

Let V denote an \mathbb{F} -module
on which X is diagonalizable.

We define the X -diagram for V as follows.

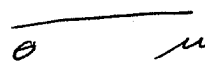
nodes

these are the eigenspaces of X on V

We label each node with the corresp eigenvalue

arcs


For nodes θ, μ

 whenever $\theta\mu = 1$

 whenever $\theta\mu = q^{-2}$

We allow loops:

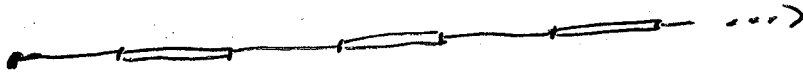
 whenever $\theta^2 = 1$

 whenever $\theta^2 = q^{-2}$

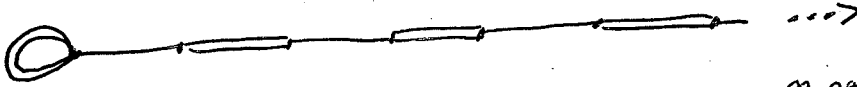
Possible X-diagrams



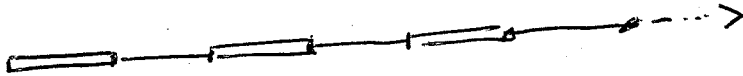
∞ path



∞ path



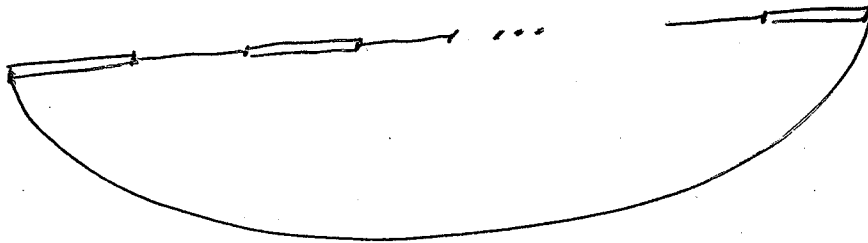
∞ path



∞ path



∞ path

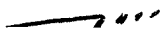
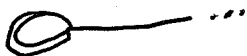


cycle



Finite path

Many versions; choices at each end are



Next goal: careful description of
an unred \hat{H}_1 -module whose X-diagram is $\infty \infty$ with

Until further notice for $0 \neq k_i \in \mathbb{F}$ $i \in \mathbb{I}$

Def functions

$$a, b, c, d : \mathbb{F} \setminus \{0, 1, -1\} \rightarrow \mathbb{F}$$

as follows.

$$a(\theta) = \frac{\theta(k_3 + k_3^{-1}) - k_0 - k_0^{-1}}{\theta - \theta^{-1}}$$

$$d(\theta) = a(\theta^{-1})$$

$$b(\theta) = \frac{G(\theta, k_0, k_3)}{\theta - \theta^{-1}}$$

$$c(\theta) = \frac{1}{\theta^{-1} - \theta}$$

Define 2x2 matrices

$$T_3(\theta) = \begin{pmatrix} a(\theta) & b(\theta) \\ c(\theta) & d(\theta) \end{pmatrix}$$

$$T_0(\theta) = \begin{pmatrix} \theta d(\theta) & \frac{-b(\theta)}{\theta} \\ -\theta c(\theta) & \frac{a(\theta)}{\theta} \end{pmatrix}$$

One checks

matrix	trace	det
$\tau_3(\theta)$	$k_3 + k_3^{-1}$	1
$\tau_0(\theta)$	$k_0 + k_0^{-1}$	1

$$\tau_3(\theta) + \tau_3(\theta)^{-1} = (k_3 + k_3^{-1}) I$$

$$\tau_0(\theta) + \tau_0(\theta)^{-1} = (k_0 + k_0^{-1}) I$$

$$\tau_3(\theta) \tau_0(\theta) = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}$$

$$\tau_0(\theta) - \tau_3(\theta) \tau_0(\theta) \tau_3(\theta)^{-1} = \begin{pmatrix} 0 & G(\theta, k_0, k_3) \\ 1 & 0 \end{pmatrix}$$

Ref to Prop 15,

relative the basis v, Gv

the matrices representing t_3, t_0 are

$T_3(0), T_0(0)$ resp.

Define functions

$$\alpha, \beta, \gamma, \delta : \mathbb{F} \setminus \{0, 1, -1\} \rightarrow \mathbb{F}$$

$$\alpha(\theta) = \frac{\theta(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{\theta - \theta^{-1}}$$

$$\delta(\theta) = \alpha(\theta^{-1})$$

$$\beta(\theta) = \frac{G(\theta, k_1, k_2)}{\theta - \theta^{-1}}$$

$$\gamma(\theta) = \frac{1}{\theta^{-1} - \theta}$$

To go from a, b, c, d to $\alpha, \beta, \gamma, \delta$

switch $k_0 \leftrightarrow k_2$
 $k_1 \leftrightarrow k_3$

Define 2×2 matrices

$$\tau_1(\theta) = \begin{pmatrix} \alpha(\theta) & \beta(\theta) \\ \gamma(\theta) & \delta(\theta) \end{pmatrix}$$

$$\tau_2(\theta) = \begin{pmatrix} \theta \delta(\theta) & -\frac{\beta(\theta)}{\theta} \\ -\theta \gamma(\theta) & \frac{\alpha(\theta)}{\theta} \end{pmatrix}$$

One checks

matrix	trace	det
$\tau_1(\theta)$	$k_1 + k_1^{-1}$	1
$\tau_2(\theta)$	$k_2 + k_2^{-1}$	1

$$\tau_1(\theta) + \tau_1(\theta)^{-1} = (k_1 + k_1^{-1}) I$$

$$\tau_2(\theta) + \tau_2(\theta)^{-1} = (k_2 + k_2^{-1}) I$$

$$\tau_1(\theta) \tau_2(\theta) = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}$$

$$\tau_2(\theta) - \tau_1(\theta) \tau_2(\theta) \tau_1(\theta)^{-1} = \begin{pmatrix} 0 & G(\theta, k_1, k_2) \\ 1 & 0 \end{pmatrix}$$

Ref to Prop 20.

relative to the basis v_1, \dots, v_n

the matrices rep t_1, t_2 are

$$T_1(q^{-1}\theta^{-1}), \quad T_2(q^{-1}\theta^{-1})$$

resp.

For $i \in \mathbb{I}$ def a matrix τ_i

with rows/cols indexed by \mathbb{Z}

and entries in \mathbb{F} .

Fix $0 \neq \theta_0 \in \mathbb{F}$

The entries of τ_i are given below

(all entries not shown are 0)

For even $i \in \mathbb{Z}$

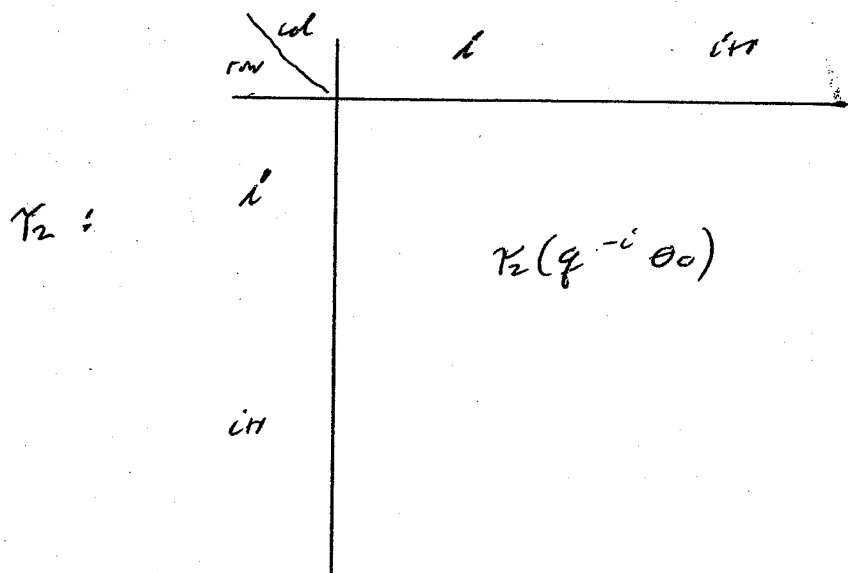
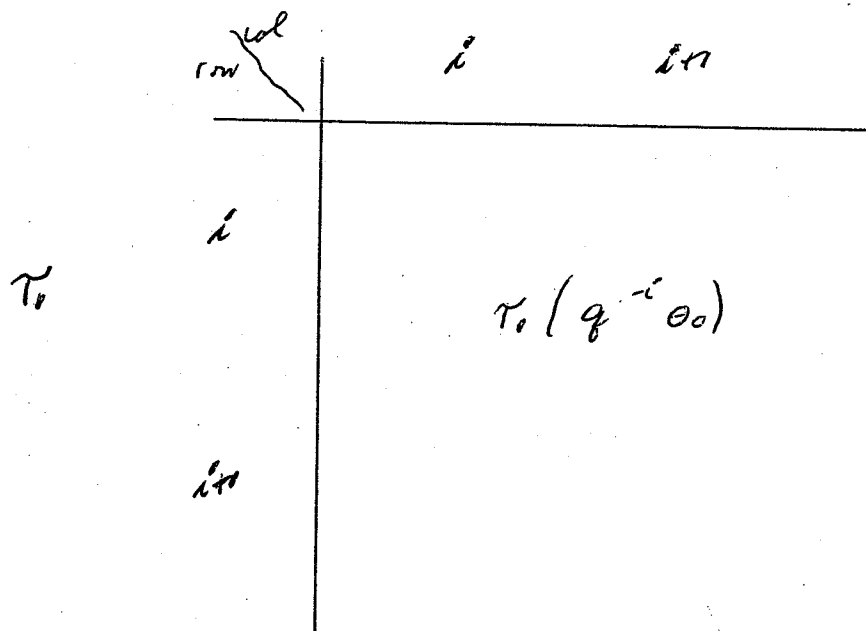
$\tau_3 :$

	col		
row		i	$i+1$
i		$\tau_3(\theta_0 q^{-i})$	
$i+1$			

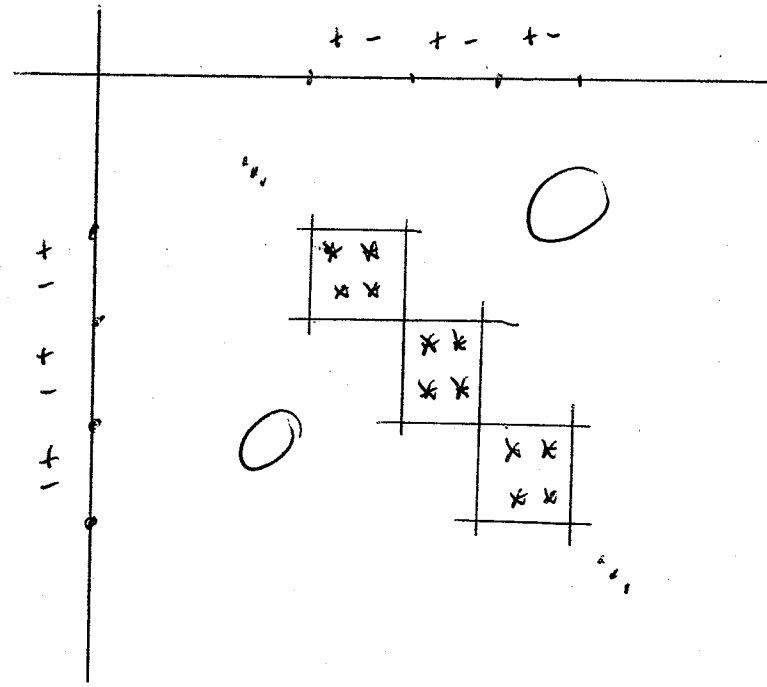
$\tau_0 :$

	col		
row		i	$i+1$
i		$\tau_0(\theta_0 q^{-i})$	
$i+1$			

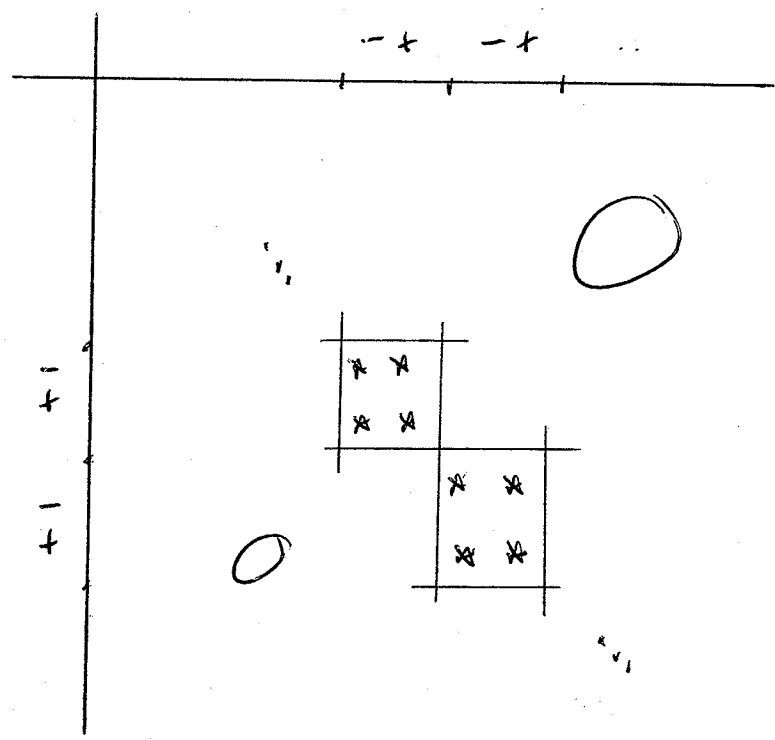
F_n odd $i \in \mathbb{Z}$



$\gamma_0, \gamma_3 \text{ :}$



$\gamma_1, \gamma_2 \text{ :}$



+ means even
- means odd

397

Show $\{\tau_i\}_{i \in \mathbb{Z}}$ satisfy the defining relations for \hat{H}_q .

By constr.

$$\tau_i + \tau_i^{-1} = (k_i + k_i^{-1}) I \quad \forall i \in \mathbb{Z}$$

central

$\tau_3 \tau_0$ is diagonal with (i, i) -entry

$$\begin{cases} \theta_0 q^{-i} & \text{if } i \text{ even} \\ \theta_0^{-1} q^{i-1} & \text{if } i \text{ odd} \end{cases} \quad i \in \mathbb{Z}$$

$\tau_1 \tau_2$ is diagonal with (i, i) -entry

$$\begin{cases} \theta_0^{-1} q^{i-1} & \text{if } i \text{ even} \\ \theta_0 q^{-i} & \text{if } i \text{ odd} \end{cases} \quad i \in \mathbb{Z}$$

By these comments

$$\tau_3 \tau_0 \tau_1 \tau_2 = q^{-1} I$$

So

$$\tau_0 \tau_1 \tau_2 \tau_3 = q^{-1} I$$

We have shown $\{\tau_i\}_{i \in \mathbb{Z}}$ satisfy the defining rels for \hat{H}_q

378

the equals of $\gamma_3 \gamma_0$ are

$$\dots q^{2\theta_0}, q^{-2\theta_0}, \theta_0, \theta_0^{-1}, q^{-2}\theta_0, q^2\theta_0^{-1}, q^{-4}\theta_0, \dots$$

*

TFAE

- (i) * are not dist
- (ii) q is not a root of 1 and $\theta_0 \notin \{ \pm q^n \mid n \in \mathbb{Z} \}$

(1)

TFAE

- (i) $G(\theta, k_0/k_3) \neq 0 \quad \forall \theta \in \mathbb{C}^*$
- (ii) $\theta_0 \notin \left\{ q^n k_0/k_3, q^n k_3/k_0, q^n k_0/k_3, \frac{q^n}{k_0 k_3} \mid n \in \mathbb{Z}, n \text{ even} \right\}$ (2)

TFAE

- (i) $G(q^{-1}\theta^{-1}, k_1/k_2) \neq 0 \quad \forall \theta \in \mathbb{C}^*$
- (ii) $\theta_0 \notin \left\{ q^n k_1/k_2, q^n k_2/k_1, q^n k_1/k_2, \frac{q^n}{k_1 k_2} \mid n \in \mathbb{Z}, n \text{ odd} \right\}$ (3)

Thm 36 Given $0 \neq k_i \in \mathbb{F}$ ($i \in \mathbb{I}$)

Given $0 \neq \theta_0 \in \mathbb{F}$ that satisfies (1)-(3)

Then \exists an \hat{H}_q -module $V = V(k_0, k_1, k_2, k_3, \theta_0)$

with the following property:

V has a basis $\{v_i\}_{i \in \mathbb{Z}}$ with resp to which the matrices rep to τ_i ($i \in \mathbb{I}$).

Moreover

• For $i \in \mathbb{Z}$, v_i is an eigenvector for X . The eigenval is

$$\begin{cases} \theta_0 q^{-i} & \text{if } i \text{ even} \\ \theta_0 q^{i\alpha} & \text{if } i \text{ odd} \end{cases}$$

• V is irreducible

• X is diagonalizable on V_i and all eigenspaces of X on V have dim 1

• the X -diagram for V is

$$\langle \dots \text{---} Fv_2 \text{---} Fv_1 \text{---} Fv_0 \text{---} Fv_1 \text{---} Fv_2 \text{---} Fv_3 \text{---} \dots \rangle$$

(∞ ∞ path)

• For $i \in \mathbb{Z}$

$$G_0 v_i = v_{i+1} \quad \text{if } i \text{ even}$$

$$G_2 v_i = v_{i+1} \quad \text{if } i \text{ odd}$$

pf Each claim already shown or routine. □

We now consider the implications of Thm 35

for the module $V = V(k_0, k_1, k_2, k_3, \theta_0)$ from Th 36

Motivation:

With ref to Th 36 assume $k_0 \neq \mathbb{F}1$.

so that

$$V = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V + \frac{t_0 - k_0}{k_0^{-1} - k_0} V \quad ds$$

For all even $i \in \mathbb{Z}$

v_i is basis for $V_x(\theta)$

$$\theta = \theta_0 z^{-i}$$

v_{i+1} --- $V_x(\theta^{-1})$

$V_x(\theta) + V_x(\theta^{-1}) = V_B(\theta + \theta^{-1})$ has basis

v_i, v_{i+1}

and another basis

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_i$$

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_i$$

So the following is a basis for

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V :$$

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_i$$

$i \in \mathbb{Z}, i \text{ even}$

Also the following is a basis for

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} V :$$

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_i$$

$i \in \mathbb{Z}, i \text{ even}$

Thm 37

With ref to Thm 36

assume $k_0 \neq \pm 1$.

Then for $i \in \mathbb{Z}$

$$A \frac{t_0 - k_0^{-i}}{k_0 - k_0^{-i}} v_{2i} =$$

term	coef
$\frac{t_0 - k_0^{-i}}{k_0 - k_0^{-i}} v_{2i}$	$\frac{G(q\theta, k_1, k_2) G(q\theta, k_0 q^{-i}, k_3)}{q^2 \theta (\theta - \theta^{-1})(q^{-i} \theta^{-1} - q\theta)}$
$\frac{t_0 - k_0^{-i}}{k_0 - k_0^{-i}} v_{2i}$	$\frac{\theta k_0 + \theta^{-1} k_0^{-i} - k_3 - k_3^{-i}}{\theta - \theta^{-1}} \frac{q^{-i} \theta^{-1} (k_1 + k_1^{-i}) - k_2 - k_2^{-i}}{q^{-i} \theta^{-1} - q\theta}$ $+$ $\frac{q^{-i} \theta (k_1 + k_1^{-i}) - k_2 - k_2^{-i}}{\theta - \theta^{-1}} \frac{k_3 + k_3^{-i} - \theta k_0^{-i} - \theta^{-1} k_0}{q^{-i} \theta - q\theta^{-1}}$ $+$ $\frac{k_1 + k_1^{-i}}{k_0}$
$\frac{t_0 k_0^i}{k_0 k_0^i} v_{2i}$	$\frac{\theta}{(\theta - \theta^{-1})(q\theta^{-1} - q^{-i}\theta)}$

where $\theta = \theta_0 q^{-2i}$

pf In 1h35 take

$$v = v_{2i}$$

So $v \in V_X(\theta)$

$$G_2 G_0 v = v_{2i+2}$$

Find rel between

$$G_2 v, v_{2i-2}$$

We have

$$G_2 G_0 v_{2i-2} = v_{2i} = v$$

So

$$G_2^2 G_0 v_{2i-2} = G_2 v$$

$$v_{2i-2} \in V_X(q^2 \theta)$$

$$G_0 v_{2i-2} \in V_X(q^{-2} \theta^{-1})$$

by Cor 9

G_2^2 acts on $V_X(q^{-2} \theta^{-1})$ as $G(q\theta, k_1, k_2) I$

So far

$$G_2 v = G(q\theta, k_1, k_2) G_0 v_{2i-2}$$

So

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} G_2 v = G(q\theta, k_1, k_2) \underbrace{\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} G_0 v_{2i-2}}_{\parallel L33}$$

$$\frac{k_3 + k_3^{-1} - q^2 \theta k_0^{-1} - q^{-2} \theta^{-1} k_0}{q^2 \theta} \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i-2}$$

183

One checks

$$\frac{\theta k_0 + \theta^{-1} k_0^{-1} - k_3 - k_3^{-1}}{(\theta - \theta^{-1})(k_0 - k_0^{-1})} \quad \frac{1}{q\theta - q^{-1}\theta^{-1}}$$

$$\times G(q\theta, k_1, k_2) \quad \frac{k_3 + k_3^{-1} - q^2 \theta k_0^{-1} - q^{-2} \theta^{-1} k_0}{q^2 \theta}$$

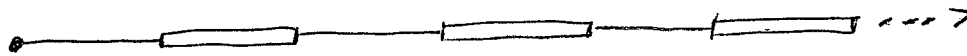
$$= \frac{G(q\theta, k_1, k_2) G(q\theta, k_0^{-1}, k_3)}{q^2 \theta (\theta - \theta^{-1})(q^2 \theta^{-1} - q\theta)}$$

Result follows

□

\mathbb{F} alg closed $0 \neq q \in \mathbb{F}$ $q^4 \neq 1$

Next goal: describe the irred \hat{H}_q modules
with X-diagram



θ_0

∞ path.

[these corresp to AW poly]

Motivation

Given \hat{H}_q module V as above

assume k_i exist $\forall i \in \mathbb{I}$

Given $0 \neq v \in V_X(\theta_0)$

By Prop 21

$$t_1 v \in \mathbb{F}v$$

$$t_2 v \in \mathbb{F}v$$

Eigvals of t_i are $k_i^{\pm 1}$

k_i defined up to reciprocal

wlog

$$t_1 v = k_1 v$$

Similarly

$$t_2 v = k_2 v$$

So

$$k_1 k_2 v = t_1 t_2 v$$

$$= q^{-1} X^{-1} v$$

$$= q^{-1} \theta_0^{-1} v$$

so

$$\theta_0 = \frac{1}{q k_1 k_2}$$

We now construct our modules

Until further notice for $0 \neq k_1 \in \mathbb{F}$ $i \in \mathbb{I}$

Define
$$\theta_0 = \frac{1}{g_{k_1, k_2}}$$

Functions $a, b, c, d : \mathbb{F} \setminus \{0, 1, -1\} \rightarrow \mathbb{F}$ as before

$\alpha, \beta, \gamma, \delta$
2x2 matrices

$$T_i(\theta) \quad i \in \mathbb{I}$$

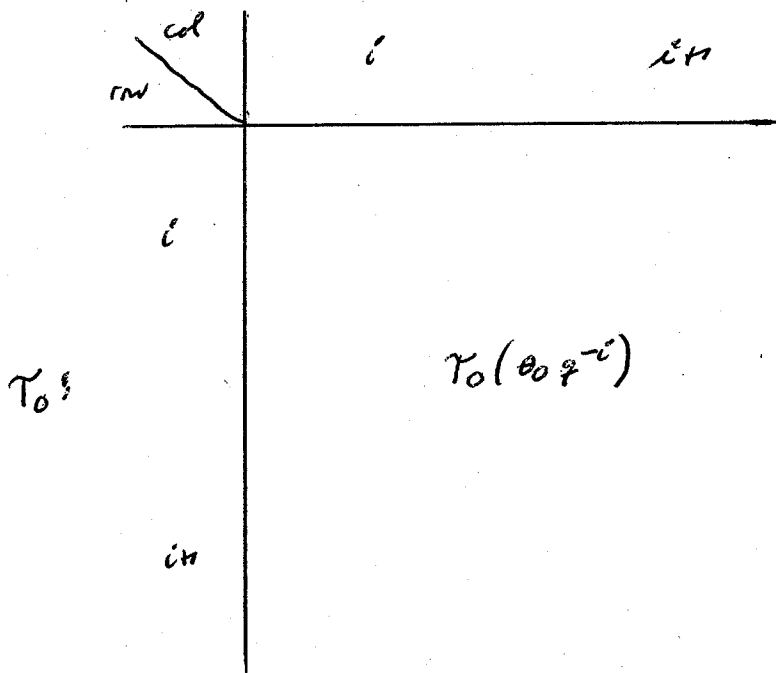
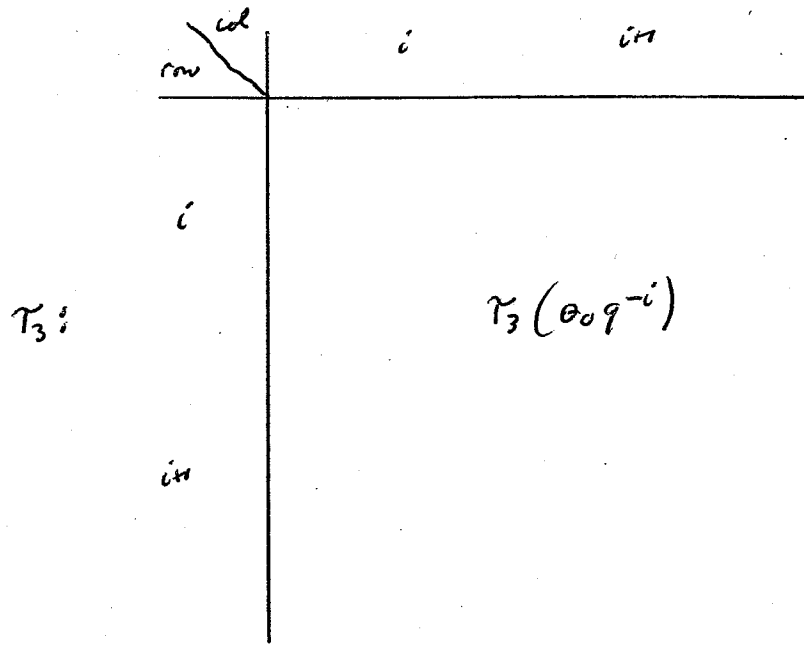
as before

We now define T_i

For $i \in \mathbb{I}$ define a matrix T_i with rows/cols indexed by nonnegative integers

The entries of T_i are given below (all entries not shown are 0)

For even $i \geq 0$



τ_1 has $(0,0)$ -entry k_1

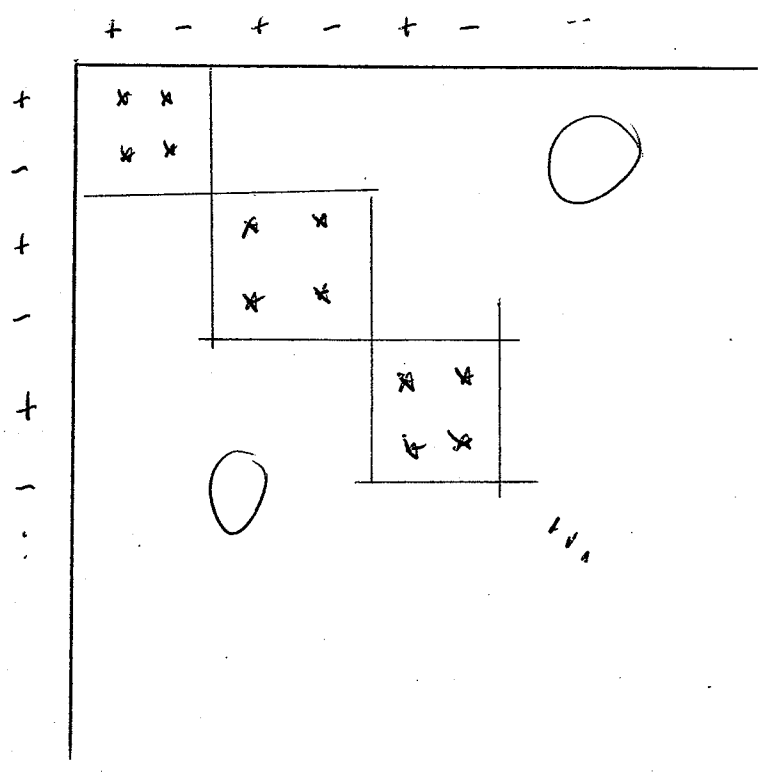
τ_2 has $(0,0)$ -entry k_2

For odd $i \geq 1$

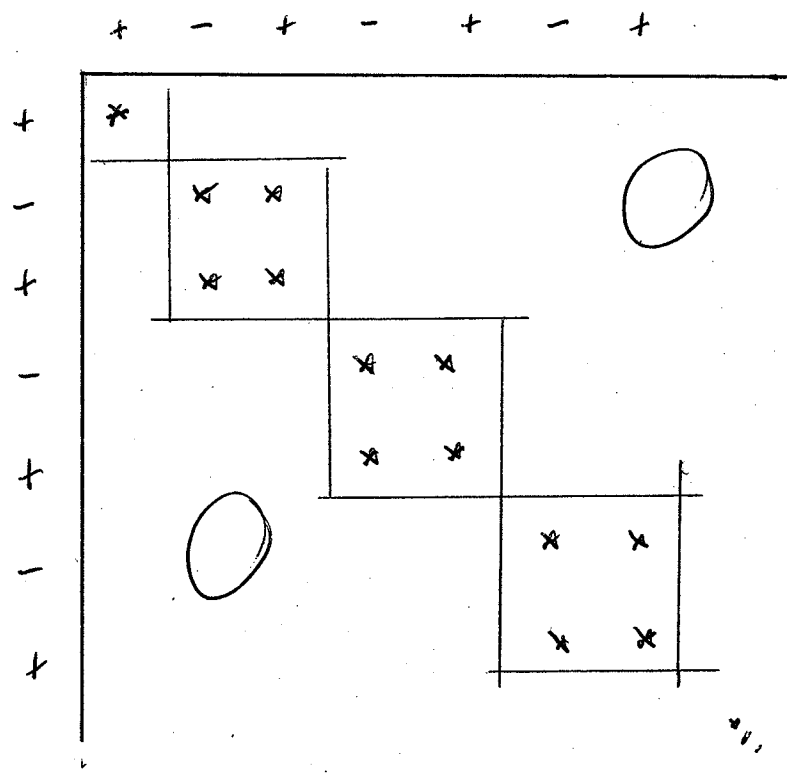
	col	i	$i+n$
row			
τ_1	i		
		$\tau_1 (0, 9^{-i})$	
	$i+n$		

	col	i	$i+n$
row			
τ_2	i		
		$\tau_2 (0, 8^{-i})$	
	$i+n$		

T_0, T_3



T_1, T_2



As in the previous example

$$\tau_i + \tau_i^{-1} = (k_i + k_i^{-1}) I \quad i \in \mathbb{I}$$

$\tau_3 \tau_0$ is diagonal with (i,i) -entry

$$\left\{ \begin{array}{ll} \theta_0 q^{-i} & \text{if } i \text{ even} \\ \theta_0^{-1} q^{i+1} & \text{if } i \text{ odd} \end{array} \right. \quad i = 0, 1, 2, \dots$$

$\tau_1 \tau_2$ is diagonal with (i,i) -entry

$$\left\{ \begin{array}{ll} \theta_0^{-1} q^{i+1} & \text{if } i \text{ even} \\ \theta_0 q^{-i} & \text{if } i \text{ odd} \end{array} \right. \quad i = 0, 1, 2, \dots$$

So as before

$$\tau_0 \tau_1 \tau_2 \tau_3 = q^{-1} I$$

So $\{\tau_i\}_{i \in \mathbb{I}}$ give a representation of \hat{H}_q .

The equals of T_0 are

$$\theta_0, \theta_0^{-1}, q^{-2}\theta_0, q^2\theta_0^{-1}, q^{-4}\theta_0, \dots$$

$(\theta_0 = \frac{1}{qk_1k_2})$

*

To make the above rep of H_2 irred. we require

- * are mutu dist (1)
- $G(\theta, k_1, k_2) \neq 0 \quad \forall \theta \in *$ (2)
- $G(q^{-2}\theta^{-1}, k_1, k_2) \neq 0 \quad \forall \theta \in * \setminus \theta_0$ (3)

Thm 38 Given $0 \neq k_i \in F$ ($i \in \mathbb{I}$)

that satisfy (1)-(3)

\exists an H_q -module $V = V(k_0, k_1, k_2, k_3)$

with the following property.

V has a basis $\{v_i\}_{i=0}^{\infty}$ with resp to which the matrix rep t_g is T_g for $g \in \mathbb{I}$

Moreover

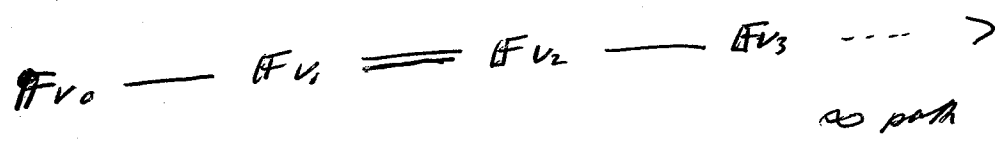
\bullet For $i=0, 1, 2, \dots$ v_i is an eigenvector for X with eigenval

$$\begin{cases} \theta_0 q^{-i} & i \text{ even} \\ \theta_0^{-1} q^{i-1} & i \text{ odd} \end{cases} \quad \theta_0 = \frac{1}{q^{k_1} k_2}$$

$\bullet V$ is irred

$\bullet X$ is diagonalizable on V and all eigenspaces of X on V have dim 1

\bullet the X -diag of V is



\bullet For $i=0, 1, 2, \dots$

$Gv_i = v_{i+1}$ if i even

$Gv_i = v_{i-1}$ if i odd

pt Each claim is already shown or routine.

Ref to the \hat{H}_2 -module $V = V(k_0, k_1, k_2, k_3)$

9

from M 38 We now relate V to the AW polynomials.

Assume $k_0 \neq \pm 1 \neq \infty$

$$V = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V + \frac{t_0 - k_0}{k_0^{-1} - k_0} V \quad \text{ds}$$

The following is a basis for $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V :$

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i} \quad i = 0, 1, 2, \dots \quad *$$

The following is a basis for $\frac{t_0 - k_0}{k_0^{-1} - k_0} V :$

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_{2i} \quad i = 0, 1, 2, \dots$$

The element $A = \gamma + \gamma^{-1}$ acts on $*$ just as in M 37 :

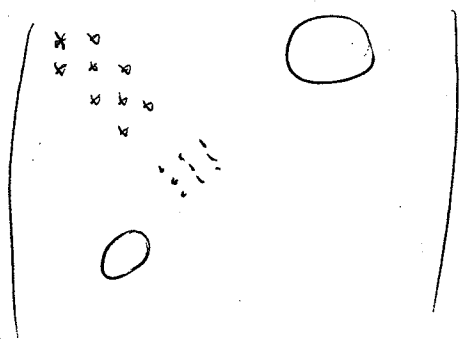
For $i = 0, 1, 2, \dots$

$$A \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i} =$$

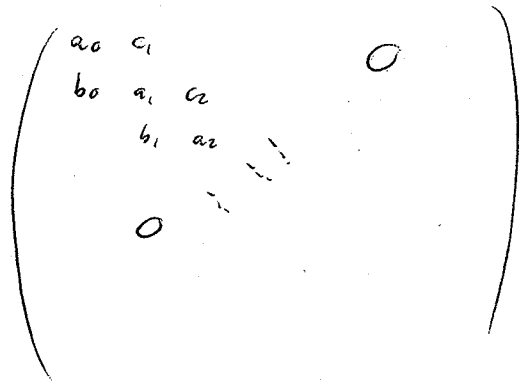
term	coef
$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i-2}$	same coef as in M 37 except that $\theta_0 = \frac{1}{qk_0k_2}$
$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i}$	
$t_0 - k_0^{-1}$	

($v_{-2} = 0$)

Rel \times matrix rep A is tri diag



Call this matrix



Define polynomials $\{f_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ by

$$x f_n = c_n f_{n+1} + a_n f_n + b_n f_{n-1}$$

$n = 0, 1, 2, \dots$

$$f_0 = 1, \quad f_{-1} = 0$$

$f_n \quad n = 0, 1, 2, \dots$

f_n has degree n

coef of x^n is $\frac{1}{b_0 b_1 \dots b_{n-1}}$

By const

$$f_n(A) = \frac{t_0 - t_0^*}{k_0 k_0^*} v_0 = \frac{t_0 - t_0^*}{k_0 k_0^*} v_{2n} \quad n = 0, 1, 2, \dots$$

394

F_n $n=0,1,2,\dots$ let F_n denote the monic
version of f_n so

$$f_n = \frac{F_n}{b_0 b_1 \dots b_{n-1}}$$

the $\{F_n\}_{n=0}^{\infty}$ satisfy the 3-term rec

$$x F_n = c_n b_n F_{n+1} + a_n F_n + F_{n-1} \quad n=0,1,2,\dots$$

$$F_0 = 1, \quad F_{-1} = 0$$

Call $\{F_n\}_{n=0}^{\infty}$ the monic polynomials that converge
to $\{k_i\}_{i \in \mathbb{I}}$

As we will see the $\{F_n\}_{n=0}^{\infty}$ are the AW poly (monic version)

The AW poly are defined using basic hypergeometric
series ${}_4\phi_3$, as we now explain

Notation: $F_a \quad a \in \mathbb{F}$

$$(a; q)_n = \underbrace{(1-a)(1-aq)(1-aq^2) \dots (1-aq^{n-1})}_{n \text{ terms}} \quad n=0,1,2,\dots$$

so $(a; q)_0 = 1$

Let a, b, c, d denote nonzero scalars in \mathbb{F}

Assume: q not a root of 1

And None of

ab, ac, ad, bc, bd, cd is an integral power of q

For $n = 0, 1, 2, \dots$ define a polynomial

$$p_n = p_n(x; a, b, c, d | q)$$

in $\mathbb{F}[x]$ by

$$p_n = q \phi_3 \left(\begin{matrix} q^{-n} & abcdq^{n-1} & ay & ay^{-1} \\ ab & ac & ad & \end{matrix} \middle| q; q \right)$$

where $x = y + y^{-1}$

$$\left[= \sum_{i=0}^{\infty} \frac{(q^{-n}; q)_i (abcdq^{n-1}; q)_i (ay; q)_i (ay^{-1}; q)_i}{(ab; q)_i (ac; q)_i (ad; q)_i (q; q)_i} q^i \right]$$

Note that

$$(q^{-n}; q)_i = 0 \text{ for } i > n$$

so above sum terminates at $i = n$.

One checks p_n really is a poly in x with degree n .

p_n is an AN polynomial

For instance

13

$$p_0 = 1$$

$$p_1 = 1 - \frac{(1-abcd)(1-ax+a^2)}{(1-ab)(1-ac)(1-ad)}$$

$$p_2 = 1 - \frac{(1+q)(1-abcdq)(1-ax+a^2)}{(1-ab)(1-ac)(1-ad)}$$

+

$$\frac{(1-abcdq)(1-abcdq^2)(1-ax+a^2)(1-axq+a^2q^2)}{q(1-ab)(1-abq)(1-ac)(1-acq)(1-ad)(1-adq)}$$

The AW polyg satisfy a 3-term rec.

For $n=0,1,2,\dots$

$$x p_n = c_n p_{n+1} + a_n p_n + b_n p_{n-1}$$

$$p_{-1} = 0$$

where

$$b_n = b_n(a, b, c, d | q)$$

$$= \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n+1})}{a(1 - abcdq^{2n+2})(1 - abcdq^{2n})}$$

$$c_n = c_n(a, b, c, d | q)$$

$$= \frac{a(1 - q^n)(1 - bcq^{n+1})(1 - bdq^{n+1})(1 - cdq^{n+1})}{(1 - abcdq^{2n+2})(1 - abcdq^{2n+1})}$$

$$a_n = a_n(a, b, c, d | q)$$

$$= a + a^{-1} - b_n - c_n$$

For $n = 0, 1, 2, \dots$

p_n has deg n

coeff of x^n is $\frac{1}{b_0 b_1 \dots b_{n-1}}$

Let $P_n = P_n(x; a, b, c, d/q)$ denote the monic version of p_n so

$$P_n = \frac{p_n}{b_0 b_1 \dots b_{n-1}}$$

Then

$$x P_n = c_n b_{n-1} P_{n+1} + a_n P_n + P_{n-1}$$

Note that

$b_{n-1} c_n$ is sym in a, b, c, d

One checks

a_n is sym in a, b, c, d

Indeed define

$$e_1 = a + b + c + d$$

$$e_2 = ab + ac + ad + bc + bd + cd$$

$$e_3 = abc + abd + acd + bcd$$

$$e_4 = abcd$$

then

$$a_n = \frac{q^{n+1} (1 - q^n - q^{n+1}) e_3 + q e_1 + q^{2n+1} e_3 e_4 - q^{n+1} (1 + q - q^{n+1}) e_1 e_4}{(1 - q^{2n+2} e_4) (1 - q^{2n} e_4)}$$

Since a_n and $b_n \rightarrow c_n$ are sym in a, b, c, d

the poles $\{P_n\}_{n=0}^{\infty}$ are sym in a, b, c, d .

Thm 39 For the H_9 module V from Thm 38

consider the corresp modic poly $\{F_n\}_{n=0}^\infty$

then

$$F_n(x) = P_n \left(x; a, b, c, d \mid q^2 \right)$$

"
 P_n

$n=0, 1, 2, \dots$

where

a, b, c, d is any perm of

$$k_0 k_1, \quad \frac{q^2 k_1}{k_0}, \quad q k_2 k_3, \quad \frac{q k_2}{k_3}$$

pf We show $\{F_n\}_{n=0}^\infty$ and $\{P_n\}_{n=0}^\infty$ satisfy

the same 3-term rec

To do this, need to show

$$b_{n+1}(a, b, c, d | q^2) c_n(a, b, c, d | q^2)$$

=

$$\frac{G(q\theta, k_1, k_2) G(q\theta, q^{-1}k_0, k_3)}{(\theta - \theta^{-1})(q^{-1}\theta^{-1} - q\theta)} \frac{1}{(q^2\theta - q^{-2}\theta^{-1})(q^2\theta^{-1} - q\theta)}$$

n = 1, 2, ...

$$a_n(a, b, c, d | q^2)$$

=

$$\frac{\theta k_0 + \theta^{-1}k_0^{-1} - k_3 k_3^{-1}}{\theta - \theta^{-1}} \frac{q^2\theta^{-1}(k_1 k_1^{-1}) - k_2 - k_2^{-1}}{q^2\theta^{-1} - q\theta}$$

+

$$\frac{q^2\theta(k_1 k_1^{-1}) - k_2 k_2^{-1}}{\theta - \theta^{-1}} \frac{k_3 k_3^{-1} - \theta k_0^{-1} - \theta^{-1}k_0}{q^2\theta - q\theta^{-1}}$$

+

$$\frac{k_1 + k_1^{-1}}{k_0}$$

n = 0, 1, 2, ...

where

$$\theta = \theta_0 q^{-2n}$$

$$\theta_0 = \frac{1}{q k_1 k_2}$$

This is a routine verification.

□ 402

Notes

$$\frac{G(q\theta, k_1, k_2) G(q\theta, q^{-1}k_0, k_3)}{(q - \theta^{-1})(q^{-1}\theta^{-1} - q\theta)} \frac{1}{(q^2\theta - q^{-2}\theta^{-1})(q^{-1}\theta^{-1} - q\theta)}$$

=

$$\frac{q^{-4} \left(\theta - \frac{k_1}{qk_2} \right) \left(\theta - \frac{k_2}{qk_1} \right) \left(\theta - \frac{k_1k_2}{q} \right) \left(\theta - \frac{1}{qk_1k_2} \right) \left(\theta - \frac{k_3}{k_0} \right) \left(\theta - \frac{1}{k_0k_3} \right) \left(\theta - \frac{k_0k_3}{q^2} \right) \left(\theta - \frac{k_0}{q^2k_3} \right)}{(q - \theta^{-1}) (q\theta - q^{-1}\theta^{-1})^2 (q^2\theta - q^{-2}\theta^{-1})}$$

$$a_n = \frac{(1 - k_1k_2) (k_1 - k_2) (k_0k_3 - q) (k_0 - qk_3)}{2 (\theta + \theta^{-1} - q - q^{-1}) q k_0 k_1 k_2 k_3}$$

+

$$\frac{(1 + k_1k_2) (k_1 + k_2) (k_0k_3 + q) (k_0 + qk_3)}{2 (\theta + \theta^{-1} + q + q^{-1}) q k_0 k_1 k_2 k_3}$$

provided char F $\neq 2$

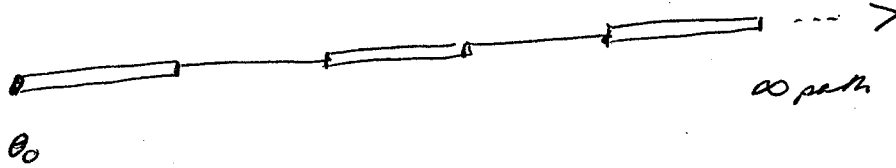
where

$$\theta = \theta_0 q^{-2n}$$

$$\theta_0 = \frac{1}{qk_1k_2}$$

\mathbb{F} alg closed $0 \neq q \in \mathbb{F}$ $q^2 \neq 1$

Next goal: describe the unred \hat{H}_q -module
with X-diagram



Motivation Given \hat{H}_q -module V as above

Assume k_i exist $\forall i \in \mathbb{I}$.

Given $0 \neq v \in V_x(\theta_0)$

$$t_0 v \in \mathbb{F}v$$

$$t_3 v \in \mathbb{F}v$$

wlog

$$t_0 v = k_0 v$$

$$t_3 v = k_3 v$$

$$k_0 k_3 v = t_0 t_3 v \\ = \chi v$$

$$\theta_0 = k_0 k_3$$

We now construct our modules

until further notice for $0 \neq k_i \in \mathbb{F}$ $i \in \mathbb{I}$

Define

$$\theta_0 = k_0 k_3$$

Functions

$$a, b, c, d : \mathbb{F} \setminus \{0, 1, -1\} \rightarrow \mathbb{F} \quad \text{as before}$$

$$\alpha, \beta, \gamma, \delta \dots$$

2x2 matrices

$$T_i(\theta) \quad i \in \mathbb{I}$$

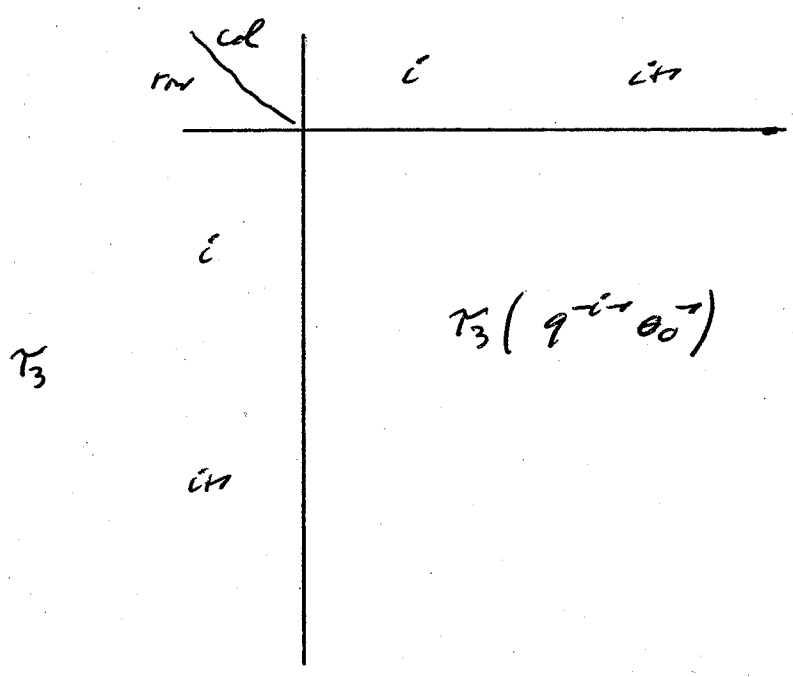
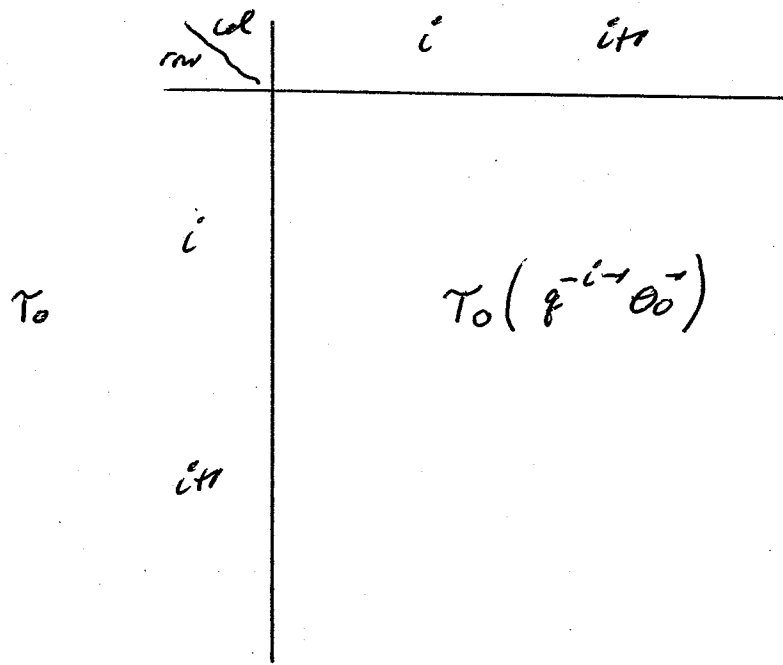
as before

For $i \in \mathbb{I}$ define matrix T_i
rows/cols indexed by nonneg integers

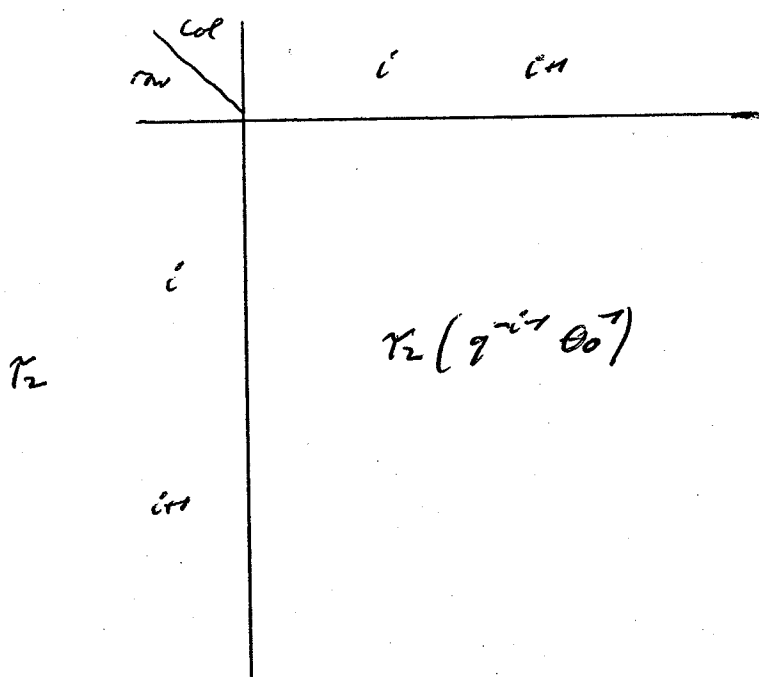
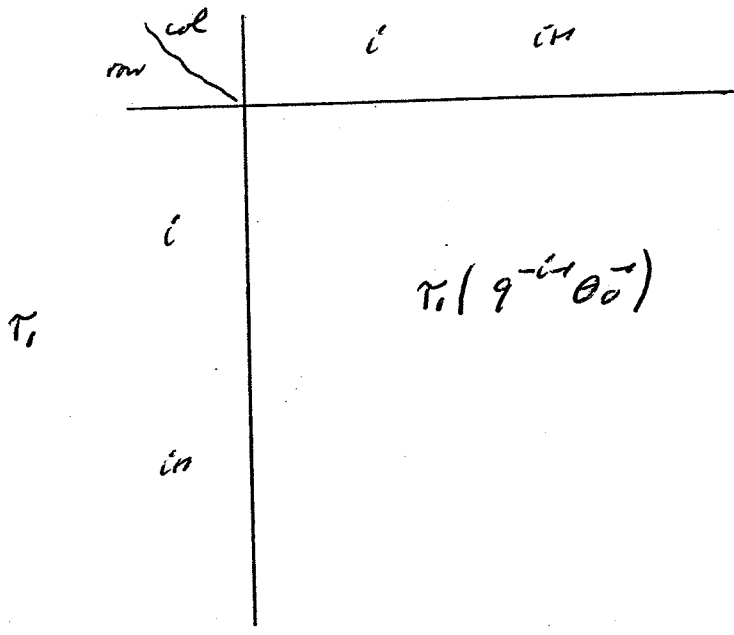
Entries of T_i given below

T_0 has $(0,0)$ -entry k_0

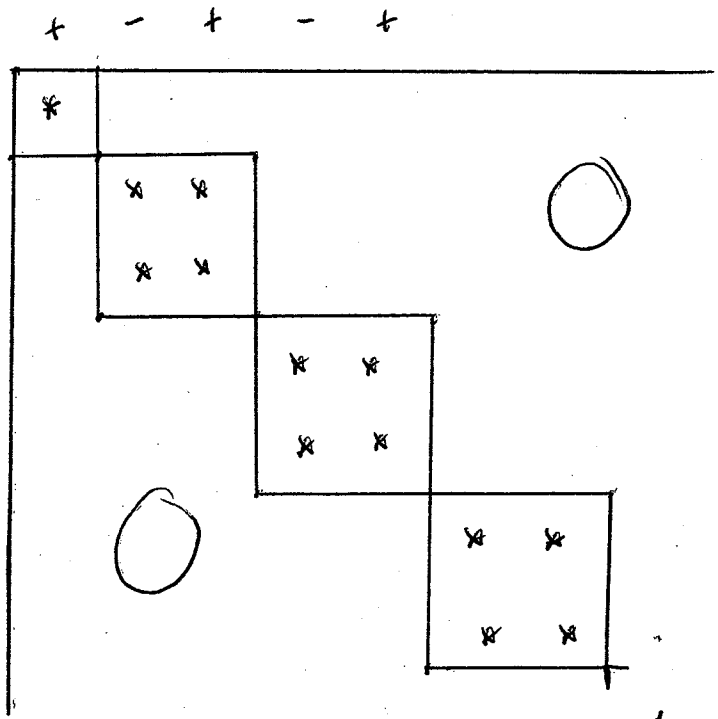
T_3 has $(0,0)$ -entry k_3



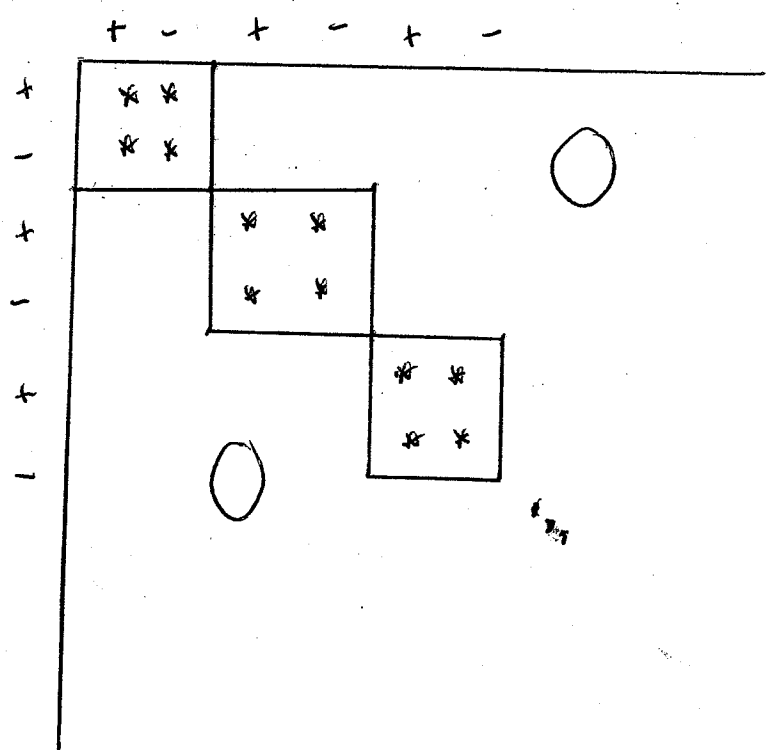
For even $i \geq 0$



T_0, T_3



T_1, T_2



As in prev examples

$$T_i + T_i = (k_i + k_i^{-1}) \quad i \in \mathbb{I}$$

$T_3 T_0$ is diag with (i,i) -entry

$$\begin{cases} \theta_0 q^i & \text{if } i \text{ even} \\ \theta_0^{-1} q^{-i-1} & \text{if } i \text{ odd} \end{cases} \quad i = 0, 1, 2, \dots$$

$T_1 T_2$ is diag with (i,i) -entry

$$\begin{cases} \theta_0 q^{-i-1} & \text{if } i \text{ even} \\ \theta_0 q^i & \text{if } i \text{ odd} \end{cases} \quad i = 0, 1, 2, \dots$$

So $T_3 T_0 T_1 T_2 = q^{-1} I$

so $T_0 T_1 T_2 T_3 = q^{-1} I$

So $\{T_i\}_{i \in \mathbb{I}}$ gives rep of H_2

Eigvals of $T_3 T_0$ are

7

$$\theta_0, q^{-2}\theta_0^{-1}, q^2\theta_0, q^{-4}\theta_0^{-1}, q^4\theta_0, q^{-6}\theta_0^{-1}, q^6\theta_0, \dots \quad *$$

$$\theta_0 = k_0 k_3$$

To make the algebra H_q -module irreducible, we require

• $*$ are mult dist

(1)

• $G(\theta, k_0, k_3) \neq 0 \quad \forall \theta \in * \setminus \theta_0$

(2)

• $G(q^{-1}\theta^{-1}, k_1, k_2) \neq 0 \quad \forall \theta \in *$

(3)

th 40 Given $0 \neq k_i \in \mathbb{F}$ ($i \in \mathbb{I}$)
 that sat (1)-(3)

\exists an \hat{H}_2 -module $V = V(k_0, k_1, k_2, k_3)$
 with the following prop.

V has a basis $\{v_i, \beta_i\}_{i \geq 0}$ with resp to which the matrix
 rep t_2 is T_2 for $i \in \mathbb{I}$

Moreover

- For $i=0,1,2,\dots$ v_i is an eigenvector for X with equal

$$\begin{cases} \theta_0 \gamma^i & \text{if even} \\ \theta_0^{-1} \gamma^{-i} & \text{if odd} \end{cases} \quad \theta_0 = k_0 k_3$$

- V is irred

• X is diagonalizable on V and all eigenspaces of X on V have dim 1

• The X -deg of V is

$$\mathbb{F}v_0 = \mathbb{F}v_1 \xrightarrow{\quad} \mathbb{F}v_2 = \mathbb{F}v_3 \xrightarrow{\quad} \mathbb{F}v_4 \xrightarrow{\quad} \dots \xrightarrow{\quad} \infty \text{ path}$$

- For $i=0,1,2,\dots$

$$G_2 v_i = v_{i+1} \quad \text{if even}$$

$$G_0 v_i = v_{i-1} \quad \text{if odd}$$

pt Each claim already shown a routine □

Ref to the \hat{H}_1 module $V = V(k_0, k_0^{-1})$ 9
 from Th 40 we now relate V to the AW polys.

Assume $k_0 \neq \pm 1$ so that

$$V = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V + \frac{t_0 - k_0}{k_0^{-1} - k_0} V \quad \text{ds}$$

The following is a basis for $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V$: *

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i} \quad i = 0, 1, 2, \dots$$

The following is a basis for $\frac{t_0 - k_0}{k_0^{-1} - k_0} V$: *

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v_{2i} \quad i = 1, 2, \dots$$

We now find the action of $A = \gamma + \gamma^{-1}$ on *

thm 41

With above notation,

10

for $i=0,1,2,\dots$

$$A \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i} =$$

term

coef

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i+2}$$

$$\frac{(t_0 - k_0^{-1} k_3)(t_0 - k_0^{-1} k_3^{-1})}{(t_0 - q^{-2} k_0^{-1} k_3)(t_0 - q^{-2} k_0^{-1} k_3^{-1})} \frac{1}{(t_0 - t_0^{-1})(q^{-1} t_0^{-1} - q t_0) q^{-1} t_0}$$

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i}$$

$$\frac{t_0 k_0 + t_0^{-1} k_0^{-1} - k_3 - k_3^{-1}}{t_0 - t_0^{-1}} \frac{q^{-1} t_0^{-1} (k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{q^{-1} t_0^{-1} - q t_0}$$

$$+ \frac{q^{-1} t_0^{-1} (k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{t_0 - t_0^{-1}} \frac{k_3 + k_3^{-1} - t_0 k_0^{-1} - t_0^{-1} k_0}{q^{-1} t_0^{-1} - q t_0}$$

$$+ \frac{k_1 + k_1^{-1}}{k_0}$$

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v_{2i-2}$$

$$\frac{\theta G(t_0^{-1}, k_0, k_3) G(q t_0^{-1}, k_1, k_2)}{(t_0 - t_0^{-1})(q t_0^{-1} - q^{-1} t_0)}$$

where $\theta = q^{2i} t_0$

$t_0 = k_0 k_3$

403