$
u_4								...

t_2^{-1}: $\frac{1}{(cd)^{1/2}a}$

	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3	u_{-4}
u_0	$ad-1+ac$	$-(1-ac)(1-ad)$						
u_{-1}	1	1						
u_1			$ad-q^{-1}+ac$	$-(1-acq)(1-adq)$				
u_{-2}			q^{-1}	q^{-1}				
u_2					$ad-q^{-2}+ac$	$-(1-acq^2)(1-adq^2)$		
u_{-3}					q^{-2}	q^{-2}		
u_3							$ad-q^{-3}+ac$	$-(1-acq^3)(1-adq^3)$
u_{-4}							q^{-3}	q^{-3}
u_4								

t_3: $\frac{1}{(q^{-1}cd)^{1/2}}$ times

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	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3	u_{-4}
u_0	$q^{-1}cd$	$-(1-ac)(1-ad)$						
u_{-1}		1					0	
u_1		$q^{-1}cd$	$q^{-1}cd$	$-(1-acq)(1-adq)$				
u_{-2}				1				
u_2				$q^{-1}cd$	$q^{-1}cd$	$-(1-acq^2)(1-adq^2)$		
u_{-3}		0				1		
u_3						$q^{-1}cd$	$q^{-1}cd$	$-(1-acq^3)(1-adq^3)$
u_{-4}								1
u_4								$q^{-1}cd$

t_3^{-1}: $\frac{1}{(q^{-1}cd)^{1/2}}$ times

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	u_0	u_{-1}	u_1	u_{-2}	u_2	u_{-3}	u_3	u_{-4}
u_0	1	$(1-ac)(1-ad)$						
u_{-1}		$q^{-1}cd$						
u_1		$-q^{-1}cd$	1	$(1-acq)(1-adq)$				
u_{-2}				$q^{-1}cd$				
u_2				$-q^{-1}cd$	1	$(1-acq^2)(1-adq^2)$		
u_{-3}						$q^{-1}cd$		
u_3						$-q^{-1}cd$	1	$(1-acq^3)(1-adq^3)$
u_{-4}								$q^{-1}cd$
u_4								$-q^{-1}cd$

Continue to discuss $\hat{H}_{q^{1/2}}$-module $\mathbb{F}[y, y^{-1}]$

Recall split basis $\{u_n\}_{n \in \mathbb{Z}}$

Recall basis $\{v_n\}_{n=0}^{\infty}$ satisfies

$$v_0 = 1$$

$$G_0 v_n = v_{n+1} \qquad n \text{ odd} \qquad n = 0, 1, 2 \ldots$$

$$G_2 v_n = v_{n+1} \qquad n \text{ even}$$

Write each v_n as lin comb of $\{u_i\}_{i \in \mathbb{Z}}$

$$v_0 = 1 = u_0$$

v_1 is a scalar mult of

term	coef
1	1
$1 - ay^{-1}$	$-\dfrac{1 - abcd}{(1-ac)(1-ad)}$

V_2 is a scaler mult of

term	coef
1	1
$1-ay^{-1}$	$\dfrac{1-q}{(1-ac)(1-ad)\,q}$
$(1-ay^{-1})(1-qay)$	$-\dfrac{1-qabcd}{(1-ac)(1-ad)(1-qab)\,q}$

V_3 is a scalar multiple of

term	coef
1	1
$1-ay^{-1}$	$-\dfrac{1-qabcd}{(1-ac)(1-ad)}$
$(1-ay^{-1})(1-qay)$	$-\dfrac{1-qabcd}{(1-ac)(1-ad)(1-qab)\,q}$
$(1-ay^{-1})(1-qay)(1-qay^{-1})$	$\dfrac{(1-qabcd)(1-q^2abcd)}{(1-ac)(1-qac)(1-ad)(1-qad)(1-qab)\,q}$

V_4 is a scalar multiple of

term	coef
1	1
$1-ay^{-1}$	$\dfrac{1-q^2}{(1-ac)(1-ad)\,q^2}$
$(1-ay^{-1})(1-qay)$	$-\dfrac{(1+q)(1-q^2abcd)}{(1-ac)(1-ad)(1-qab)\,q^2}$
$(1-ay^{-1})(1-qay)(1-qay^{-1})$	$-\dfrac{(1-q^2)(1-q^2abcd)}{(1-ac)(1-qac)(1-ad)(1-qad)(1-qab)\,q^3}$
$(1-ay^{-1})(1-qay)(1-qay^{-1})(1-q^2ay)$	$\dfrac{(1-q^2abcd)(1-q^3abcd)}{(1-ac)(1-qac)(1-ad)(1-qad)(1-qab)(1-q^2ab)\,q^3}$

V_5 is a scalar multiple of

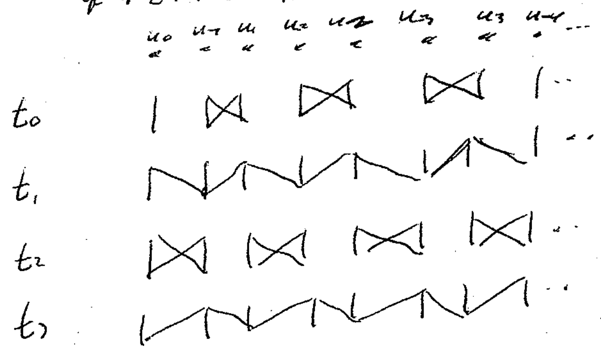
term	coef
1	1
$1-ay^{-1}$	$-\dfrac{1-q^2abcd}{(1-ac)(1-ad)}$
$(1-ay^{-1})(1-qay)$	$-\dfrac{(1+q)(1-q^2abcd)}{(1-ac)(1-ad)(1-qab)\,q^2}$
$(1-ay^{-1})(1-qay)(1-qay^{-1})$	$\dfrac{(1+q)(1-q^2abcd\quad)(1-q^3abcd\quad)}{(1-ac)(1-qac)(1-ad)(1-qad)(1-qab)\,q^2}$
$(1-ay^{-1})(1-qay)(1-qay^{-1})(1-q^2ay)$	$\dfrac{(1-q^2abcd\quad)(1-q^3abcd\quad)}{(1-ac)(1-qac)(1-ad)(1-qad)(1-qab)(1-q^2ab)\,q^3}$
$(1-ay^{-1})(1-qay)(1-qay^{-1})(1-q^2ay)(1-q^2ay^{-1})$	$-\dfrac{(1-q^2abcd\quad)(1-q^3abcd\quad)(1-q^4abcd\quad)}{(1-ac)(1-qac)(1-q^2ac)(1-ad)(1-qad)(1-q^2ad)(1-qab)(1-q^2ab)\,q^3}$

Problem 81 For $n = 0, 1, \ldots, 5$ we just expressed v_n as a sum. Can this sum be interpreted as a basic hypergeometric series?

Problem 82 Recall that the basis $\{v_n\}_{n=0}^{\infty}$ of $\mathbb{F}[y, y^{-1}]$ diagonalizes the generator $X = t_3 t_0$.
Is there an element $0 \neq A \in \hat{H}_{q^{1/2}}$ that is diagonalized by the split basis $\{u_n\}_{n \in \mathbb{Z}}$?

Problem 83 We saw that for the split basis $\{u_n\}_{n \in \mathbb{Z}}$ of $\mathbb{F}[y, y^{-1}]$ the matrix rep $Y = t_0 t_1$ is lower triang and the matrix rep $X = t_3 t_0$ is upper triang.
Does this feature characterize the split basis? (up to normalization)

Problem 84 We saw that for the split basis $\{u_n\}_{n \in \mathbb{Z}}$ of $\mathbb{F}[y, y^{-1}]$ the matrices rep t_0, t_1, t_2, t_3 look like



Does this feature characterize the split basis (up to normalization)?

Next topic: the Askey-Wilson q-difference operator

Given any seq $\{p_n\}_{n=0}^{\infty}$ of AW polys

$$p_n = p_n(x; a,b,c,d \mid q) = {}_4\phi_3\left(\begin{matrix} q^{-n} & abcdq^{n-1} & ay & ay^{-1} \\ ab & ac & ad & \end{matrix} \,\middle|\, q, q\right)$$

$x = y + y^{-1}$

$\{p_n\}_{n=0}^{\infty}$ basis for $\mathbb{F}[x] \subseteq \mathbb{F}[y, y^{-1}]$

Earlier we defined an $\mathbb{F}$-lin trans

$$B : \mathbb{F}[x] \to \mathbb{F}[x]$$

s.t.

$$B\, p_n = \theta_n^* p_n \qquad \theta_n^* = q^{-n} + abcdq^{n-1} \qquad n = 0, 1, \ldots$$

Given any $f \in \mathbb{F}[x]$ what does B do to f?

We now answer this question.

Consider action of $\hat{H}_{q^{1/2}}$ on $\mathbb{F}[y, y^{-1}]$

Compare

- above action of B on $\mathbb{F}[x]$
- restriction of $X + X^{-1}$ to $\mathbb{F}[x]$.

We saw earlier

$$(X + X^{-1})\, p_n = \left(k_0 k_3 q^n + \frac{1}{k_0 k_3 q^n}\right) p_n \qquad n = 0,1,2\ldots$$

$$k_0 = (ab)^{1/2}$$
$$k_3 = (q^{-1} cd)^{1/2}$$

$$k_0 k_3 q^n + \frac{1}{k_0 k_3 q^n} = \frac{1}{(q^{-1} abcd)^{1/2}}\, \theta_n^* \qquad n = 0,1,2\ldots$$

So the following actions agree on $\mathbb{F}[x]$:

- the above action of B
- $(q^{-1} abcd)^{1/2}(X + X^{-1})$ $\quad (X = t_3 t_0)$

Let τ denote the $\mathbb{F}$-alg aut of $\mathbb{F}[y, y^{-1}]$ that sends
$$y \to qy$$

Thm 85 The following coincide:

- the map B
- the restriction of

$$\phi(y)(\tau - I) + \phi(y^{-1})(\tau^{-1} - I) + (1 + abcdq^{-1})I$$

to $\mathbb{F}[x]$ where

$$I = \text{identity}$$

$$\phi(y) = \frac{(1-ay)(1-by)(1-cy)(1-dy)}{(1-y^2)(1-qy^2)}$$

In other words, for all $f \in \mathbb{F}[y, y^{-1}]$ s.t. $f(y) = f(y^{-1})$,

$$B.f(y) =$$

$$\frac{(1-ay)(1-by)(1-cy)(1-dy)}{(1-y^2)(1-qy^2)}\Big(f(qy) - f(y)\Big)$$

$$+$$

$$\frac{(1-ay^{-1})(1-by^{-1})(1-cy^{-1})(1-dy^{-1})}{(1-y^{-2})(1-qy^{-2})}\Big(f(q^{-1}y) - f(y)\Big)$$

$$+$$

$$(1 + q^{-1}abcd)\, f(y).$$

pf on $\mathbb{F}[x]$

$$B = (q^{-1}abcd)^{1/2}(X + X^{-1})$$

$$X = t_3 t_0, \qquad X^{-1} = t_0^{-1} t_3^{-1} = (T_0 - t_0)(T_3 - t_3)$$

T_0 acts on $\mathbb{F}[y, y^{-1}]$ as

$(k_0 + k_0^{-1}) I$ $\qquad k_0 = (ab)^{1/2}$

T_3 acts on $\mathbb{F}[y, y^{-1}]$ as

$(k_3 + k_3^{-1}) I$ $\qquad k_3 = (q^{-1}cd)^{1/2}$

t_0 acts on $\mathbb{F}[y, y^{-1}]$ as

$$t_0 . f(y) = (ab)^{1/2} f(y) + \frac{(y-a)(y-b)}{(ab)^{1/2} y} \frac{f(y) - f(y^{-1})}{y - y^{-1}}$$

t_3 acts on $\mathbb{F}[y, y^{-1}]$ as

$$t_3 . f(y) = (q^{-1}cd)^{1/2} f(y) - \frac{(y - c^{-1})(y - d^{-1})}{(qc^{-1}d^{-1})^{1/2} y} \frac{f(y) - f(q^{-1}y^{-1})}{y - q^{-1}y^{-1}}$$

Using these facts the result is routinely checked. □

Def 86 In view of Th 85 the map $B : \mathbb{F}[x] \to \mathbb{F}[x]$ is called the <u>Askey-Wilson q-difference operator.</u>

The non symmetric TD rels

Back to $\hat{H}_q$ Recall that

$A = Y + Y^{-1}$ $B = X + X^{-1}$

satisfy the TD rels

TD1 $$A^3 B - [3]_q A^2 B A + [3]_q A B A^2 - B A^3 = -(q^2 - q^{-2})^2 (AB - BA)$$

TD2 $$B^3 A - [3]_q B^2 A B + [3]_q B A B^2 - A B^3 = -(q^2 - q^{-2})^2 (BA - AB)$$

$$[3]_q = \frac{q^3 - q^{-3}}{q - q^{-1}}$$

These can be rephrased as:

TD1: A commutes with

$$A^2 B - (q^2 + q^{-2}) ABA + BA^2 + (q^2 - q^{-2})^2 B$$

TD2: B commutes with

$$B^2 A - (q^2 + q^{-2}) BAB + AB^2 + (q^2 - q^{-2})^2 A$$

Next goal: display similar eqs involving just X, Y

Recall $\{C_i\}_{i \in I}$ from Def 28

$$C_0 = q\left(qYX - q^{-1}XY\right)$$

$$C_1 = -\left(q^{-1}YX^{-1} - qX^{-1}Y\right)$$

$$C_2 = q^{-1}\left(qY^{-1}X^{-1} - q^{-1}X^{-1}Y^{-1}\right)$$

$$C_3 = -\left(q^{-1}Y^{-1}X - qXY^{-1}\right)$$

Recall from Prop 30:

	t_0T_2	t_1T_3	t_2T_0	t_3T_1	T_0T_2	T_1T_3
C_0	q	1	q^{-1}	1	$-q^{-1}$	-1
C_1	1	q	1	q^{-1}	-1	$-q^{-1}$
C_2	q^{-1}	1	q	1	$-q^{-1}$	-1
C_3	1	q^{-1}	1	q	-1	$-q^{-1}$

↑ treating this as coef matrix,

we can solve for $t_0T_2,\ t_1T_3,\ t_2T_0,\ t_3T_1$ in terms of $C_0, C_1, C_2, C_3,\ T_0T_2,\ T_1T_3$

this gives the following.

Prop 86 We have

(i) $$\frac{qC_0 - C_1 + q^{-1}C_2 - C_3}{q-q^{-1}} = (qt_0 + q^{-1}t_0^{-1})T_2 - T_1T_3$$

(ii) $$\frac{qC_1 - C_2 + q^{-1}C_3 - C_0}{q-q^{-1}} = (qt_1 + q^{-1}t_1^{-1})T_3 - T_0T_2$$

(iii) $$\frac{qC_2 - C_3 + q^{-1}C_0 - C_1}{q-q^{-1}} = (qt_2 + q^{-1}t_2^{-1})T_0 - T_1T_3$$

(iv) $$\frac{qC_3 - C_0 + q^{-1}C_1 - C_2}{q-q^{-1}} = (qt_3 + q^{-1}t_3^{-1})T_1 - T_0T_2$$

pf ✓

Note LHS of Prop 86 (i) is

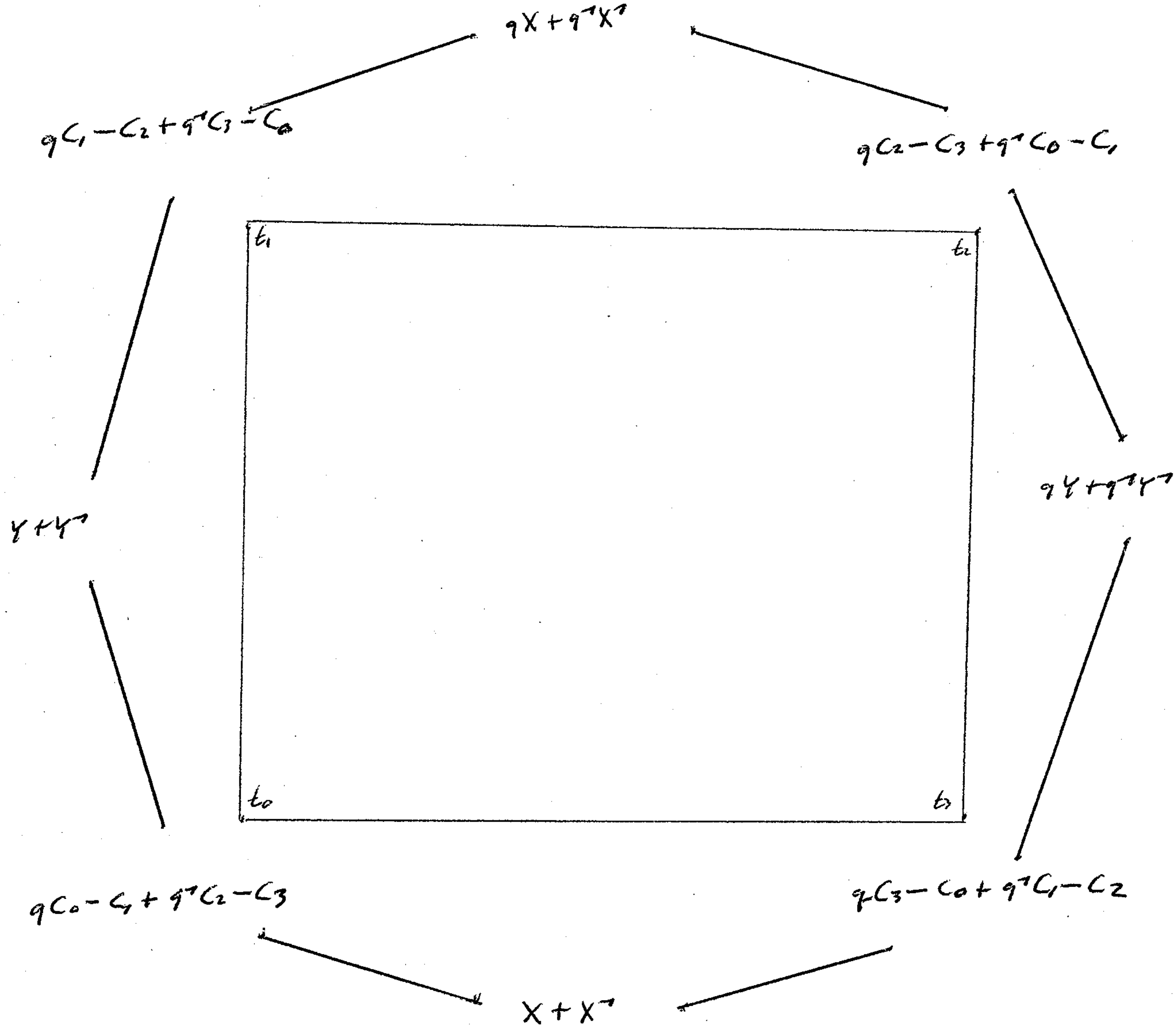
$$\frac{q^{-1}(Y+Y^{-1})(X+X^{-1}) - q(X+X^{-1})(Y+Y^{-1})}{q-q^{-1}} + (q+q^{-1})(qYX + q^{-1}X^{-1}Y^{-1})$$

To get (ii) – (iv) apply $\mathbb{Z}_4$-sym

$$X \to Y \to q^{-1}X^{-1} \to q^{-1}Y^{-1} \to X$$

Thm 87 Recall the elements $X = t_3 t_0$, $Y = t_0 t_1$ in $\hat{H}_q$

In the diagram below r ——— s means $rs = sr$



"Non sym TD rels"

pf $qC_0 - C_1 + q^{-1}C_2 - C_3$ commutes with $X + X^{-1}$ and $Y + Y^{-1}$ by Prop 86 (i) and since t_0, T_1, T_2, T_3 commutes with $X + X^{-1}$ and $Y + Y^{-1}$. Rest is sim.

□

Cor 88 For the elements

$$X = t_3 t_0 \qquad Y = t_0 t_1$$

in $\hat{H}_q$,

(i) $$(X + X^{-1})^2 Y - (q+q^{-1})(X+X^{-1}) Y (qX+q^{-1}X^{-1}) + Y(qX+q^{-1}X^{-1})^2 = -(q-q^{-1})^2 Y$$

(ii) $$(Y + Y^{-1})^2 X^{-1} - (q+q^{-1})(Y+Y^{-1}) X^{-1}(qY+q^{-1}Y^{-1}) + X^{-1}(qY+q^{-1}Y^{-1})^2 = -(q-q^{-1})^2 X^{-1}$$

(iii) $$(qX + q^{-1}X^{-1})^2 Y^{-1} - (q+q^{-1})(qX+q^{-1}X^{-1}) Y^{-1}(X+X^{-1}) + Y^{-1}(X+X^{-1})^2 = -(q-q^{-1})^2 Y^{-1}$$

(iv) $$(qY+q^{-1}Y^{-1})^2 X - (q+q^{-1})(qY+q^{-1}Y^{-1}) X (Y+Y^{-1}) + X(Y+Y^{-1})^2 = -(q-q^{-1})^2 X$$

pf (i) Using Th 87

$$0 = [X+X^{-1}, \; qC_0 - C_1 + q^{-1}C_2 - C_3] \qquad (*)$$

$$0 = [X+X^{-1}, \; qC_3 - C_0 + q^{-1}C_1 - C_2] \qquad (**)$$

Add $(*)$ and $q^{-1}(**)$ to find

$$0 = [X+X^{-1}, \; qC_0 - C_1] \qquad (***)$$

Now eval $(***)$ using the def of C_0, C_1

(ii)–(iv) Apply $\mathbb{Z}_4$-symmetry to (i)

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□

The End