Let V denote an irred $\hat{H}_q$-module on which X is diagonalyable.

We define the X-diagram for V as follows.

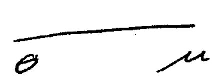
<u>nodes</u>

these are the eigenspaces of X on V

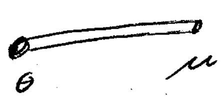
We label each node with the corresp eigvalue

<u>arcs</u>

For nodes θ, μ



whenever $\theta\mu = 1$



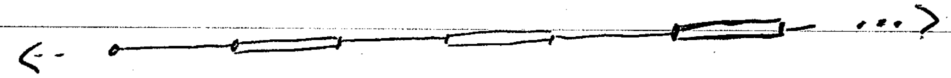
whenever $\theta\mu = q^{-2}$

We allow loops:

whenever $\theta^2 = 1$

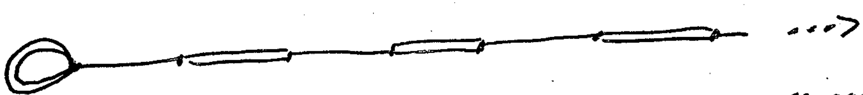
whenever $\theta^2 = q^{-2}$

Possible X-diagrams

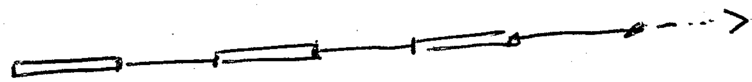


$\infty\infty$ path

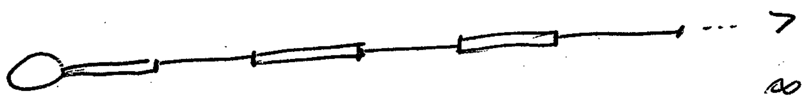
∞ path



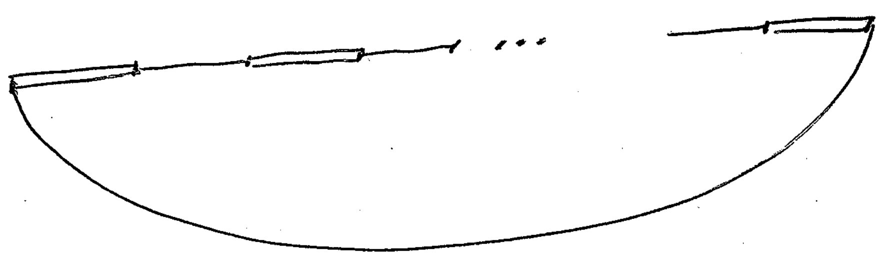
∞ path



∞ path



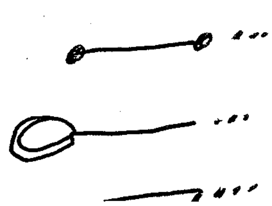
∞ path



cycle

Finite path

Many versions; choices at each end are



Next goal: careful description of an irred $\hat{H}_q$-module whose X-diagram is $\infty\infty$ path

Until further notice fix $0 \neq k_i \in \mathbb{F}$ $\quad i \in \mathbb{I}$

Def functions

$$a, b, c, d : \mathbb{F} \setminus \{0, 1, -1\} \to \mathbb{F}$$

as follows.

$$a(\theta) = \frac{\theta(k_3 + k_3^{-1}) - k_0 - k_0^{-1}}{\theta - \theta^{-1}}$$

$$d(\theta) = a(\theta^{-1})$$

$$b(\theta) = \frac{G(\theta, k_0, k_3)}{\theta - \theta^{-1}}$$

$$c(\theta) = \frac{1}{\theta^{-1} - \theta}$$

Define 2 x 2 matrices

$$T_3(\theta) = \begin{pmatrix} a(\theta) & b(\theta) \\ c(\theta) & d(\theta) \end{pmatrix}$$

$$T_0(\theta) = \begin{pmatrix} \theta d(\theta) & \frac{-b(\theta)}{\theta} \\ -\theta c(\theta) & \frac{a(\theta)}{\theta} \end{pmatrix}$$

One checks

matrix	trace	det
$\tau_3(\theta)$	$k_3+k_3^{-1}$	1
$\tau_0(\theta)$	$k_0+k_0^{-1}$	1

$$\tau_3(\theta) + \tau_3(\theta)^{-1} = (k_3+k_3^{-1})\, I$$

$$\tau_0(\theta) + \tau_0(\theta)^{-1} = (k_0+k_0^{-1})\, I$$

$$\tau_3(\theta)\,\tau_0(\theta) = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}$$

$$\tau_0(\theta) - \tau_3(\theta)\,\tau_0(\theta)\,\tau_3(\theta)^{-1} = \begin{pmatrix} 0 & G(\theta, k_0, k_3) \\ 1 & 0 \end{pmatrix}$$

Ref to Prop 15,

relative the basis v, $G_0 v$

the matrices representing t_3, t_0 are

$\tau_3(\theta)$, $\tau_0(\theta)$ resp.

Fix $0 \neq k_i \in \mathbb{F}$ [illegible]

Define functions

$$\alpha, \beta, \gamma, \delta \; : \; \mathbb{F} \setminus \{0, 1, -1\} \to \mathbb{F}$$

$$\alpha(\theta) = \frac{\theta(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{\theta - \theta^{-1}}$$

$$\delta(\theta) = \alpha(\theta^{-1})$$

$$\beta(\theta) = \frac{G(\theta, k_1, k_2)}{\theta - \theta^{-1}}$$

$$\gamma(\theta) = \frac{1}{\theta^{-1} - \theta}$$

To go from a, b, c, d to $\alpha, \beta, \gamma, \delta$

switch $k_0 \leftrightarrow k_2$

$k_1 \leftrightarrow k_3$

Define 2×2 matrices

$$T_1(\theta) = \begin{pmatrix} \alpha(\theta) & \beta(\theta) \\ \gamma(\theta) & \delta(\theta) \end{pmatrix}$$

$$T_2(\theta) = \begin{pmatrix} \theta\,\delta(\theta) & -\dfrac{\beta(\theta)}{\theta} \\ -\theta\,\gamma(\theta) & \dfrac{\alpha(\theta)}{\theta} \end{pmatrix}$$

One checks

matrix	trace	det
$T_1(\theta)$	$k_1 + k_1^{-1}$	1
$T_2(\theta)$	$k_2 + k_2^{-1}$	1

$$T_1(\theta) + T_1(\theta)^{-1} = (k_1 + k_1^{-1}) I$$

$$T_2(\theta) + T_2(\theta)^{-1} = (k_2 + k_2^{-1}) I$$

$$T_1(\theta)\, T_2(\theta) = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}$$

$$T_2(\theta) - T_1(\theta)\, T_2(\theta)\, T_1(\theta)^{-1} = \begin{pmatrix} 0 & G(\theta, k_1, k_2) \\ 1 & 0 \end{pmatrix}$$

Ref to Prop 20.

relative to the basis v, $G_2 v$

the matrices resp t_1, t_2 are

$T_1(q^{-1}\theta^{-1})$, $T_2(q^{-1}\theta^{-1})$

resp.

For $i \in \mathbb{I}$ def a matrix T_i

with rows / cols indexed by $\mathbb{Z}$

and entries in $\mathbb{F}$.

Fix $0 \neq \theta_0 \in \mathbb{F}$

The entries of T_i are given below

(all entries not shown are 0)

For even $i \in \mathbb{Z}$

$$T_3: \quad \begin{array}{c|cc} \text{row} \diagdown \text{col} & i & i+1 \\ \hline i & \multicolumn{2}{c}{\multirow{2}{*}{$T_3(\theta_0 q^{-i})$}} \\ i+1 & & \end{array}$$

$$T_0: \quad \begin{array}{c|cc} \text{row} \diagdown \text{col} & i & i+1 \\ \hline i & \multicolumn{2}{c}{\multirow{2}{*}{$T_0(\theta_0 q^{-i})$}} \\ i+1 & & \end{array}$$

For odd $i \in \mathbb{Z}$

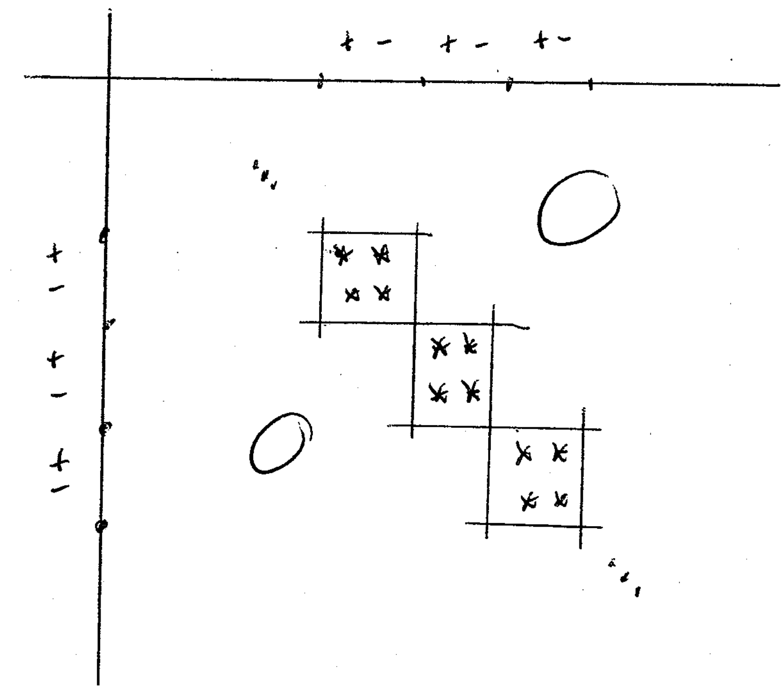
τ_1

<table>
<tr><th>row \ col</th><th>i</th><th>$i+1$</th></tr>
<tr><td>i</td><td colspan="2" rowspan="2">$\tau_1(q^{-i}\theta_0)$</td></tr>
<tr><td>$i+1$</td></tr>
</table>

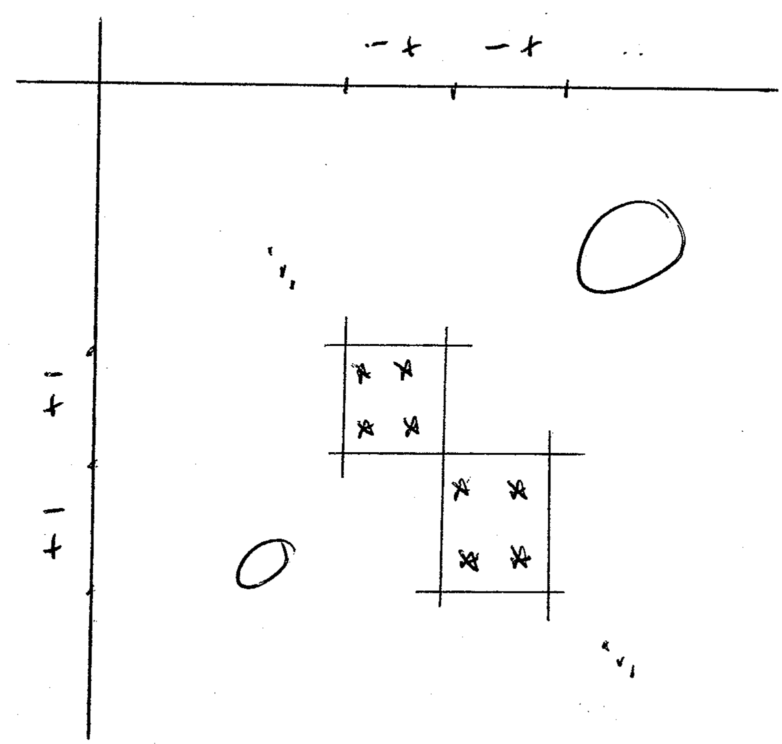
τ_2 :

<table>
<tr><th>row \ col</th><th>i</th><th>$i+1$</th></tr>
<tr><td>i</td><td colspan="2" rowspan="2">$\tau_2(q^{-i}\theta_0)$</td></tr>
<tr><td>$i+1$</td></tr>
</table>

τ_0, τ_3:



τ_1, τ_2:



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$+$ means even
$-$ means odd

Show $\{T_i\}_{i \in I}$ satisfy the defining relations for $\hat{H}_q$.

By constr

$$T_i + T_i = (k_i + k_i^{-1}) I \qquad \forall i \in I$$

central

$T_3 T_0$ is diagonal with (i,i)-entry

$$\begin{cases} \theta_0 q^{-i} & \text{if } i \text{ even} \\ \theta_0^{-1} q^{i-1} & \text{if } i \text{ odd} \end{cases} \qquad i \in \mathbb{Z}$$

$T_1 T_2$ is diagonal with (i,i)-entry

$$\begin{cases} \theta_0^{-1} q^{i-1} & \text{if } i \text{ even} \\ \theta_0 q^{-i} & \text{if } i \text{ odd} \end{cases} \qquad i \in \mathbb{Z}$$

By these comments

$$T_3 T_0 T_1 T_2 = q^{-1} I$$

So

$$T_0 T_1 T_2 T_3 = q^{-1} I$$

We have shown $\{T_i\}_{i \in I}$ satisfy the defining rels for $\hat{H}_q$

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Comments on the construction

the eigvals of $T_3 T_0$ are

$$\dots\ q^2\theta_0,\quad q^{-2}\theta_0,\quad \theta_0,\quad \theta_0^{-1},\quad q^{-2}\theta_0,\quad q^2\theta_0^{-1},\quad q^{-4}\theta_0,\ \dots \qquad *$$

TFAE

(i) $*$ are mut dist

(ii) q is not a root of 1 and

$$\theta_0 \notin \{\pm q^n \mid n \in \mathbb{Z}\} \qquad (1)$$

TFAE

(i) $G(\theta, k_0, k_3) \neq 0 \quad \forall\ \theta$ in $*$

(ii) $$\theta_0 \notin \left\{ q^n k_0/k_3,\ q^n k_3/k_0,\ q^n k_0 k_3,\ \frac{q^n}{k_0 k_3} \;\middle|\; n \in \mathbb{Z},\ n \text{ even} \right\} \qquad (2)$$

TFAE

(i) $G(q^{-1}\theta^{-1}, k_1, k_2) \neq 0 \quad \forall\ \theta$ in $*$

(ii) $$\theta_0 \notin \left\{ q^n k_1/k_2,\ q^n k_2/k_1,\ q^n k_1 k_2,\ \frac{q^n}{k_1 k_2} \;\middle|\; n \in \mathbb{Z},\ n \text{ odd} \right\} \qquad (3)$$

thm 36 Given $0 \neq k_i \in \mathbb{F}$ $(i \in \mathbb{I})$

Given $0 \neq \theta_0 \in \mathbb{F}$ that satisfies (1)—(3)

Then $\exists$ an $\hat{H}_q$-module $V = V(k_0, k_1, k_2, k_3, \theta_0)$

with the following property:

V has a basis $\{v_i\}_{i \in \mathbb{Z}}$ with resp to which the matrix

rep t_i is T_i $(\forall i \in \mathbb{I})$.

Moreover

- For $i \in \mathbb{Z}$, v_i is an eigenvector for X. The eigval is

$$\begin{cases} \theta_0 q^{-i} & \text{if } i \text{ even} \\ \theta_0^{-1} q^{i-1} & \text{if } i \text{ odd} \end{cases}$$

- V is irred
- X is diagonalizable on V, and all eigenspaces of X on V have dim 1
- the X-diagram for V is

$$\cdots \; \mathbb{F}v_{-2} \;\text{—}\; \mathbb{F}v_{-1} \;=\!=\; \mathbb{F}v_0 \;\text{—}\; \mathbb{F}v_1 \;=\!=\; \mathbb{F}v_2 \;\text{—}\; \mathbb{F}v_3 \; \cdots$$

($\infty\infty$ path)

- For $i \in \mathbb{Z}$

$$G_0 v_i = v_{i+1} \quad \text{if } i \text{ even}$$

$$G_2 v_i = v_{i+1} \quad \text{if } i \text{ odd}$$

pf Each claim already shown or routine. $\square$

We now consider the implications of Thm 35

for the module $V = V(k_0, k_1, k_2, k_3, \theta_0)$ from Th 36

Motivation:

With ref to Th 36 assume $k_0 \neq \mp 1$.

so that

$$V = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V + \frac{t_0 - k_0}{k_0^{-1} - k_0} V \qquad \text{ds}$$

For all even $i \in \mathbb{Z}$

v_i is basis for $V_X(\theta)$ $\qquad \theta = \theta_0 q^{-i}$

v_{i+1} ... $V_X(\theta^{-1})$

$V_X(\theta) + V_X(\theta^{-1}) = V_B(\theta + \theta^{-1})$ has basis

$v_i, \quad v_{i+1}$

and another basis

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_i, \qquad \frac{t_0 - k_0}{k_0^{-1} - k_0} v_i$$

So the following is a basis for $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V$:

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_i \qquad i \in \mathbb{Z}, \ i \text{ even}$$

Also the following is a basis for $\frac{t_0 - k_0}{k_0^{-1} - k_0} V$:

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_i \qquad i \in \mathbb{Z}, \ i \text{ even}$$

Thm 37 With ref to thm 36 assume $k_0 \neq \pm 1$.

Then for $i \in \mathbb{Z}$

$$A \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i} =$$

term	coef
$\frac{t_0-k_0^{-1}}{k_0-k_0^{-1}} v_{2i-2}$	$\frac{G(q\theta, k_1, k_2)\, G(q\theta, k_0 q^{-1}, k_3)}{q^2\theta(\theta-\theta^{-1})(q^{-1}\theta^{-1}-q\theta)}$
$\frac{t_0-k_0^{-1}}{k_0-k_0^{-1}} v_{2i}$	$\frac{\theta k_0 + \theta^{-1}k_0^{-1} - k_3 - k_3^{-1}}{\theta-\theta^{-1}} \frac{q^{-1}\theta^{-1}(k_1+k_1^{-1}) - k_2 - k_2^{-1}}{q^{-1}\theta^{-1} - q\theta} + \frac{q^{-1}\theta(k_1+k_1^{-1}) - k_2 - k_2^{-1}}{\theta-\theta^{-1}} \frac{k_3+k_3^{-1} - \theta k_0^{-1} - \theta^{-1}k_0}{q^{-1}\theta - q\theta^{-1}} + \frac{k_1+k_1^{-1}}{k_0}$
$\frac{t_0-k_0^{-1}}{k_0-k_0^{-1}} v_{2i+2}$	$\frac{\theta}{(\theta-\theta^{-1})(q\theta^{-1}-q^{-1}\theta)}$

where $\theta = \theta_0 q^{-2i}$

pf In Th 35 take

$$v = v_{2i}$$

So

$$v \in V_X(\theta)$$

$$G_2 G_0 v = v_{2i+2}$$

Find rel between

$$G_2 v, \quad v_{2i-2}$$

We have

$$G_2 G_0 v_{2i-2} = v_{2i} = v$$

so

$$G_2^2 \underbrace{G_0 v_{2i-2}} = G_2 v$$

$$v_{2i-2} \in V_X(q^2\theta)$$

$$G_0 v_{2i-2} \in V_X(q^{-2}\theta^{-1})$$

by Cor 9

G_2^2 acts on $V_X(q^{-2}\theta^{-1})$ as $G(q\theta, k_1, k_2) I$

So far

$$G_2 v = G(q\theta, k_1, k_2) \; G_0 v_{2i-2}$$

so

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} G_2 v = G(q\theta, k_1, k_2) \underbrace{\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} G_0 v_{2i-2}}_{\text{// L33}}$$

$$\frac{k_3 + k_3^{-1} - q^2\theta k_0^{-1} - q^{-2}\theta^{-1} k_0}{q^2\theta} \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i-2}$$

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One checks

$$\frac{\theta k_0 + \theta^{-1} k_0^{-1} - k_3 - k_3^{-1}}{(\theta - \theta^{-1})(k_0 - k_0^{-1})} \quad \frac{1}{q\theta - q^{-1}\theta^{-1}}$$

$$\times \quad G(q\theta, k_1, k_2) \quad \frac{k_3 + k_3^{-1} - q^{2}\theta k_0^{-1} - q^{-2}\theta^{-1} k_0}{q^2 \theta}$$

$$= \quad \frac{G(q\theta, k_1, k_2)\, G(q\theta, k_0 q^{-1}, k_3)}{q^2 \theta \,(\theta - \theta^{-1})(q^{-1}\theta^{-1} - q\theta)}$$

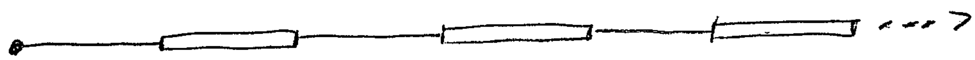
Result follows $\square$

Lecture 33 Friday Nov 10

$\mathbb{F}$ alg closed $0 \neq q \in \mathbb{F}$ $q^4 \neq 1$

Next goal: describe the irred $\hat{H}_q$-modules

with X-diagram



θ_0 ∞ path.

[these corresp to AW polys]

Motivation Given $\hat{H}_q$-module V as above

assume k_i exist $\forall i \in \mathbb{I}$

Given $0 \neq v \in V_X(\theta_0)$

By Prop 21

$$t_1 v \in \mathbb{F}v$$

$$t_2 v \in \mathbb{F}v$$

Eigvals of t_1 are $k_1^{\pm 1}$

k_1 defined up to reciprocal

WLOG

$$t_1 v = k_1 v$$

Similarly

$$t_2 v = k_2 v$$

So

$$\begin{aligned} k_1 k_2 v &= t_1 t_2 v \\ &= q^{-1} X^{-1} v \\ &= q^{-1} \theta_0^{-1} v \end{aligned}$$

So

$$\theta_0 = \frac{1}{q k_1 k_2}$$

We now construct our modules

Until further notice fix $0 \neq k_i \in \mathbb{F}$ $i \in \mathbb{I}$

Define $$\theta_0 = \frac{1}{q k_1 k_2}$$

Functions $a, b, c, d : \mathbb{F} \setminus \{0, 1, -1\} \to \mathbb{F}$ as before
$\alpha, \beta, \gamma, \delta$ ...

2x2 matrices

$$T_i(\theta) \qquad i \in \mathbb{I}$$

as before

We now define T_i

For $i \in \mathbb{I}$ define a matrix T_i with rows/cols indexed by nonneg integers

The entries of T_i are given below (all entries not shown are 0.)

For even $i \geq 0$

τ_3:

row \ col	i	$i+1$
i	$\tau_3(\theta_0 q^{-i})$	
$i+1$		

τ_0:

row \ col	i	$i+1$
i	$\tau_0(\theta_0 q^{-i})$	
$i+1$		

τ_1 has $(0,0)$-entry k_1

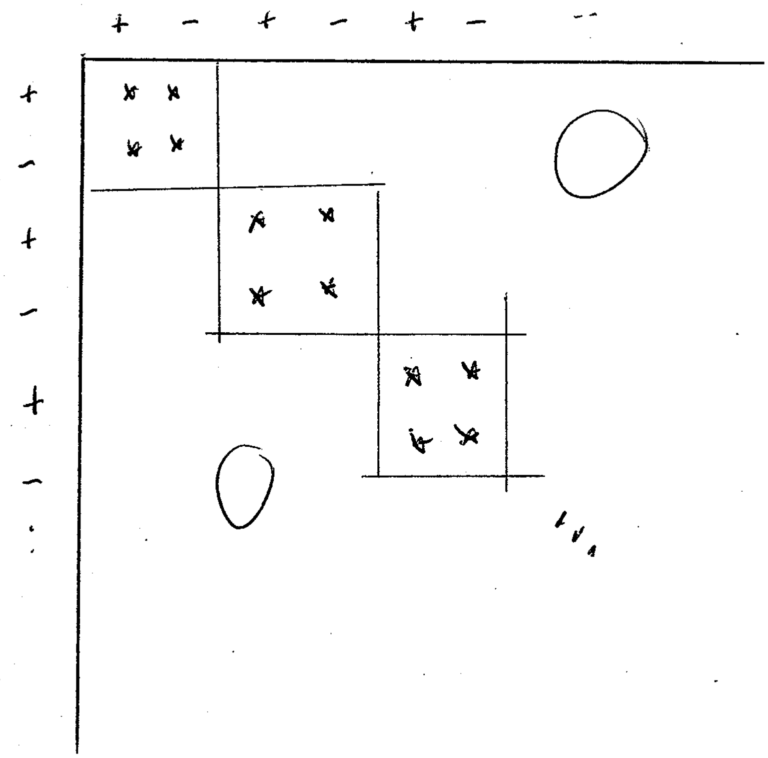
τ_2 has $(0,0)$-entry k_2

For odd $i \geq 1$

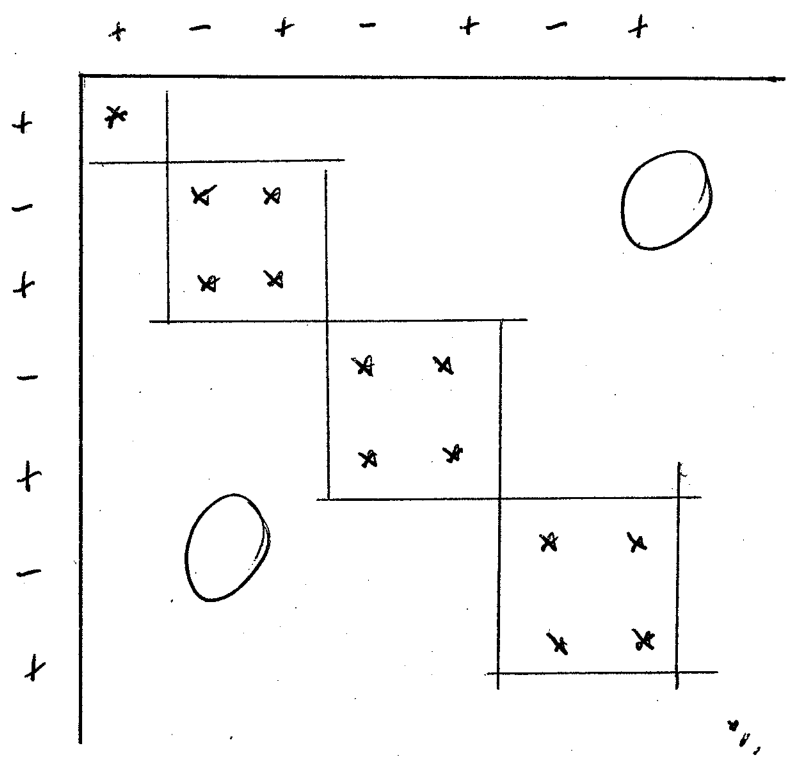
$$\tau_1 \qquad \begin{array}{c|cc} {}_{\text{row}} \backslash {}^{\text{col}} & i & i+1 \\ \hline i & & \\ & \tau_1(\theta_0 q^{-i}) & \\ i+1 & & \end{array}$$

$$\tau_2 \qquad \begin{array}{c|cc} {}_{\text{row}} \backslash {}^{\text{col}} & i & i+1 \\ \hline i & & \\ & \tau_2(\theta_0 q^{-i}) & \\ i+1 & & \end{array}$$

T_0, T_3



T_1, T_2



As in the previous example

$$T_i + T_i^{-1} = (k_i + k_i^{-1})\, I \qquad i \in \mathbb{I}$$

$T_3 T_0$ is diagonal with (i,i)-entry

$$\begin{cases} \theta_0 q^{-i} & \text{if } i \text{ even} \\ \theta_0^{-1} q^{i+1} & \text{if } i \text{ odd} \end{cases} \qquad i = 0, 1, 2 \ldots$$

$T_1 T_2$ is diagonal with (i,i)-entry

$$\begin{cases} \theta_0^{-1} q^{i+1} & \text{if } i \text{ even} \\ \theta_0 q^{-i} & \text{if } i \text{ odd} \end{cases} \qquad i = 0, 1, 2, \ldots$$

So as before

$$T_0 T_1 T_2 T_3 = q^{-1} I$$

So $\{T_i\}_{i \in \mathbb{I}}$ give a representation of $\hat{H}_q$.

The eigenvals of $T_3 T_0$ are

$$\theta_0, \quad \theta_0^{-1}, \quad q^{-2}\theta_0, \quad q^2\theta_0^{-1}, \quad q^{-4}\theta_0, \ \dots \qquad * \qquad \left(\theta_0 = \frac{1}{qk_1k_2}\right)$$

To make the above rep of $\hat{H}_q$ irred, we ~~assume~~

- $*$ are mut dist (1)
- $G(\theta, k_0, k_3) \neq 0 \quad \forall \theta \in *$ (2)
- $G(q^{-1}\theta^{-1}, k_1, k_2) \neq 0 \quad \forall \theta \in * \setminus \theta_0$ (3)

thm 38 Given $0 \neq k_i \in \mathbb{F}$ $(i \in \mathbb{I})$

that satisfy (1)–(3)

$\exists$ an $\hat{H}_q$-module $V = V(k_0, k_1, k_2, k_3)$

with the following property.

V has a basis $\{v_i\}_{i=0}^{\infty}$ with resp to which the

matrix rep t_j is T_j for $j \in \mathbb{I}$

Moreover

- For $i = 0,1,2\ldots$ v_i is an eigenvector for X with eigval

$$\begin{cases} \theta_0 q^{-i} & i \text{ even} \\ \theta_0^{-1} q^{i-1} & i \text{ odd} \end{cases} \qquad \theta_0 = \frac{1}{q k_1 k_2}$$

- V is irred
- X is diagonalable on V and all eigenspaces of X on V have dim 1
- the X-diag of V is

$$\mathbb{F}v_0 \text{ --- } \mathbb{F}v_1 = \mathbb{F}v_2 \text{ --- } \mathbb{F}v_3 \; \cdots >$$

∞ path

- For $i = 0,1,2\ldots$

$G_0 v_i = v_{i+1}$ if i even

$G_2 v_i = v_{i+1}$ i odd

pf Each claim is already shown or routine. $\square$ 392

Ref to the $\hat{H}_q$-module $V = V(k_0, k_1, k_2, k_3)$

from M 38 We now relate V to the AW polynomials.

Assume $k_0 \neq \pm 1$ so

$$V = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V + \frac{t_0 - k_0}{k_0^{-1} - k_0} V \qquad ds$$

The following is a basis for $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V$:

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i} \qquad i = 0,1,2 \ldots \qquad \ast$$

The following is a basis for $\frac{t_0 - k_0}{k_0^{-1} - k_0} V$:

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_{2i} \qquad i = 0,1,2 \ldots$$

The element $A = Y + Y^{-1}$ acts on $\ast$ just as in M 37 :

For $i = 0,1,2 \ldots$

$$A \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i} =$$

term	coef
$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i-2}$	$(v_{-2} = 0)$ same coef
$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i}$	as in M 37 except now $\theta_0 = \frac{1}{q k_1 k_2}$
$\frac{t_0 - k_0^{-1}}{\ }$	

Rel $*$ matrix rep A is tridiag

$$\begin{pmatrix} * & x & & & \\ x & x & x & & 0 \\ & x & x & x & \\ & & x & \ddots & \ddots \\ & 0 & & \ddots & \ddots \end{pmatrix}$$

Call this matrix

$$\begin{pmatrix} a_0 & c_1 & & & 0 \\ b_0 & a_1 & c_2 & & \\ & b_1 & a_2 & \ddots & \\ & 0 & \ddots & \ddots & \end{pmatrix}$$

Define polynomials $\{f_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ by

$$x f_n = c_n f_{n-1} + a_n f_n + b_n f_{n+1} \qquad n = 0, 1, 2 \ldots$$

$$f_0 = 1, \qquad f_{-1} = 0$$

Fa $n = 0, 1, 2 \ldots$

f_n has degree n

coef of x^n is $\dfrac{1}{b_0 b_1 \cdots b_{n-1}}$

By constr

$$f_n(A) \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_0 = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2n} \qquad n = 0, 1, 2 \ldots$$

For $n=0,1,2\ldots$ let F_n denote the monic version of f_n so

$$f_n = \frac{F_n}{b_0 b_1 \cdots b_{n-1}}$$

the $\{F_n\}_{n=0}^{\infty}$ satisfy the 3-term rec

$$x F_n = c_n b_{n-1} F_{n-1} + a_n F_n + F_{n+1} \qquad n=0,1,2\ldots$$

$$F_0 = 1, \qquad F_{-1} = 0$$

Call $\{F_n\}_{n=0}^{\infty}$ the monic polynomials that corresp to $\{k_i\}_{i\in I}$

As we will see the $\{F_n\}_{n=0}^{\infty}$ are the AW polys (monic version)

The AW polys are defined using basic hypergeometric series ${}_4\phi_3$, as we now explain

Notation: For $a \in \mathbb{F}$

$$(a;q)_n = \underbrace{(1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})}_{n \text{ terms}} \qquad n=0,1,2\ldots$$

So $(a;q)_0 = 1$

Let a, b, c, d denote nonzero scalars in $\mathbb{F}$

Assume: q not a root of 1

And None of

ab, ac, ad, bc, bd, cd is an integral power of q

For $n = 0, 1, 2, \ldots$ define a polynomial

$$p_n = p_n(x; a, b, c, d \mid q)$$

in $\mathbb{F}[x]$ by

$$p_n = {}_4\phi_3\left(\begin{matrix} q^{-n} & abcdq^{n-1} & ay & ay^{-1} \\ ab & ac & ad & \end{matrix} \middle| q; q\right)$$

where $x = y + y^{-1}$

$$\left[= \sum_{i=0}^{\infty} \frac{(q^{-n};q)_i \, (abcdq^{n-1};q)_i \, (ay;q)_i \, (ay^{-1};q)_i}{(ab;q)_i \, (ac;q)_i \, (ad;q)_i \, (q;q)_i} q^i \right]$$

Note that $(q^{-n};q)_i = 0$ for $i > n$

so above sum terminates at $i = n$.

One checks p_n really is a poly in x with degree n.

p_n is nth <u>AW polynomial</u>

For instance

$$p_0 = 1$$

$$p_1 = 1 - \frac{(1-abcd)(1-ax+a^2)}{(1-ab)(1-ac)(1-ad)}$$

$$p_2 = 1 - \frac{(1+q^{-1})(1-abcdq)(1-ax+a^2)}{(1-ab)(1-ac)(1-ad)}$$

$$+ \frac{(1-abcdq)(1-abcdq^2)(1-ax+a^2)(1-axq+a^2q^2)}{q(1-ab)(1-abq)(1-ac)(1-acq)(1-ad)(1-adq)}$$

The AW polys satisfy a 3-term rec.

For $n=0,1,2,\dots$

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1}$$

$$p_{-1}=0$$

where

$$b_n = b_n(a,b,c,d\,|\,q) = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})}$$

$$c_n = c_n(a,b,c,d\,|\,q) = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}$$

$$a_n = a_n(a,b,c,d\,|\,q) = a+a^{-1}-b_n-c_n$$

For $n = 0, 1, 2 \ldots$

p_n has deg n

coeff of x^n is $\dfrac{1}{b_0 b_1 \cdots b_{n-1}}$

Let $P_n = P_n(x; a,b,c,d \mid q)$ denote the monic

version of p_n so

$$p_n = \frac{P_n}{b_0 b_1 \cdots b_{n-1}}$$

Then

$$x P_n = c_n b_{n-1} P_{n-1} + a_n P_n + P_{n+1}$$

Note that

$b_{n-1} c_n$ is sym in a, b, c, d

One checks

a_n is sym in a, b, c, d

Indeed define

$$e_1 = a + b + c + d$$

$$e_2 = ab + ac + ad + bc + bd + cd$$

$$e_3 = abc + abd + acd + bcd$$

$$e_4 = abcd$$

then

$$a_n = q^{n-1} \frac{(1-q^n-q^{n+1})e_3 + q e_1 + q^{2n-1} e_3 e_4 - q^{n-1}(1+q-q^{n+1}) e_1 e_4}{(1-q^{2n-2}e_4)(1-q^{2n}e_4)}$$

Since a_n and $b_{n-1}c_n$ are sym in a, b, c, d

the polys $\{P_n\}_{n=0}^{\infty}$ are sym in a, b, c, d.

thm 39 For the $\hat{H}_q$-module V from thm 38

consider the corresp monic polys $\{F_n\}_{n=0}^{\infty}$

Then

$$F_n(x) = P_n\left(x;\, a,b,c,d \,|\, q^2\right) \qquad n=0,1,2\ldots$$

$$\overset{\shortparallel}{P_n}$$

where a,b,c,d is any perm of

$$k_0k_1, \quad \frac{q^2k_1}{k_0}, \quad qk_2k_3, \quad \frac{qk_2}{k_3}$$

pf We show $\{F_n\}_{n=0}^{\infty}$ and $\{P_n\}_{n=0}^{\infty}$ satisfy

the same 3-term rec

To do this, need to show

$$b_{n-1}(a,b,c,d\,|\,q^2)\, c_n(a,b,c,d\,|\,q^2)$$

$$= \frac{G(q\theta, k_1, k_2)\,G(q\theta, q^{-1}k_0, k_3)}{(\theta-\theta^{-1})(q^{-1}\theta^{-1}-q\theta)} \; \frac{1}{(q^2\theta-q^{-2}\theta^{-1})(q^{-1}\theta^{-1}-q\theta)}$$

$n = 1, 2 \ldots$

$$a_n(a,b,c,d\,|\,q^2)$$

$$= \frac{\theta k_0 + \theta^{-1}k_0^{-1} - k_3 - k_3^{-1}}{\theta-\theta^{-1}} \; \frac{q^{-1}\theta^{-1}(k_1+k_1^{-1}) - k_2 - k_2^{-1}}{q^{-1}\theta^{-1}-q\theta}$$

$$+ \frac{q^{-1}\theta(k_1+k_1^{-1}) - k_2 - k_2^{-1}}{\theta-\theta^{-1}} \; \frac{k_3+k_3^{-1} - \theta k_0^{-1} - \theta^{-1}k_0}{q^{-1}\theta - q\theta^{-1}}$$

$$+ \frac{k_1+k_1^{-1}}{k_0}$$

$n = 0, 1, 2 \ldots$

where

$$\theta = \theta_0 q^{-2n} \qquad \theta_0 = \frac{1}{q k_1 k_2}$$

This is a routine verification. $\square$ 402

Notes

$$\frac{G(q\theta, k_1, k_2)\, G(q\theta, q^{-1}k_0, k_3)}{(\theta-\theta^{-1})(q^{-1}\theta^{-1}-q\theta)} \quad \frac{1}{(q^2\theta-q^{-2}\theta^{-1})(q^{-1}\theta^{-1}-q\theta)}$$

$$=$$

$$\frac{q^4\theta^{-4}\left(\theta-\frac{k_1}{qk_2}\right)\left(\theta-\frac{k_2}{qk_1}\right)\left(\theta-\frac{k_1k_2}{q}\right)\left(\theta-\frac{1}{qk_1k_2}\right)\left(\theta-\frac{k_3}{k_0}\right)\left(\theta-\frac{1}{k_0k_3}\right)\left(\theta-\frac{k_0k_3}{q^2}\right)\left(\theta-\frac{k_0}{q^2k_3}\right)}{\left(\theta-\theta^{-1}\right)\left(q\theta-q^{-1}\theta^{-1}\right)^2\left(q^2\theta-q^{-2}\theta^{-1}\right)}$$

$$a_n = \frac{(1-k_1k_2)(k_1-k_2)(k_0k_3-q)(k_0-qk_3)}{2(\theta+\theta^{-1}-q-q^{-1})\,qk_0k_1k_2k_3}$$

$$+$$

$$\frac{(1+k_1k_2)(k_1+k_2)(k_0k_3+q)(k_0+qk_3)}{2(\theta+\theta^{-1}+q+q^{-1})\,qk_0k_1k_2k_3}$$

provided char $F \neq 2$

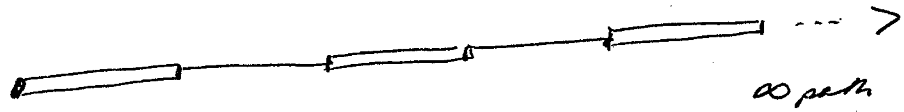
where

$$\theta = \theta_0 q^{-2n} \qquad \theta_0 = \frac{1}{qk_1k_2}$$

$\mathbb{F}$ alg closed $\quad 0 \neq q \in \mathbb{F} \quad q^4 \neq 1$

Next goal: describe the irred $\hat{H}_q$-module

with X-diagram



∞ path

θ_0

Motivation Given $\hat{H}_q$-module V as above

Assume k_i exist $\forall i \in \mathbb{I}$.

Given $\quad 0 \neq v \in V_X(\theta_0)$

$t_0 v \in \mathbb{F} v$

$t_3 v \in \mathbb{F} v$

WLOG $\quad t_0 v = k_0 v$

$t_3 v = k_3 v$

$k_0 k_3 v = t_3 t_0 v$

$= X v$

$\theta_0 = k_0 k_3$

We now construct our modules

Until further notice fix $0 \neq k_i \in \mathbb{F}$ $\quad i \in \mathbb{I}$

Define $\quad \theta_0 = k_0 k_3$

Functions $\quad a, b, c, d : \mathbb{F} \setminus \{0, 1, -1\} \to \mathbb{F}$ $\quad$ as before

$\alpha, \beta, \gamma, \delta$ ..

2x2 matrices $\quad T_i(\theta) \quad i \in \mathbb{I}$

as before

For $i \in \mathbb{I}$ define matrix T_i

rows/cols indexed by nonneg integers

Entries of T_i given below

T_0 has $(0,0)$-entry k_0

T_3 has $(0,0)$-entry k_3

T_0

row \ col	i	$i+1$
i	$T_0(q^{-i-1}\theta_0^{-1})$	
$i+1$		

T_3

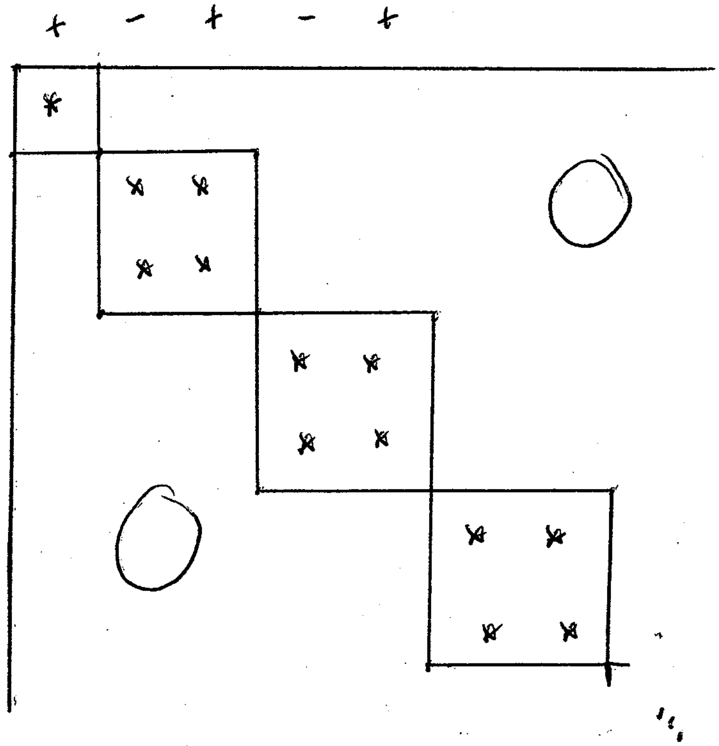
row \ col	i	$i+1$
i	$T_3(q^{-i-1}\theta_0^{-1})$	
$i+1$		

For even $i \geq 0$

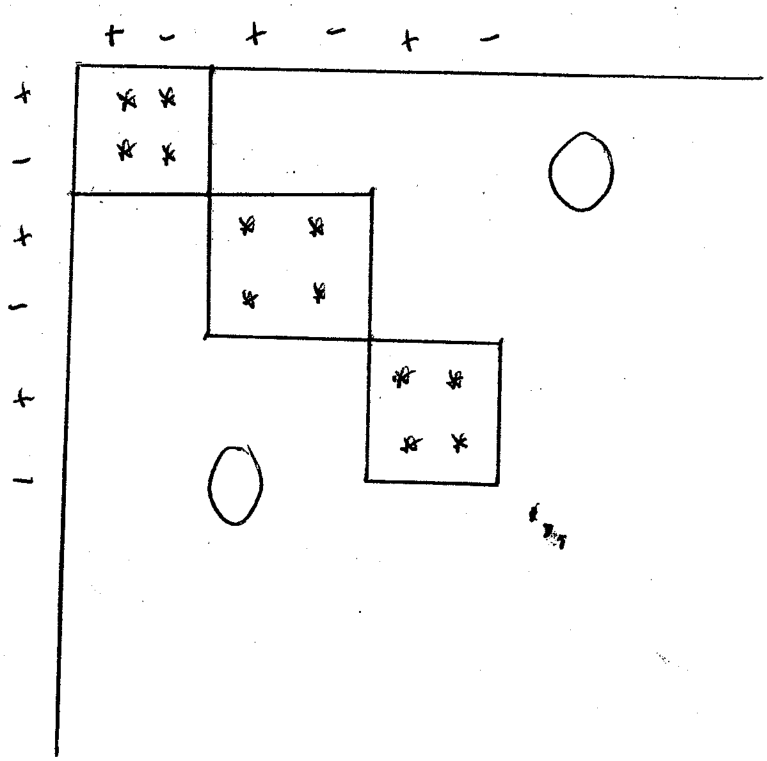
$$\tau_1 \quad \begin{array}{c|cc} {}_{\text{row}}\backslash{}^{\text{col}} & i & i+1 \\ \hline i & & \tau_1(q^{-i-1}\theta_0^{-1}) \\ i+1 & & \end{array}$$

$$\tau_2 \quad \begin{array}{c|cc} {}_{\text{row}}\backslash{}^{\text{col}} & i & i+1 \\ \hline i & & \tau_2(q^{-i-1}\theta_0^{-1}) \\ i+1 & & \end{array}$$

γ_0, γ_3:



γ_1, γ_2



As in prev examples

$$T_i + T_i = (k_i + k_i^{-1}) \qquad i \in \mathbb{I}$$

$T_3 T_0$ is diag with (i,i)-entry

$$\begin{cases} \theta_0 q^i & \text{if } i \text{ even} \\ \theta_0^{-1} q^{-i-1} & \text{if } i \text{ odd} \end{cases} \qquad i = 0,1,2\ldots$$

$T_1 T_2$ is diag with (i,i)-entry

$$\begin{cases} \theta_0^{-1} q^{-i-1} & \text{if } i \text{ even} \\ \theta_0 q^i & \text{if } i \text{ odd} \end{cases} \qquad i = 0,1,2\ldots$$

So $\quad T_3 T_0 T_1 T_2 = q^{-1} I$

so $\quad T_0 T_1 T_2 T_3 = q^{-1} I$

So $\{T_i\}_{i \in \mathbb{I}}$ gives rep of $\hat{H}_q$

Eigvals of $T_3 T_0$ are

$$\theta_0,\quad q^{-2}\theta_0^{-1},\quad q^{2}\theta_0,\quad q^{-4}\theta_0^{-1},\quad q^{4}\theta_0,\quad q^{-6}\theta_0^{-1},\quad q^{6}\theta_0, \dots \qquad *$$

$$\theta_0 = k_0 k_3$$

To make the above $\hat{H}_q$-module **irred**, we require

- $*$ are mut dist (1)
- $G(\theta, k_0, k_3) \neq 0 \quad \forall\, \theta \in * \setminus \theta_0$ (2)
- $G(q^{-1}\theta^{-1}, k_1, k_2) \neq 0 \quad \forall\, \theta \in *$ (3)

th 40 Given $0 \neq k_i \in \mathbb{F}$ $(i \in \mathbb{I})$
that sat (1)–(3)

$\exists$ an $\hat{H}_q$-module $V = V(k_0, k_1, k_2, k_3)$
with the following prop.
V has a basis $\{v_i\}_{i=0}^{\infty}$ with resp to which the matrix
rep t_g is τ_g for $g \in \mathbb{I}$

Moreover
- For $i = 0, 1, 2 \ldots$ v_i is an eigvector for X with eigval

$$\begin{cases} \theta_0 q^i & \text{if } i \text{ even} \\ \theta_0^{-1} q^{-i-1} & \text{if } i \text{ odd} \end{cases} \qquad \theta_0 = k_0 k_3$$

- V is irred
- X is diagonalizable on V and all eigenspaces of X on V have dim 1
- the X-diag of V is

$$\mathbb{F}v_0 = \mathbb{F}v_1 - \mathbb{F}v_2 = \mathbb{F}v_3 - \mathbb{F}v_4 \dashrightarrow$$

∞ path

- For $i = 0, 1, 2 \ldots$

$$G_2 v_i = v_{i+1} \quad \text{if } i \text{ even}$$
$$G_0 v_i = v_{i+1} \quad \text{if } i \text{ odd}$$

$\square$

pf Each claim already shown or routine.

Ref to the $\hat{H}_1$-module $V = V(k_0, k_1, k_2, k_3)$

from th 40 we now relate V to the AW polys.

Assume $k_0 \neq \mp 1$ so that

$$V = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V + \frac{t_0 - k_0}{k_0^{-1} - k_0} V \qquad \text{ds}$$

The following is a basis for $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V$: $\quad$ *

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i} \qquad i = 0, 1, 2 \ldots$$

The following is a basis for $\frac{t_0 - k_0}{k_0^{-1} - k_0} V$: $\quad$ **

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_{2i} \qquad i = 1, 2 \ldots$$

We now find the action of $A = Y + Y^{-1}$ on * thm 41 With above notation, for $i = 0, 1, 2 \ldots$

$$A \, \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} \, v_{2i} =$$

term	coef
$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V_{2i+2}$	$\frac{(\theta - k_0^{-1}k_3)(\theta - k_0^{-1}k_3^{-1})}{(\theta - q^{-2}k_0^{-1}k_3)(\theta - q^{-2}k_0^{-1}k_3^{-1})} \, \frac{1}{(\theta - \theta^{-1})(q^{-1}\theta^{-1} - q\theta)\, q^{4}\theta}$
$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V_{2i}$	$\frac{\theta k_0 + \theta^{-1}k_0^{-1} - k_3 - k_3^{-1}}{\theta - \theta^{-1}} \, \frac{q^{-1}\theta^{-1}(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{q^{-1}\theta^{-1} - q\theta}$ $+ \frac{q^{-1}\theta(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{\theta - \theta^{-1}} \, \frac{k_3 + k_3^{-1} - \theta k_0^{-1} - \theta^{-1}k_0}{q^{-1}\theta - q\theta^{-1}}$ $+ \frac{k_1 + k_1^{-1}}{k_0}$
$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V_{2i-2}$	$\frac{\theta\, G(\theta^{-1}, k_0, k_3)\, G(q\theta^{-1}, k_1, k_2)}{(\theta - \theta^{-1})(q\theta^{-1} - q^{-1}\theta)}$

where $\theta = q^{2i}\theta_0 \qquad \theta_0 = k_0 k_3$

pf Find coef of $\frac{-}{k_0 - k_0^{-1}}$ v_{2i+2}

Use th 35

$$V_{2i+2} = G_0 G_2 \, v_{2i}$$

$$G_0 V_{2i+2} = G_0^2 \quad G_2 v_{2i}$$

$$\left[G_2 v_{2i} \in V_X (q^{-2}\theta^{-1}) \right]$$

$$= G(q^{-2}\theta^{-1}, k_0, k_3) \, G_2 v_{2i}$$

$$\underbrace{\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} \; G_0 V_{2i+2}}_{} = G(q^{-2}\theta^{-1}, k_0, k_3) \; \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} \; G_2 v_{2i}$$

|| L 33, $v_{2i+2} \in V_X(q^2\theta)$

$$\frac{k_3 + k_3^{-1} - q^2\theta k_0^{-1} - q^{-2}\theta^{-1} k_0}{q^2\theta} \qquad \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} \; V_{2i+2}$$

Now use th 35, and note that

$$\frac{\theta k_0 + \theta^{-1} k_0^{-1} - k_3 - k_3^{-1}}{(\theta - \theta^{-1})(q\theta - q^{-1}\theta^{-1})} \quad \frac{k_3 + k_3^{-1} - q^2\theta k_0^{-1} - q^{-2}\theta^{-1} k_0}{q^2\theta \quad G(q^{-2}\theta^{-1}, k_0, k_3)}$$

$$= \frac{(\theta - k_0^{-1} k_3)(\theta - k_0^{-1} k_3^{-1})}{(\theta - q^2 k_0^{-1} k_3)(\theta - q^{-2} k_0^{-1} k_3^{-1})} \; \frac{1}{(\theta - \theta^{-1})(q^{-1}\theta^{-1} - q\theta)\, q^4 \theta}$$

Find coef of $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V_{2i-2}$:

$$V_{2i} = G_0 G_2 V_{2i-2}$$

$$G_2 G_0 V_{2i} = \underbrace{G_2 G_0^2 \underbrace{G_2 V_{2i-2}}_{V_X(\theta^{-1})}}$$

$$= G(\theta^{-1}, k_0, k_3) \; G_2^2 \underbrace{V_{2i-2}}_{V_X(q^{-2}\theta)}$$

$$= G(\theta^{-1}; k_0, k_3) \; G(q\theta^{-1}, k_1, k_2) \; V_{2i-2}$$

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} G_2 G_0 V_{2i} = G(\theta^{-1}, k_0, k_3) \, G(q\theta^{-1}, k_1, k_2) \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V_{2i-2}$$

$$\frac{\theta}{(\theta - \theta^{-1})(q\theta^{-1} - q^{-1}\theta)} G(\theta^{-1}, k_0, k_3) \, G(q\theta^{-1}, k_1, k_2)$$

is coef of $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V_{2i-2}$

Find coef of $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V_{2i}$: from M 35

$\square$

Th 42 For the $\hat{H}_q$-module V

from Th 40

Consider the monic polys $\{F_n\}_{n=0}^{\infty}$ corresp

corresp the basis $*$:

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i} \qquad i = 0, 1, 2 \ldots$$

Then

monic AW polys

$$F_n(x) = P_n(x; a, b, c, d \mid q^2) \qquad n = 0, 1, 2 \ldots$$

where a, b, c, d is any perm of

$$k_0 k_1, \quad k_0 k_1^{-1}, \quad q k_2 k_3, \quad q k_2^{-1} k_3$$

pf We show $\{F_n\}_{n=0}^{\infty}$ and $\{P_n\}_{n=0}^{\infty}$ satisfy the same 3-term rec,

To do this, need to show

$$b_{n-1}(a,b,c,d/q^2)\,c_n(a,b,c,d/q^2)$$

$$= \frac{G(q^{-1}\theta,\, q^{-1}k_0,\, k_3)}{(\theta-\theta^{-1})(q^{-2}\theta-q^2\theta^{-1})}\;\frac{G(q^{-1}\theta,\, k_1,\, k_2)}{(q\theta^{-1}-q^{-1}\theta)^2}$$

$n=1,2,\ldots$

$$a_n(a,b,c,d/q^2) =$$

$$\frac{\theta k_0+\theta^{-1}k_0^{-1}-k_3-k_3^{-1}}{\theta-\theta^{-1}}\;\frac{q^{-1}\theta^{-1}(k_1+k_1^{-1})-k_2-k_2^{-1}}{q^{-1}\theta^{-1}-q\theta}$$

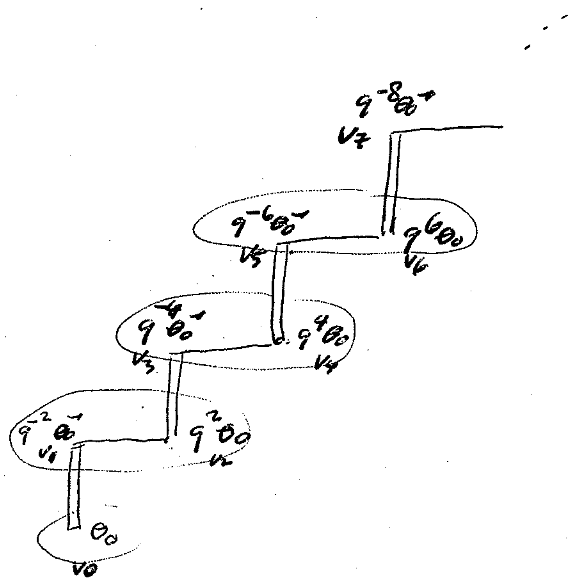
$$+\;\frac{q^{-1}\theta(k_1+k_1^{-1})-k_2-k_2^{-1}}{\theta-\theta^{-1}}\;\frac{k_3+k_3^{-1}-\theta k_0^{-1}-\theta^{-1}k_0}{q^{-1}\theta-q\theta^{-1}}$$

$$+\;\frac{k_1+k_1^{-1}}{k_0}$$

$n=0,1,2,\ldots$

where $\theta=\theta_0 q^{2n}$ $\quad\theta_0=k_0k_3$

Pf Routinely checked. $\square$



$\theta_0 = k_0 k_3$

$G_0 G_2 \, v_{2i} = v_{2i+2}$

$$b_i = \frac{(\theta - k_0^{-1}k_3)(\theta - k_0^{-1}k_3^{-1})}{(\theta - q^{-2}k_0^{-1}k_3)(\theta - q^{-2}k_0^{-1}k_3^{-1})} \frac{1}{(\theta-\theta^{-1})(q^{-1}\theta^{-1} - q\theta)q^{4}\theta}$$

$$\theta = q^{2i}\theta_0 \qquad \theta_0 = k_0 k_3$$

$$c_i = \frac{\theta G(\theta^{-1}; k_0, k_3) G(q\theta^{-1}; k_1, k_2)}{(\theta-\theta^{-1})(q\theta^{-1} - q^{-1}\theta)}$$

$$b_{i-1}c_i = \frac{(q^{-2}\theta - k_0^{-1}k_3)(q^{-2}\theta - k_0^{-1}k_3^{-1})}{(q^{-2}\theta - q^{-2}k_0^{-1}k_3)(q^{-2}\theta - q^{-2}k_0^{-1}k_3^{-1})} \frac{1}{(q^{-2}\theta - q^{2}\theta^{-1})(q\theta^{-1} - q^{-1}\theta)q^{2}\theta}$$

$$\frac{\theta G(\theta^{-1}; k_0, k_3) G(q\theta^{-1}; k_1, k_2)}{(\theta-\theta^{-1})(q\theta^{-1} - q^{-1}\theta)}$$

$$= \frac{G(q^{-1}\theta, q^{-1}k_0, k_3)\, G(q^{-1}\theta, k_1, k_2)}{(q^{-2}\theta - q^{2}\theta^{-1})(q\theta^{-1} - q^{-1}\theta)^{2}(\theta-\theta^{-1})}$$

$\mathbb{F}$ alg closed

$0 \neq q \in \mathbb{F} \qquad q^4 \neq 1$

Ref to $\hat{H}_q$-module $V = V(k_0, k_1, k_2, k_3)$ from Th 40

Recall following is basis for $\dfrac{t_0 - k_0}{k_0^{-1} - k_0} V$:

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_{2i} \qquad i = 1, 2, 3 \ldots \qquad \ast\ast$$

Next goal: Find action of $A = Y + Y^{-1}$ on $\ast\ast$

Th 43 With above rotation

For $i=1,2,3\ldots$

$$A\,\frac{t_0-k_0}{k_0^{-1}-k_0}\,v_{2i} =$$

term	coef
$\frac{t_0-k_0}{k_0^{-1}-k_0}\,v_{2i+2}$	$\frac{(\theta-k_0k_3)(\theta-k_0k_3^{-1})}{(\theta-q^{-2}k_0k_3)(\theta-q^{-2}k_0k_3^{-1})}\;\frac{1}{(\theta-\theta^{-1})(q^{-1}\theta^{-1}-q\theta)\,q^{4}\theta}$
$\frac{t_0-k_0}{k_0^{-1}-k_0}\,v_{2i}$	$\frac{\theta k_0^{-1}+\theta^{-1}k_0-k_3-k_3^{-1}}{\theta-\theta^{-1}}\;\frac{q^{-1}\theta^{-1}(k_1+k_1^{-1})-k_2-k_2^{-1}}{q^{-1}\theta^{-1}-q\theta}$ $+\;\frac{q^{-1}\theta(k_1+k_1^{-1})-k_2-k_2^{-1}}{\theta-\theta^{-1}}\;\frac{k_3+k_3^{-1}-\theta k_0-\theta^{-1}k_0^{-1}}{q^{-1}\theta-q\theta^{-1}}$ $+\;\frac{k_1+k_1^{-1}}{k_0^{-1}}$
$\frac{t_0-k_0}{k_0^{-1}-k_0}\,v_{2i-2}$	$\frac{\theta\,G(\theta^{-1},k_0,k_3)\,G(q\theta^{-1},k_1,k_2)}{(\theta-\theta^{-1})(q\theta^{-1}-q^{-1}\theta)}$

where $\theta = q^{2i}\theta_0 \qquad \theta_0 = k_0k_3$

pf Similar to pf of Th 41 $\square$

Thm 44 For the H_q-module V from Thm 40

Consider the monic polys $\{F_n\}_{n=0}^{\infty}$ corresp the basis ~~xx~~:

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_{2i} \qquad i = 1, 2, 3 \ldots$$

Then

$$F_n(x) = P_n(x; a, b, c, d \mid q^2) \qquad n = 0, 1, 2 \ldots$$

where a, b, c, d is any perm of

$$q k_2 k_3, \quad \frac{q k_3}{k_2}, \quad q^2 k_0 k_1, \quad q^2 \frac{k_0}{k_1}$$

pf $F_0 = P_0 = 1$

show $\{F_n\}_{n=0}^{\infty}$ $\{P_n\}_{n=0}^{\infty}$ sat same 3-term rec.

Require

$$b_{n-2}\left(a,b,c,d/q^2\right) c_{n-1}\left(a,b,c,d/q^2\right)$$

$$= \frac{G(q^{-1}\theta,\ q^{-1}k_0^{-1},\ k_3)}{(\theta-\theta^{-1})(q^{-2}\theta-q^{2}\theta^{-1})}\ \frac{G(q^{-1}\theta,\ k_1,\ k_2)}{(q\theta^{-1}-q^{-1}\theta)^2}$$

$n=2,3\ldots$

$$a_{n-1}\left(a,b,c,d/q^2\right) =$$

$$\frac{\theta k_0^{-1}+\theta^{-1}k_0-k_3-k_3^{-1}}{\theta-\theta^{-1}}\ \frac{q^{-1}\theta^{-1}(k_1+k_1^{-1})-k_2-k_2^{-1}}{q^{-1}\theta^{-1}-q\theta}$$

$$+\ \frac{q^{-1}\theta(k_1+k_1^{-1})-k_2-k_2^{-1}}{\theta-\theta^{-1}}\ \frac{k_3+k_3^{-1}-\theta k_0-\theta^{-1}k_0^{-1}}{q^{-1}\theta-q\theta^{-1}}$$

$$+\ \frac{k_1+k_1^{-1}}{k_0^{-1}}$$

$n=1,2\ldots$

where $\theta = \theta_0 q^{2n}$ $\quad\theta_0 = k_0 k_3$

Routinely checked. $\square$

Lecture 56 Monday Nov 28 11/28/11

1

$\mathbb{F}$ alg closed

$0 \neq q \in \mathbb{F}$ $q^4 \neq 1$

We have been discussing irred $\hat{H}_q$-modules

associated with AW polys.

We now find a basis for these modules said to be "split".

With resp to a split basis

the matrix rep A is lower bidiag

" " B " upper "

Motivation to motivate the split basis we

consider some features of the AW polys.

Start with arb sequence of AW polys

$\{p_n\}_{n=0}^{\infty}$

$$p_n = p_n(x; a,b,c,d/q)$$

$$= {}_4\varphi_3\left(\begin{matrix} q^{-n} & abcd\,q^{n-1} & ay & ay^{-1} \\ ab & ac & ad & \end{matrix} \;\Big/\; q;q\right)$$

$$= \sum_{i=0}^{n} \frac{(q^{-n};q)_i\,(abcd\,q^{n-1};q)_i\,(ay;q)_i\,(ay^{-1};q)_i\;q^i}{(ab;q)_i\,(ac;q)_i\,(ad;q)_i\,(q;q)_i} \qquad *$$

$$x = y + y^{-1}$$

Def 45 Define

$$\theta_n = a q^n + a^{-1} q^{-n} \qquad n = 0, 1, 2, \ldots$$

$$\theta^*_n = q^{-n} + abcd\, q^{n-1} \qquad \text{"}$$

$$\varphi_n = a^{-1} q^{1-2n} (1 - q^n)(1 - ab q^{n-1})(1 - ac q^{n-1})(1 - ad q^{n-1})$$

$$n = 1, 2, \ldots$$

One checks

$$\theta_r - \theta_s = (1 - q^{r-s})(1 - a^2 q^{r+s})\, a^{-1} q^{-r}$$
$$= -(1 - q^{s-r})(1 - a^2 q^{r+s})\, a^{-1} q^{-s}$$

$$x - \theta_s = -(1 - ay q^s)(1 - a y^{-1} q^s)\, a^{-1} q^{-s}$$

$$\theta^*_r - \theta^*_s = (1 - q^{r-s})(1 - abcd\, q^{r+s-1})\, q^{-r}$$
$$= -(1 - q^{s-r})(1 - abcd\, q^{r+s-1})\, q^{-s}$$

LEM 46 With above notation, for $0 \le i \le n$ the ith summand in $*$ is

$$\frac{(\theta_n^* - \theta_0^*)(\theta_n^* - \theta_1^*) \cdots (\theta_n^* - \theta_{i-1}^*)(x - \theta_0)(x - \theta_1) \cdots (x - \theta_{i-1})}{\varphi_1 \varphi_2 \cdots \varphi_i} \qquad **$$

pf let

$t_i = i$th summand in $*$ $\qquad 0 \le i \le n$

obs $t_0 = 1$

let $\tilde{t}_i =$ expression in $**$ $\qquad 0 \le i \le n$

obs $\tilde{t}_0 = 1$

Show $\dfrac{t_i}{t_{i-1}} = \dfrac{\tilde{t}_i}{\tilde{t}_{i-1}} \qquad 1 \le i \le n$

observe:

$$t_i/t_{i-1} =$$

$$\frac{(1-q^{i-1-n})(1-abcdq^{i+n-2})(1-ayq^{i-1})(1-ay^{-1}q^{i-1})\,q}{(1-abq^{i-1})(1-acq^{i-1})(1-adq^{i-1})(1-q^{i})}$$

$$\tilde{t}_i/\tilde{t}_{i-1} =$$

$$\frac{(\theta^*_n-\theta^*_{i-1})(x-\theta_{i-1})}{\varphi_i}$$

Using this we checks

$$t_i/t_{i-1} = \tilde{t}_i/\tilde{t}_{i-1}$$

$\square$

Cor 47 With the above notation

$$p_n = \sum_{i=0}^{n} \frac{(\theta_n^* - \theta_0^*)(\theta_n^* - \theta_1^*) \cdots (\theta_n^* - \theta_{i-1}^*)(x-\theta_0)(x-\theta_1) \cdots (x - \theta_{i-1})}{\varphi_1 \varphi_2 \cdots \varphi_i}$$

pf By Lem 46 and def of p_n $\square$

Def 48 For the above AW polys $\{p_n\}_{n=0}^{\infty}$

the corresp parameter array is the sequence

$$\left(\{\theta_n\}_{n=0}^{\infty}, \{\theta_n^*\}_{n=0}^{\infty}, \{\varphi_n\}_{n=1}^{\infty} \right)$$

By Cor 47 the p_n are determined by their parameter array

For our AW polys $\{p_n\}_{n=0}^{\infty}$ recall the 3-term rec

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0,1,2,\ldots$$

$$p_{-1} = 0$$

where

$$c_n = c_n(a,b,c,d \mid q) \quad \text{etc.}$$

We now express

$$c_n,\ a_n,\ b_n$$

in terms of the parameter array.

thm 49 With above notation

$$b_n = \varphi_{n+1} \frac{(\theta_n^* - \theta_0^*)(\theta_n^* - \theta_1^*) \cdots (\theta_n^* - \theta_{n-1}^*)}{(\theta_{n+1}^* - \theta_0^*)(\theta_{n+1}^* - \theta_1^*) \cdots (\theta_{n+1}^* - \theta_n^*)} \qquad n = 0, 1, 2, \ldots$$

$$a_n = \theta_n + \frac{\varphi_n}{\theta_n^* - \theta_{n-1}^*} + \frac{\varphi_{n+1}}{\theta_n^* - \theta_{n+1}^*} \qquad n = 1, 2, \ldots$$

$$a_0 = \theta_0 + \frac{\varphi_1}{\theta_0^* - \theta_1^*}$$

$$c_n = \theta_0 - a_n - b_n \qquad n = 0, 1, 2, \ldots$$

pf b_n: By Cor 47

$$\text{coef of } x^n \text{ in } p_n = \frac{(\theta_n^* - \theta_0^*) \cdots (\theta_n^* - \theta_{n-1}^*)}{\varphi_1 \varphi_2 \cdots \varphi_n}$$

We saw earlier

$$\text{coef of } x^n \text{ in } p_n = \frac{1}{b_0 b_1 \cdots b_{n-1}}$$

a_n: Consider coef of X^n in p_{n+1}

Using 3-term rec,

$$p_{n+1} = \frac{X^{n+1} - (a_0 + a_1 + \cdots + a_n) X^n + \cdots LT}{b_0 b_1 \cdots b_n}$$

$$\text{coef of } X^n = -\frac{a_0 + a_1 + \cdots + a_n}{b_0 b_1 \cdots b_n}$$

$$= -(a_0 + a_1 + \cdots + a_n) \frac{(\theta_{n+1}^* - \theta_0^*) \cdots (\theta_{n+1}^* - \theta_n^*)}{\varphi_1 \varphi_2 \cdots \varphi_{n+1}} \qquad (1)$$

Using Cor 47

$$p_{n+1} = (X - \theta_0) \cdots (X - \theta_n) \frac{(\theta_{n+1}^* - \theta_0^*) \cdots (\theta_{n+1}^* - \theta_n^*)}{\varphi_1 \cdots \varphi_{n+1}}$$

$$+ (X - \theta_0) \cdots (X - \theta_{n-1}) \frac{(\theta_{n+1}^* - \theta_0^*) \cdots (\theta_{n+1}^* - \theta_{n-1}^*)}{\varphi_1 \cdots \varphi_n}$$

$$+ LT$$

So

coef of X^n in p_{n+1} =

$$-(\theta_0 + \theta_1 + \cdots + \theta_n) \frac{(\theta_{n+1}^* - \theta_0^*) \cdots (\theta_{n+1}^* - \theta_n^*)}{\varphi_1 \cdots \varphi_{n+1}}$$

$$+ \frac{(\theta_{n+1}^* - \theta_0^*) \cdots (\theta_{n+1}^* - \theta_{n-1}^*)}{\varphi_1 \cdots \varphi_n} \qquad (2)$$

Comparing (1), (2)

$$a_0 + a_1 + \cdots + a_n = \theta_0 + \theta_1 + \cdots + \theta_n - \frac{\varphi_{n+1}}{\theta_{n+1}^{*} - \theta_n^{*}}$$

result follows

c_n: Recall

$$c_n + a_n + b_n = a + a^{-1} \qquad n = 0, 1, 2 \ldots$$

$$\| \quad \theta_0$$

□

AW polys $\{p_n\}_{n=0}^{\infty}$ as above

Consider $\mathbb{F}$-vector space

$$V = \mathbb{F}[x]$$

View $\{p_n\}_{n=0}^{\infty}$ as basis for V

We define $\mathbb{F}$-lin trans

$$A : V \to V \qquad B : V \to V$$

as follows.

$$A : \begin{array}{ccc} V & \longrightarrow & V \\ f(x) & \longrightarrow & x f(x) \end{array}$$ "mult by x"

$$B p_n = \theta_n^* p_n \qquad n = 0, 1, 2 \ldots$$

Rel the basis $\{p_n\}_{n=0}^{\infty}$ the matrices rep A, B are:

$$A : \begin{pmatrix} a_0 & c_1 & & & \\ b_0 & a_1 & c_2 & & \mathbf{0} \\ & b_1 & & & \\ & & \ddots & \ddots & \ddots \\ & \mathbf{0} & & & \end{pmatrix}$$

$c_n = c_n(a, b, c, d | q)$ etc

$$B : \operatorname{diag}(\theta_0^*, \theta_1^*, \ldots \quad)$$

Earlier we saw how certain $\hat{H}_q$-modules give

the AW polys.

In $\hat{H}_q$ the elements $A = Y + Y^{-1}$ and $B = X + X^{-1}$ satisfy

the $\mathbb{Z}_3$-sym AW rels

These A, B roughly corresp to above lin trans $A: V \to V$, $B: V \to V$

but the normalization is different.

So we expect above $A: V \to V$, $B: V \to V$ to satisfy

some relations resembling the AW relations.

thm 50 Ref to the above lin trans

$A: V \to V$, $\qquad B: V \to V$

$$A^2B - (q+q^{-1})ABA + BA^2 + (q-q^{-1})^2 B = \omega A + \zeta I,$$

$$B^2A - (q+q^{-1})BAB + AB^2 + abcdq^{-1}(q-q^{-1})^2 A = \omega B + \zeta^* I,$$

$$\omega = -q^{-1}(q-1)^2 e_1 - q^{-2}(q-1)^2 e_3$$

$$\zeta = q^{-1}(q+1)(q-1)^2 + q^{-2}(q+1)(q-1)^2 e_2 + q^{-3}(q+1)(q-1)^2 e_4$$

$$\zeta^* = q^{-3}(q+1)(q-1)^2 e_1 e_4 + q^{-2}(q+1)(q-1)^2 e_3$$

where

$e_1 = a+b+c+d$

$e_2 = ab+ac+ad+bc+bd+cd$

$e_3 = abc+abd+acd+bcd$

$e_4 = abcd$

pf Matrix mult, using identities on next page

$$\theta^*_{n-1} - (q+q^{-1})\theta^*_n + \theta^*_{n+1} = 0 \qquad n=1,2,\ldots$$

$$\theta^{*2}_{n-1} - (q+q^{-1})\theta^*_{n-1}\theta^*_n + \theta^{*2}_n + (q-q^{-1})^2 abcd q^{-1} = 0$$

$n=1,2,\ldots$

$$a_n\left(\theta^{*2}_n(2-q-q^{-1}) + abcdq^{-1}(q-q^{-1})^2\right) = w\theta^*_n + \eta^*$$

$n=0,1,2\ldots$

$$(\theta^*_n - \theta^*_{n-1})(\theta^*_n - \theta^*_{n+1}) = \theta^{*2}_n(2-q-q^{-1}) + abcd\, q^{-1}(q-q^{-1})^2$$

$n=1,2,\ldots$

$$w = a_n(\theta^*_n - \theta^*_{n+1}) + a_{n-1}(\theta^*_{n-1} - \theta^*_{n-2}) \qquad n=2,3\ldots$$

$0 =$

term	coef
$b_{n-1}c_n$	$2\theta^*_n - (q+q^{-1})\theta^*_{n-1}$
$b_n c_{n+1}$	$2\theta^*_n - (q+q^{-1})\theta^*_{n+1}$
a_n^2	$2\theta^*_n - (q+q^{-1})\theta^*_n$
a_n	$-w$
1	$-\eta + (q-q^{-1})^2\theta^*_n$

$\mathbb{F}$ alg closed

$0 \neq q \in \mathbb{F}$ $q^4 \neq 1$

Continue to discuss AW polys $\{p_n\}_{n=0}^{\infty}$

$$p_n = p_n(x; a, b, c, d \mid q)$$

with PA $(\{\theta_n\}_{n=0}^{\infty}, \{\theta_n^*\}_{n=0}^{\infty}, \{\varphi_n\}_{n=1}^{\infty})$

Recall $\{p_n\}_{n=0}^{\infty}$ is a basis for $V = \mathbb{F}[x]$

Recall lin trans

$$\begin{aligned} A : V &\to V \\ f &\to xf \end{aligned}$$

$$B p_n = \theta_n^* p_n \qquad n = 0, 1, 2 \ldots$$

Obs that the following is a basis for V:

$$(x-\theta_0)(x-\theta_1)\cdots(x-\theta_{n-1}) \qquad n = 0, 1, 2, \ldots \qquad *$$

thm 51 With respect to the bases *
the matrices representing A, B are:

$$A: \begin{pmatrix} \theta_0 & & & & & \\ 1 & \theta_1 & & & & 0 \\ & 1 & \theta_2 & & & \\ & & \ddots & \ddots & & \\ & 0 & & & & \end{pmatrix}$$

Lower bidiag

$$B: \begin{pmatrix} \theta_0^{\flat} & \psi_1 & & & \\ & \theta_1^{\flat} & \psi_2 & & 0 \\ & & \theta_2^{\flat} & & \\ & & & \ddots & \ddots \\ & 0 & & & \end{pmatrix}$$

upper bidiag

pf For $n = 0, 1, 2 \ldots$ define

$$v_n = (x-\theta_0)(x-\theta_1) \cdots (x-\theta_{n-1})$$

and note

$$(x-\theta_n)\, v_n = v_{n+1}$$

A: show

$$\underbrace{(A-\theta_n)\, v_n}_{\shortparallel} \overset{?}{=} v_{n+1} \qquad n = 0, 1, 2 \ldots$$

$(x-\theta_n)\, v_n$

$\shortparallel$

v_{n+1} ✓

B: show

$$(B-\theta_n^*)\, v_n = \varphi_n v_{n-1} \qquad n = 0, 1, 2 \ldots \qquad (1)$$

$\varphi_0 = 0 \qquad v_{-1} = 0$

Use ind on n.

<u>$n = 0$:</u>

$$B p_0 = \theta_0^* p_0$$

$$p_0 = 1 = v_0$$

so

$$B v_0 = \theta_0^* v_0 \quad ✓$$

<u>$n \geq 1$:</u>

By Cor 47.

$$p_n = \sum_{i=0}^{n} \frac{(\theta_n^* - \theta_0^*) \cdots (\theta_n^* - \theta_{i-1}^*)}{\varphi_1 \cdots \varphi_i} v_i$$

Apply $B - \theta_n^* I$ to both sides and use induction

$$0 = \sum_{i=0}^{n-1} \frac{(\theta_n^* - \theta_0^*) \cdots (\theta_n^* - \theta_{i-1}^*)}{\varphi_1 \cdots \varphi_i} \Big((\theta_i^* - \theta_n^*) v_i + \varphi_i v_{i-1} \Big)$$

$$+ \frac{(\theta_n^* - \theta_0^*) \cdots (\theta_n^* - \theta_{n-1}^*)}{\varphi_1 \cdots \varphi_n} \left(B - \theta_n^* I \right) v_n$$

=

term	coef	
v_0	$\theta_0^* - \theta_n^* + \frac{\theta_n^* - \theta_0^*}{\varphi_1} \varphi_1$	$\longrightarrow 0$
v_1	$\frac{(\theta_n^* - \theta_0^*)(\theta_1^* - \theta_n^*)}{\varphi_1} + \frac{(\theta_n^* - \theta_0^*)(\theta_n^* - \theta_1^*)}{\varphi_1 \varphi_2} \varphi_2$	$\longrightarrow 0$
v_2	$\frac{(\theta_n^* - \theta_0^*)(\theta_n^* - \theta_1^*)}{\varphi_1 \varphi_2} (\theta_2^* - \theta_n^*)$	
$\vdots$	$\vdots$	
v_{n-2}		$\longrightarrow 0$
v_{n-1}	$\frac{(\theta_n^* - \theta_0^*) \cdots (\theta_n^* - \theta_{n-2}^*)}{\varphi_1 \cdots \varphi_{n-1}} (\theta_{n-1}^* - \theta_n^*)$	
$(B - \theta_n^* I) v_n$	$\frac{(\theta_n^* - \theta_0^*) \cdots (\theta_n^* - \theta_{n-1}^*)}{\varphi_1 \cdots \varphi_n}$	

$4\varphi^2$

$$= \underbrace{\frac{(\theta_n^* - \theta_0^*)\cdots(\theta_n^* - \theta_{n-1}^*)}{\varphi_1 \cdots \varphi_n}}_{\neq 0} \Big(\underbrace{(B - \theta_n^* I)\, v_n - \varphi_n v_{n-1}}_{\text{must be } 0} \Big)$$

So

$$(B - \theta_n^* I)\, v_n = \varphi_n v_{n-1} \qquad \checkmark \qquad \square$$

— o —

In pr 50 we displayed two relations sat by the lin trans $A : V \to V$, $B : V \to V$

If we represent A, B by the matrices in pr 51 each entry gives an identity involving the parameter array.

The resulting identities are given below

LEM 52 For our AW polys $\{p_n\}_{n=0}^{\infty}$

the corresp PA satisfies

$$0 = \theta_{n-1} - (q+q^{-1})\theta_n + \theta_{n+1} \qquad n=1,2,\ldots$$

$$0 = \theta_{n-1}^2 - (q+q^{-1})\theta_{n-1}\theta_n + \theta_n^2 + (q-q^{-1})^2 \qquad n=1,2,\ldots$$

$$0 = \theta^*_{n-1} - (q+q^{-1})\theta^*_n + \theta^*_{n+1} \qquad \text{''}$$

$$0 = \theta^{*2}_{n-1} - (q+q^{-1})\theta^*_{n-1}\theta^*_n + \theta^{*2}_n + abcd\,q^{-1}(q-q^{-1})^2 \qquad \text{''}$$

$$\omega = \varphi_{n-1} - (q+q^{-1})\varphi_n + \varphi_{n+1} + \theta_{n-1}\theta^*_n + \theta_n\theta^*_{n-1} \qquad n=1,2\ldots$$
$$+ (1-q-q^{-1})(\theta_{n-1}\theta^*_{n-1} + \theta_n\theta^*_n)$$

$(\varphi_0 = 0)$

$$\eta = \varphi_n(\theta_n - \theta_{n-1}) + \varphi_{n+1}(\theta_n - \theta_{n+1}) + (q-q^{-1})^2\theta^*_n \qquad \text{''}$$
$$- q^{-1}(q-1)^2\theta_n^2\theta^*_n - \omega\theta_n$$

$$\eta^* = \varphi_n(\theta^*_n - \theta^*_{n-1}) + \varphi_{n+1}(\theta^*_n - \theta^*_{n+1}) + abcd\,q^{-1}(q-q^{-1})^2\theta_n \qquad \text{''}$$
$$- q^{-1}(q-1)^2\theta_n\theta^{*2}_n - \omega\theta^*_n$$

pf Routine verification using def of $\theta_n, \theta^*_n, \varphi_n$ $\square$

For our AW polys $\{p_n\}_{n=0}^{\infty}$ we saw that corresp lin trans $A: V \to V$ $B: V \to V$ satisfy the eqs of th 50.

So there should be a module str on V for the univ AW algebra $\Delta = \Delta_{q^{1/2}}$

We now display this module.
UNTIL Further notice fix square roots
$q^{1/2}, a^{1/2}, b^{1/2}, c^{1/2}, d^{1/2}$

and define

$$\tilde{\theta}_n^* = (abcdq^{-1})^{-1/2} q^{-n} + (abcdq^{-1})^{1/2} q^n \qquad n = 0, 1, 2, \ldots$$

$$= (abcdq^{-1})^{-1/2} \theta_n^*$$

Define

$$\tilde{B} = (abcdq^{-1})^{-1/2} B$$

So

$$\tilde{B} p_n = \tilde{\theta}_n^* p_n \qquad n = 0, 1, 2, \ldots$$

LEM 53 with above notation

$$\frac{A^2 \tilde{B} - (q+q^{-1}) A \tilde{B} A + \tilde{B} A^2 + (q-q^{-1})^2 \tilde{B} + (q^{1/2}-q^{-1/2})^2 A \gamma}{(q^{1/2}-q^{-1/2})(q-q^{-1})} = \beta,$$

$$\frac{\tilde{B}^2 A - (q+q^{-1}) \tilde{B} A \tilde{B} + A \tilde{B}^2 + (q-q^{-1})^2 A + (q^{1/2}-q^{-1/2})^2 \tilde{B} \gamma}{(q^{1/2}-q^{-1/2})(q-q^{-1})} = \alpha$$

where

$$\alpha = \frac{z^*}{(q^{1/2}-q^{-1/2})(q-q^{-1})\, abcd\, q^{-1}}$$

$$= q^{-1/2} e_1 + q^{1/2} e_3/e_4$$

$$= q^{-1/2}(a+b+c+d) + q^{1/2}(a^{-1}+b^{-1}+c^{-1}+d^{-1})$$

$$\beta = \frac{z}{(q^{1/2}-q^{-1/2})(q-q^{-1})(abcd\,q^{-1})^{1/2}}$$

$$= q\, e_4^{-1/2} + q^{-1} e_4^{1/2} + e_2/e_4^{1/2}$$

$$= \frac{q}{(abcd)^{1/2}} + \frac{(abcd)^{1/2}}{q} + \left(\frac{ab}{cd}\right)^{1/2} + \left(\frac{ac}{bd}\right)^{1/2} + \left(\frac{ad}{bc}\right)^{1/2}$$

$$+ \left(\frac{cd}{ab}\right)^{1/2} + \left(\frac{bd}{ac}\right)^{1/2} + \left(\frac{bc}{ad}\right)^{1/2}$$

$$\gamma = -\frac{\omega}{(q^{1/2}-q^{-1/2})^2\,(abcdq^{-1})^{1/2}}$$

$$= \frac{q^{1/2}e_1 + q^{-1/2}e_3}{e_4^{\,1/2}}$$

$$= q^{\frac{1}{2}}\frac{(a+b+c+d)}{(abcd)^{1/2}} + q^{-1/2}(a^{-1}+b^{-1}+c^{-1}+d^{-1})(abcd)^{1/2}$$

pf Routine adjustment of Th 5.0 $\square$

We now put the equations of L53 in $\mathbb{Z}_3$-symmetric form. Define

$$C = \frac{\gamma I}{q^{1/2} + q^{-1/2}} - \frac{q^{1/2} A\widetilde{B} - q^{-1/2}\widetilde{B}A}{q - q^{-1}}$$

Thm 54 With above notation

$$A + \frac{q^{1/2}\widetilde{B}C - q^{-1/2}C\widetilde{B}}{q - q^{-1}} = \frac{\alpha}{q^{1/2} + q^{-1/2}} I$$

$$\widetilde{B} + \frac{q^{1/2}CA - q^{-1/2}AC}{q - q^{-1}} = \frac{\beta}{q^{1/2} + q^{-1/2}} I$$

$$C + \frac{q^{1/2}A\widetilde{B} - q^{-1/2}\widetilde{B}A}{q - q^{-1}} = \frac{\gamma}{q^{1/2} + q^{-1/2}} I$$

where α, β, γ are from L53

pf This is L53 with a change of variables. $\square$

thm 54 gives a module structure
on $V = \mathbb{F}[x]$ for the univ Askey-Wilson algebra
$\Delta_{q^{1/2}}$

Recall the H_q-module V from ....

Diag is

∞ path

On $\dfrac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V$ $\qquad$ t_0 acts as k_0

Recall from Th 39 $\qquad$ a, b, c, d is perm of

$$k_0 k_1, \quad \frac{q^2 k_1}{k_0}, \quad q k_2 k_3, \quad \frac{q k_2}{k_3}$$

Consider the actions of the following on V:

$$\alpha = (q^{-1} t_0 + q t_0^{-1}) T_1 + T_2 T_3$$

$$\beta = (q^{-1} t_0 + q t_0^{-1}) T_3 + T_1 T_2$$

$$\gamma = (q^{-1} t_0 + q t_0^{-1}) T_2 + T_1 T_3$$

2 ways to compute action on $\dfrac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V$

(1) α acts as

$$(q^{-1} k_0 + q k_0^{-1})(k_1 + k_1^{-1}) + (k_2 + k_2^{-1})(k_3 + k_3^{-1})$$

(2) α acts as in LS3

Similar for β, γ.

Check actions are same

$$e_1 = a + b + c + d$$

$$= q k_1 (q k_0^{-1} + q^{-1} k_0) + q k_2 (k_3 + k_3^{-1})$$

$$e_2 = q^2 K_1^2 + q^2 K_2^2 + q^2 K_1 K_2 (K_3 + K_3^{-1})(q^{-1}K_0 + qK_0^{-1})$$

$$e_4 = abcd$$

$$= q^4 K_1^2 K_2^2$$

$$\frac{e_3}{e_4} = a^{-1} + b^{-1} + c^{-1} + d^{-1}$$

$$= q^{-1}K_1^{-1}(qK_0^{-1} + q^{-1}K_0) + q^{-1}K_2^{-1}(K_3 + K_3^{-1})$$