II Representation theory of UBAHA

until further notice:

$\mathbb{F}$ alg closed

$0 \neq q \in \mathbb{F}$

$q^4 \neq 1$

LEM 1 Let V denote a f.d. irred $\hat{H}_q$-module. Then for $i \in \mathbb{I}$ $\exists$ $0 \neq k_i \in \mathbb{F}$ s.t.

$$\left(T_i - (k_i + k_i^{-1}) I \right) V = 0$$

"ith corresp parameter"

k_i is defined up to reciprocal.

pf. Since $\mathbb{F}$ is alg closed and $\dim V < \infty$ by Schurs lemma $\exists$ $K_i \in \mathbb{F}$ s.t

$$(T_i - K_i) V = 0$$

Since $\mathbb{F}$ is alg closed the poly

$$\lambda^2 - K_i \lambda + 1$$

has a root in $\mathbb{F}$, call it k_i. obs

$$k_i + k_i^{-1} = K_i$$

Result follows.

$\square$

$\hat{H}_q$-modules vs $H(k_0, k_1, k_2, k_3; q)$-modules

Given nonzero $\{k_i\}_{i \in I}$ in $\mathbb{F}$

Let V denote a $H(k_0, k_1, k_2, k_3; q)$-module.

The module structure induces an $\mathbb{F}$-algebra hom

$$H(k_0, k_1, k_2, k_3; q) \longrightarrow \mathrm{End}(V)$$

Consider comp

$$\hat{H}_q \longrightarrow H(k_0, k_1, k_2, k_3; q) \longrightarrow \mathrm{End}(V)$$

$$T_i \longrightarrow k_i t k_i^{-1}$$

This gives a $\hat{H}_q$-module str on V.

Conversely let V denote a f.d. irred $\hat{H}_q$-module

with corresp params $\{k_i\}_{i \in I}$

By L1 $\hat{H}_q$ module str on V induces a

$H(k_0, k_1, k_2, k_3; q)$ module str on V

Conclusion In the f.d. irred case

the rep theory of DAHA and UDAHA

is the same.

Let V denote a $\hat{H}_q$-module

[not nec ~~fin-d.~~]

For $i \in \mathbb{I}$ and $0 \neq k_i \in \mathbb{F}$

We say V <u>has parameter k_i</u>

or <u>the parameter k_i exists</u> whenever

$$\left(T_i - (k_i + k_i^{-1})1\right) V = 0$$

Let V denote a $\hat{H}_q$-module

For $h \in \hat{H}_q$ and $\theta \in \mathbb{F}$ define

$$V_h(\theta) = \{ v \in V \mid hv = \theta v \}$$

eigenspace for h with eigenval θ

LEM 2 Given a $\hat{H}_q$-module V with parameter k_0. Assume $k_0 \neq \mp 1$. Then

$$V = V_{t_0}(k_0) + V_{t_0}(k_0^{-1}) \qquad \text{(ds sum)}$$

Moreover each of $V_{t_0}(k_0^{\pm 1})$ is invariant under $\hat{H}_q^+$

pf on V

$$t_0 + t_0^{-1} = T_0 = k_0 + k_0^{-1}$$

so $$(t_0 - k_0)(t_0 - k_0^{-1}) = 0$$

$k_0 \neq \mp 1$ so $k_0 \neq k_0^{-1}$ so

t_0 is diagonalizable on V with all eigenvalues of t_0 in $\{k_0, k_0^{-1}\}$.

To get the last assertion of the lemma recall t_0 commutes with everything in $\hat{H}_q^+$ $\square$

Note Ref to Lem 2 we will see:

each of $V_{t_0}(k_0^{\pm 1})$ is irred as a $\hat{H}_q^+$-module

Moreover if $\dim V < \infty$ then A, B act on each

of $V_{t_0}(k_0^{\pm 1})$ as a Leonard pair.

DEF 3 Define

$$G_0 = t_0 - t_3 t_0 t_3^{-1}$$

$$G_1 = t_1 - t_0 t_1 t_0^{-1}$$

$$G_2 = t_2 - t_1 t_2 t_1^{-1}$$

$$G_3 = t_3 - t_2 t_3 t_2^{-1}$$

Recall the aut of $\hat{H}_9$ that sends

$$t_0 \to t_1 \to t_2 \to t_3 \to t_0$$

This aut sends

$$G_0 \to G_1 \to G_2 \to G_3 \to G_0$$

LEM 8 We have

(i) $X G_0 = G_0 X^{-1}$ $\qquad X^{-1} G_0 = G_0 X$

(ii) $Y G_1 = G_1 Y^{-1}$ $\qquad Y^{-1} G_1 = G_1 Y$

(iii) $X G_2 = q^{-2} G_2 X^{-1}$ $\qquad X^{-1} G_2 = q^{2} G_2 X$

(iv) $Y G_3 = q^{-2} G_3 Y^{-1}$ $\qquad Y^{-1} G_3 = q^{2} G_3 Y$

pf (i) Recall

$$t_0 (t_0 t_3 - t_3 t_0) = (t_0 t_3 - t_3 t_0) t_0^{-1}$$
$$t_3 (t_0 t_3 - t_3 t_0) = (t_0 t_3 - t_3 t_0) t_3^{-1}$$

so

$$t_3 t_0 (t_0 t_3 - t_3 t_0) = (t_0 t_3 - t_3 t_0) t_3^{-1} t_0^{-1}$$
$$= (t_0 - t_3 t_0 t_3^{-1}) t_0^{-1}$$

Mult each term on right by t_3^{-1}:

$$\underbrace{t_3 t_0}_{X} (\underbrace{t_0 - t_3 t_0 t_3^{-1}}_{G_0}) = t_3 t_0 (t_0 t_3 - t_3 t_0) t_3^{-1}$$
$$= \underbrace{(t_0 - t_3 t_0 t_3^{-1})}_{G_0} \underbrace{t_0^{-1} t_3^{-1}}_{X^{-1}}$$

(ii)–(iv) apply $\mathbb{Z}_4$-action to (i)

□

LEM 5 Let V denote a $\hat{H}_q$ module.

(i) $V_X(0) = 0$ $\quad V_Y(0) = 0$

(ii) For $0 \neq \theta \in \mathbb{F}$

$$G_0 V_X(\theta) \subseteq V_X(\theta^{-1})$$

$$G_2 V_X(\theta) \subseteq V_X(q^{-2}\theta^{-1})$$

[Caution: poss $\theta = \theta^{-1}$ or $\theta = q^{-2}\theta^{-1}$]

*

(iii) For $0 \neq \theta \in \mathbb{F}$

$$G_1 V_Y(\theta) \subseteq V_Y(\theta^{-1})$$

$$G_3 V_Y(\theta) \subseteq V_Y(q^{-2}\theta^{-1})$$

pf (i) X^{-1}, Y^{-1} exist.

(ii) Show * Given $v \in V_X(\theta)$

$Xv = \theta v$ $\quad X^{-1}v = \theta^{-1}v$

$$X G_0 = G_0 X^{-1}$$

so

$$X G_0 v = G_0 \underbrace{X^{-1} v}_{\theta^{-1} v}$$

$$X_0 G_0 v = \theta^{-1} G_0 v$$

$$G_0 v \in V_X(\theta^{-1})$$

Gives *

(iii) Sim to (ii)

□

Cor 6 Let V denote a $\hat{H}_q$-module.

Given $\theta \in \mathbb{F}$ s.t. $V_X(\theta) \neq 0$

(i) Either

$G_0 V_X(\theta) = 0$ or $V_X(\theta^{-1}) \neq 0$

(ii) Either

$G_2 V_X(\theta) = 0$ or $V_X(q^{-2}\theta^{-1}) \neq 0$

[Sim results hold for Y]

LEM 7 We have

(i) $G_0^2 = \beta^2 - \beta T_0 T_3 + T_0^2 + T_3^2 - 4$

$\beta = X + X^{-1}$

(ii) $G_1^2 = A^2 - A T_0 T_1 + T_0^2 + T_1^2 - 4$

$A = Y + Y^{-1}$

(iii) $G_2^2 = (qX + q^{-1}X^{-1})^2 - (qX + q^{-1}X^{-1}) T_1 T_2 + T_1^2 + T_2^2 - 4$

(iv) $G_3^2 = (qY + q^{-1}Y^{-1})^2 - (qY + q^{-1}Y^{-1}) T_2 T_3 + T_2^2 + T_3^2 - 4$

pf (i) By L 70, 71

(ii)–(iv) Apply $\mathbb{Z}_4$-sym to (i) $\square$

We now consider the form of the polys in L 7.

LEM 8 For indets a, b, λ

$$(\lambda+\lambda^{-1})^2 - (\lambda+\lambda^{-1})(a+a^{-1})(b+b^{-1}) + (a+a^{-1})^2 + (b+b^{-1})^2 - 4$$

$$= \lambda^{-2}(\lambda - a/b)(\lambda - b/a)(\lambda - ab)(\lambda - 1/ab)$$

pf (ex)

details : Eval RHS

$$\lambda^{-1}(\lambda - a/b)(\lambda - b/a) = \lambda + \lambda^{-1} - a/b - b/a$$

$$\lambda^{-1}(\lambda - ab)(\lambda - 1/ab) = \lambda + \lambda^{-1} - ab - 1/ab$$

$$RHS = (\lambda + \lambda^{-1} - a/b - b/a)(\lambda + \lambda^{-1} - ab - 1/ab)$$

$$= (\lambda+\lambda^{-1})^2 - (\lambda+\lambda^{-1})(\underbrace{a/b + b/a + ab + 1/ab}_{(a+a^{-1})(b+b^{-1})})$$

$$+ \underbrace{(a/b + b/a)(ab + \tfrac{1}{ab})}_{\substack{\shortparallel \\ a^2+b^2+a^{-2}+b^{-2} \\ \shortparallel \\ (a+a^{-1})^2 + (b+b^{-1})^2 - 4}}$$

Cor 9 Let V denote a $\hat{H}_q$-module

Given $0 \neq \theta \in \mathbb{F}$

(i) G_0^2 acts on $V_X(\theta)$ as

$$\theta^{-2}\left(\theta - \frac{k_0}{k_3}\right)\left(\theta - \frac{k_3}{k_0}\right)\left(\theta - k_0k_3\right)\left(\theta - \frac{1}{k_0k_3}\right) I$$

provided the parameters k_0, k_3 exist

(ii) G_2^2 acts on $V_X(\theta)$ as

$$q^2\theta^{-2}\left(\theta - \frac{k_1}{qk_2}\right)\left(\theta - \frac{k_2}{qk_1}\right)\left(\theta - \frac{k_1k_2}{q}\right)\left(\theta - \frac{1}{qk_1k_2}\right) I$$

provided k_1, k_2 exist

(iii) G_1^2 acts on $V_Y(\theta)$ as

$$\theta^{-2}\left(\theta - \frac{k_0}{k_1}\right)\left(\theta - \frac{k_1}{k_0}\right)\left(\theta - k_0k_1\right)\left(\theta - \frac{1}{k_0k_1}\right) I$$

provided k_0, k_1 exist

(iv) G_3^2 acts on $V_Y(\theta)$ as

$$q^2\theta^{-2}\left(\theta - \frac{k_2}{qk_3}\right)\left(\theta - \frac{k_3}{qk_2}\right)\left(\theta - \frac{k_2k_3}{q}\right)\left(\theta - \frac{1}{qk_2k_3}\right)$$

provided k_2, k_3 exist.

pf Combine L7, L8. $\square$

Prop 10 With ref to Cor 9

Assume the param k_0, k_3 exist and

$$\theta \notin \{ k_0/k_3,\ k_3/k_0,\ k_0k_3,\ 1/k_0k_3 \}$$

then

(i) G_0^2 is invertible on $V_x(\theta) + V_x(\theta^{-1})$

(ii) G_0 swaps

$$V_x(\theta) \qquad V_x(\theta^{-1})$$

(iii) $$\dim V_x(\theta) = \dim V_x(\theta^{-1})$$

pf (i) By Cor 9 (i)

(ii) By (i) and since

$$G_0 V_x(\theta) \subseteq V_x(\theta^{-1})$$

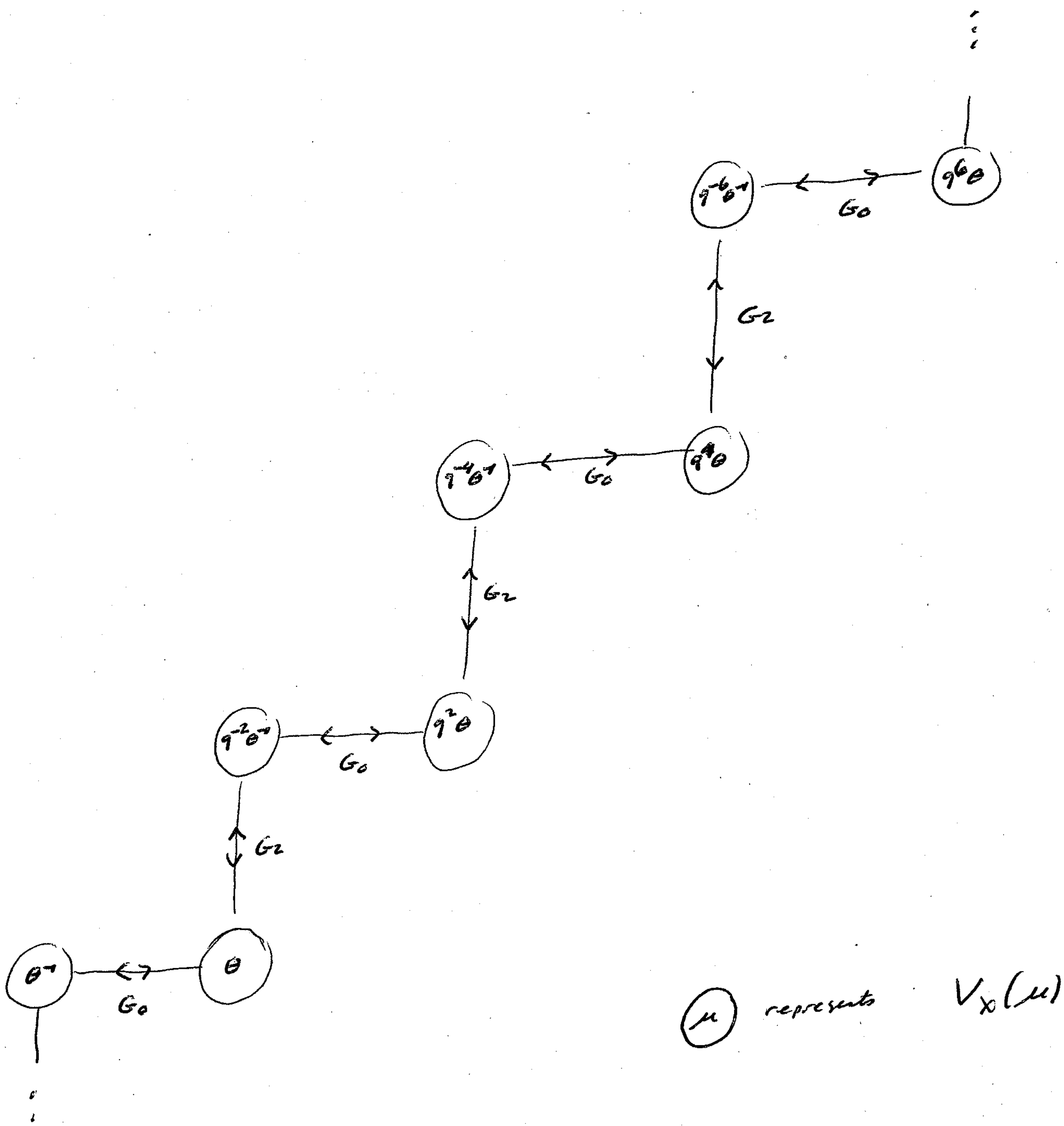
$$G_0 V_x(\theta^{-1}) \subseteq V_x(\theta)$$

(iii) By (ii) $\square$

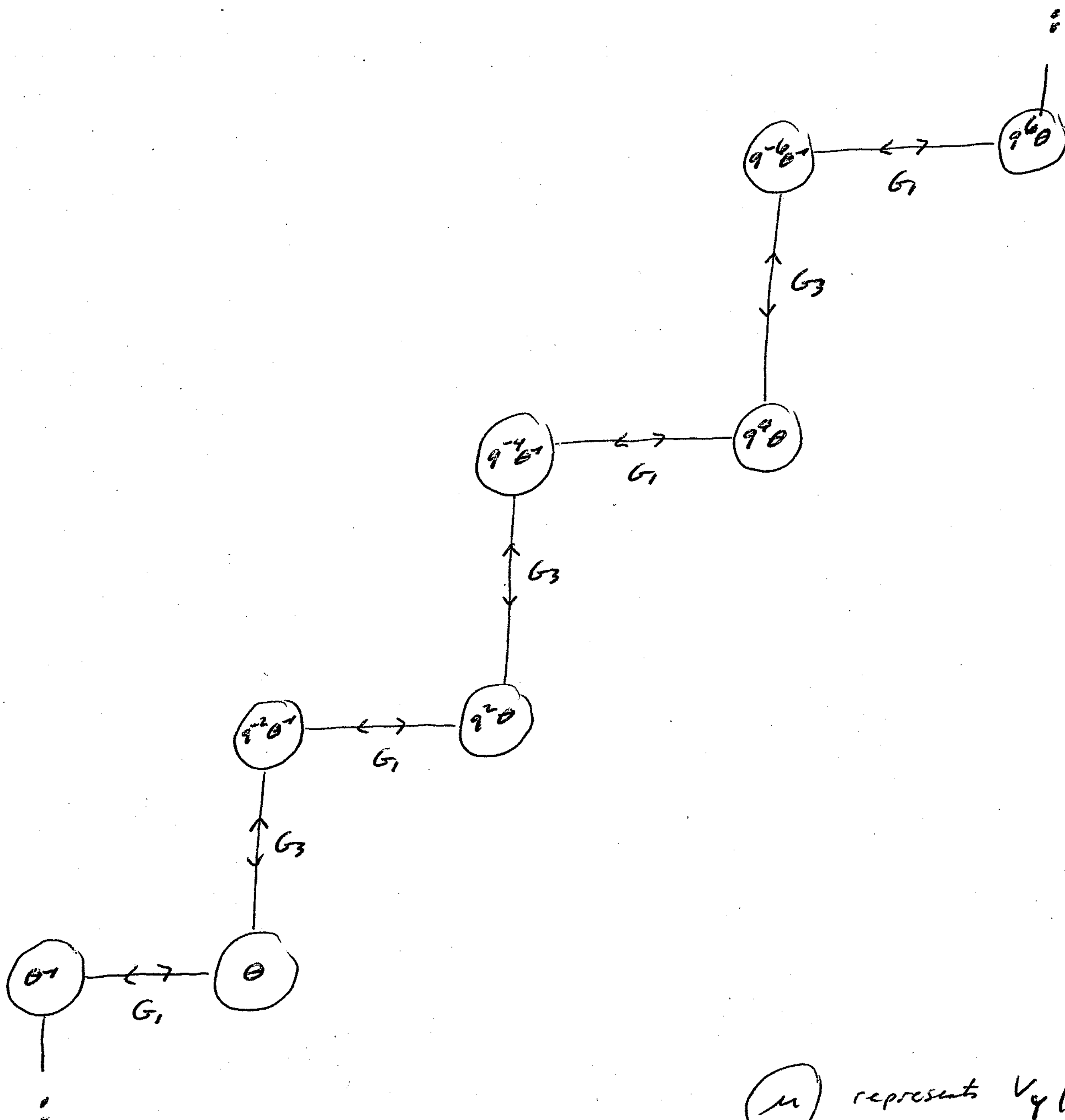
Prop 10 is about G_0
similar results hold for G_1, G_2, G_3

Summary

Actions of G_0, G_2 on the eigenspaces of X



Actions of G_1, G_3 on the eigenspaces of Ψ



μ represents $V_\Psi(\mu)$

$\mathbb{F}$ alg closed

$0 \neq q \in \mathbb{F}$ $\quad q^4 \neq 1$

Describe the $\hat{H}_q$ modules

LEM 11

(i) $G_0 = t_0(1 - X^{-2}) + T_3 X^{-1} - T_0$

(ii) $G_0 = t_3(X^{-1} - X) + T_3 X - T_0$

(iii) $G_2 = t_2(1 - q^2 X^2) + q T_1 X - T_2$

(iv) $G_2 = t_1(q X - q^{-1} X^{-1}) + q^{-1} T_1 X^{-1} - T_2$

pf (i) $G_0 = t_0 - t_3 t_0 t_3^{-1}$

$X = t_3 t_0$

$t_3 = X t_0^{-1}$ $\quad t_3^{-1} = t_0 X^{-1}$

$G_0 = t_0 - \underline{X t_0} X^{-1}$

$t_0 X^{-1} = \underline{X t_0} - X T_0 + T_3$

$G_0 = t_0 - (t_0 X^{-1} + X T_0 - T_3) X^{-1}$

$= t_0(1 - X^{-2}) + T_3 X^{-1} - T_0$

(ii) Use

$$t_0 = t_3^{-1} X$$
$$= T_3 X - t_3 X$$

(iii), (iv) Apply $\mathbb{Z}_4$-sym to (i), (ii)

$$G_0 \leftrightarrow G_2$$
$$t_0 \leftrightarrow t_2$$
$$t_1 \leftrightarrow t_3$$
$$X \leftrightarrow q^{-1} X^{-1}$$

$\square$

LEM 12 Let V denote a $\hat{H}_q$ module.

For $0 \neq \theta \in \mathbb{F}$ and $0 \neq v \in V_X(\theta)$

(i) $$(\theta - \theta^{-1}) t_0 v = (\theta T_0 - T_3) v + \theta G_0 v$$

(ii) $$(\theta - \theta^{-1}) t_3 v = (\theta T_3 - T_0) v - G_0 v$$

pf In L 11 (i),(ii) apply both sides to v and use $Xv = \theta v$ $\square$

Ref to Lem 12

Cases

$$\theta = \theta^{-1} \qquad (\text{ie } \theta = \mp 1)$$

$$\theta \neq \theta^{-1} \qquad G_0 v \neq 0$$

$$\theta \neq \theta^{-1} \qquad G_0 v = 0$$

LEM 13 With ref to L12 assume $\theta = \mp 1$

(i) $G_0 v = (\theta T_3 - T_0) v$

(ii) Assume k_0, k_3 exist. then

$$G_0 v = (\theta(k_3 + k_3^{-1}) - k_0 - k_0^{-1}) v$$

$$\in \mathbb{F} v$$

$\square$

pf By L12

LEM 14   With ref to L 12 assume

$$\theta \neq \theta^{-1}$$

Then

(i) $$t_0 v = \frac{\theta T_0 - T_3}{\theta - \theta^{-1}} v + \frac{\theta}{\theta - \theta^{-1}} G_0 v$$

(ii) $$t_0 G_0 v = \frac{\theta^{-1} T_0 - T_3}{\theta^{-1} - \theta} G_0 v + \frac{\theta^{-1}}{\theta^{-1} - \theta} G_0^2 v$$

(iii) $$t_3 v = \frac{\theta T_3 - T_0}{\theta - \theta^{-1}} v + \frac{1}{\theta^{-1} - \theta} G_0 v$$

(iv) $$t_3 G_0 v = \frac{\theta^{-1} T_3 - T_0}{\theta^{-1} - \theta} G_0 v + \frac{1}{\theta - \theta^{-1}} G_0^2 v$$

pf (i) By L 12 (i)

(ii) Apply (i) above to $v' = G_0 v$ $\theta' = \theta^{-1}$

(iii) By L 12 (i)

(iv) Apply (iii) above to $v' = G_0 v$, $\theta' = \theta^{-1}$

$\square$

Prop 15. With ref to Lem 12 assume

$$\theta \neq \theta^{-1}, \qquad G_0 v \neq 0$$

and that k_0, k_3 exist.

Then $v, G_0 v$ is a basis for a subspace of

V that is inv under $t_0^{\pm 1}, t_3^{\pm 1}$.

With resp to this basis:

$$X: \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}$$

$$G_0: \begin{pmatrix} 0 & \theta^{-2}\left(\theta - \frac{k_0}{k_3}\right)\left(\theta - \frac{k_3}{k_0}\right)\left(\theta - k_0 k_3\right)\left(\theta - \frac{1}{k_0 k_3}\right) \\ 1 & 0 \end{pmatrix}$$

$$t_0 : \begin{pmatrix} \dfrac{\theta(k_0+k_0^{-1}) - k_3 - k_3^{-1}}{\theta - \theta^{-1}} & \dfrac{(\theta - k_0k_3^{-1})(\theta - k_0^{-1}k_3)(\theta - k_0k_3)(\theta - k_0^{-1}k_3^{-1})}{\theta^3(\theta^{-1} - \theta)} \\ \dfrac{\theta}{\theta - \theta^{-1}} & \dfrac{\theta^{-1}(k_0+k_0^{-1}) - k_3 - k_3^{-1}}{\theta^{-1} - \theta} \end{pmatrix}$$

$$t_0^{-1} : \begin{pmatrix} \dfrac{\theta^{-1}(k_0+k_0^{-1}) - k_3 - k_3^{-1}}{\theta^{-1} - \theta} & \dfrac{(\theta - k_0k_3^{-1})(\theta - k_0^{-1}k_3)(\theta - k_0k_3)(\theta - k_0^{-1}k_3^{-1})}{\theta^3(\theta - \theta^{-1})} \\ \dfrac{\theta}{\theta^{-1} - \theta} & \dfrac{\theta(k_0+k_0^{-1}) - k_3 - k_3^{-1}}{\theta - \theta^{-1}} \end{pmatrix}$$

$$t_3: \begin{pmatrix} \dfrac{\theta(k_3+k_3^{-1}) - k_0 - k_0^{-1}}{\theta - \theta^{-1}} & \dfrac{(\theta - k_0 k_3^{-1})(\theta - k_0^{-1}k_3)(\theta - k_0 k_3)(\theta - k_0^{-1}k_3^{-1})}{\theta^2(\theta - \theta^{-1})} \\ \\ \dfrac{1}{\theta^{-1} - \theta} & \dfrac{\theta^{-1}(k_3+k_3^{-1}) - k_0 - k_0^{-1}}{\theta^{-1} - \theta} \end{pmatrix}$$

$$t_3^{-1}: \begin{pmatrix} \dfrac{\theta^{-1}(k_3+k_3^{-1}) - k_0 - k_0^{-1}}{\theta^{-1} - \theta} & \dfrac{(\theta - k_0 k_3^{-1})(\theta - k_0^{-1}k_3)(\theta - k_0 k_3)(\theta - k_0^{-1}k_3^{-1})}{\theta^2(\theta^{-1} - \theta)} \\ \\ \dfrac{1}{\theta - \theta^{-1}} & \dfrac{\theta(k_3+k_3^{-1}) - k_0 - k_0^{-1}}{\theta - \theta^{-1}} \end{pmatrix}$$

pf $\quad v, \; G_0 v$ lin indep since

$$0 \neq v \in V_X(\theta), \qquad 0 \neq G_0 v \in V_X(\theta^{-1})$$

$$\theta \neq \theta^{-1}$$

Define $\quad W = \mathrm{Span}\{v, G_0 v\}$

$t_0 W \subseteq W$ by Lem 12 (i), (ii) and since

$G_0^2 v \in \mathrm{Span}(v)$

Similarly $\quad t_3 W \subseteq W$

Find matrices

X ✓

G_0 ✓

t_0: By L 1# (i), (ii) and Cor 9 (i)

t_0^{-1}: $\quad t_0^{-1} = T_0 - t_0$

t_3: By L# (iii), (iv) and Cor 9 (i)

t_3^{-1}: $\quad t_3^{-1} = T_3 - t_3$ $\square$

Note Ref to Prop 15

For $i=0,3$ the matrix rep t_i has trace $k_i + k_i^{-1}$ and det 1.

Prop 16 With ref to L 12 assume

$$\theta \neq \theta^{-1}, \qquad G_0 v = 0.$$

and that k_0, k_3 exist.

Then

(i) $$t_0 v = \frac{\theta(k_0 + k_0^{-1}) - k_3 - k_3^{-1}}{\theta - \theta^{-1}} v$$

(ii) $$t_0^{-1} v = \frac{\theta^{-1}(k_0 + k_0^{-1}) - k_3 - k_3^{-1}}{\theta^{-1} - \theta} v$$

(iii) $$t_3 v = \frac{\theta(k_3 + k_3^{-1}) - k_0 - k_0^{-1}}{\theta - \theta^{-1}} v$$

(iv) $$t_3^{-1} v = \frac{\theta^{-1}(k_3 + k_3^{-1}) - k_0 - k_0^{-1}}{\theta^{-1} - \theta} v$$

(v) $$\theta \in \left\{ \frac{k_0}{k_3}, \frac{k_3}{k_0}, k_0 k_3, \frac{1}{k_0 k_3} \right\}$$

pf (i) By L 19 (i)

(ii) use $t_0^{-1} = T_0 - t_0$

(iii) By L 19 (iii)

(iv) Use $t_3^{-1} = T_3 - t_3$

(v) By Prop 10

$\square$

$\mathbb{F}$ alg closed

$0 \neq q \in \mathbb{F} \quad q^4 \neq 1$

Continue to desc the $\hat{H}_q$ modules

Lem 17 Let V denote a $\hat{H}_q$ module.

For $0 \neq \theta \in \mathbb{F}$ and $0 \neq v \in V_X(\theta)$

(i) $(q^{-1}\theta^{-1} - q\theta)\, t_2 v = (q^{-1}\theta^{-1} T_2 - T_0)v + q^{-1}\theta^{-1} G_2 v$

(ii) $(q^{-1}\theta^{-1} - q\theta)\, t_0 v = (q^{-1}\theta^{-1} T_1 - T_2)v - G_2 v$

pf In L 11 (iii), (iv) apply each side to v and use $Xv = \theta v$ $\square$

Ref to Lem 17

Cases: $q\theta = q^{-1}\theta^{-1}$ (ie $\theta = \pm q^{-1}$)

$q\theta \neq q^{-1}\theta^{-1}$ $G_2 v \neq 0$

$q\theta \neq q^{-1}\theta^{-1}$ $G_2 v = 0$

LEM 18 With ref to L 17 assume $\theta = \mp q^{-1}$

then

(i) $G_2 v = (q^{-1}\theta^{-1}T_1 - T_2)v$

(ii) Assume k_1, k_2 exist. then

$$G_2 v = (q^{-1}\theta^{-1}(k_1 + k_1^{-1}) - k_2 - k_2^{-1})v$$

$\in \mathbb{F}v$

$\square$

pf by L 17

LEM 19 with ref to L 17 assume

$$q\theta \neq q^{-1}\theta^{-1}$$

Then

$$(i) \quad t_2 v = \frac{q^{-1}\theta^{-1} T_2 - T_1}{q^{-1}\theta^{-1} - q\theta} v + \frac{q^{-1}\theta^{-1}}{q^{-1}\theta^{-1} - q\theta} G_2 v$$

$$(ii) \quad t_2 G_2 v = \frac{q\theta T_2 - T_1}{q\theta - q^{-1}\theta^{-1}} G_2 v + \frac{q\theta}{q\theta - q^{-1}\theta^{-1}} G_2^2 v$$

$$(iii) \quad t_1 v = \frac{q^{-1}\theta^{-1} T_1 - T_2}{q^{-1}\theta^{-1} - q\theta} v + \frac{1}{q\theta - q^{-1}\theta^{-1}} G_2 v$$

$$(iv) \quad t_1 G_2 v = \frac{q\theta T_1 - T_2}{q\theta - q^{-1}\theta^{-1}} G_2 v + \frac{1}{q^{-1}\theta^{-1} - q\theta} G_2^2 v$$

$\square$

pf Sim to pf of L 14.

Prop 20 With ref to L17 assume

$$q\theta \neq q^{-1}\theta^{-1}, \qquad G_2 v \neq 0$$

and that k_1, k_2 exist.

Then $v, G_2 v$ is a basis for a subspace of V that is invariant under $t_1^{\pm 1}, t_2^{\pm 1}$.

Rel the basis $v, G_2 v$,

$$X: \begin{pmatrix} \theta & 0 \\ 0 & q^{-2}\theta^{-1} \end{pmatrix}$$

$$G_2: \begin{pmatrix} 0 & q^2\theta^{-2}\left(\theta - \frac{k_1}{qk_2}\right)\left(\theta - \frac{k_2}{qk_1}\right)\left(\theta - \frac{k_1k_2}{q}\right)\left(\theta - \frac{1}{qk_1k_2}\right) \\ 1 & 0 \end{pmatrix}$$

$$t_2: \begin{pmatrix} \dfrac{q^{-1}\theta^{-1}(k_2+k_2^{-1}) - k_1 - k_1^{-1}}{q^{-1}\theta^{-1} - q\theta} & \dfrac{q^3\left(\theta - \frac{k_1}{qk_2}\right)\left(\theta - \frac{k_2}{qk_1}\right)\left(\theta - \frac{k_1k_2}{q}\right)\left(\theta - \frac{1}{qk_1k_2}\right)}{\theta(q\theta - q^{-1}\theta^{-1})} \\ \dfrac{q^{-1}\theta^{-1}}{q^{-1}\theta^{-1} - q\theta} & \dfrac{q\theta(k_2+k_2^{-1}) - k_1 - k_1^{-1}}{q\theta - q^{-1}\theta^{-1}} \end{pmatrix}$$

$$t_2^{-1}: \begin{pmatrix} \dfrac{q\theta(k_2+k_2^{-1}) - k_1 - k_1^{-1}}{q\theta - q^{-1}\theta^{-1}} & \dfrac{q^3\left(\theta - \frac{k_1}{qk_2}\right)\left(\theta - \frac{k_2}{qk_1}\right)\left(\theta - \frac{k_1k_2}{q}\right)\left(\theta - \frac{1}{qk_1k_2}\right)}{\theta(q^{-1}\theta^{-1} - q\theta)} \\ \dfrac{q^{-1}\theta^{-1}}{q\theta - q^{-1}\theta^{-1}} & \dfrac{q^{-1}\theta^{-1}(k_2+k_2^{-1}) - k_1 - k_1^{-1}}{q^{-1}\theta^{-1} - q\theta} \end{pmatrix}$$

$$t_1: \begin{pmatrix} \dfrac{q^{-1}\theta^{-1}(k_1+k_1^{-1}) - k_2 - k_2^{-1}}{q^{-1}\theta^{-1} - q\theta} & \dfrac{q^2\left(\theta - \frac{k_1}{qk_2}\right)\left(\theta - \frac{k_2}{qk_1}\right)\left(\theta - \frac{k_1k_2}{q}\right)\left(\theta - \frac{1}{qk_1k_2}\right)}{\theta^2(q^{-1}\theta^{-1} - q\theta)} \\ \\ \dfrac{1}{q\theta - q^{-1}\theta^{-1}} & \dfrac{q\theta(k_1+k_1^{-1}) - k_2 - k_2^{-1}}{q\theta - q^{-1}\theta^{-1}} \end{pmatrix}$$

$$t_1^{-1}: \begin{pmatrix} \dfrac{q\theta(k_1+k_1^{-1}) - k_2 - k_2^{-1}}{q\theta - q^{-1}\theta^{-1}} & \dfrac{q^2\left(\theta - \frac{k_1}{qk_2}\right)\left(\theta - \frac{k_2}{qk_1}\right)\left(\theta - \frac{k_1k_2}{q}\right)\left(\theta - \frac{1}{qk_1k_2}\right)}{\theta^2(q\theta - q^{-1}\theta^{-1})} \\ \\ \dfrac{1}{q^{-1}\theta^{-1} - q\theta} & \dfrac{q^{-1}\theta^{-1}(k_1+k_1^{-1}) - k_2 - k_2^{-1}}{q^{-1}\theta^{-1} - q\theta} \end{pmatrix}$$

pf sim to pf of P15 $\square$

Prop 21 With ref to L17 assume

$$q\theta \neq q^{-1}\theta^{-1}, \qquad G_2 v = 0$$

and that k_1, k_2 exist.
Then

(i) $$t_2 v = \frac{q^{-1}\theta^{-1}(k_2+k_2^{-1}) - k_1 - k_1^{-1}}{q^{-1}\theta^{-1} - q\theta} v$$

(ii) $$t_2^{-1} v = \frac{q\theta(k_2+k_2^{-1}) - k_1 - k_1^{-1}}{q\theta - q^{-1}\theta^{-1}} v$$

(iii) $$t_1 v = \frac{q^{-1}\theta^{-1}(k_1+k_1^{-1}) - k_2 - k_2^{-1}}{q^{-1}\theta^{-1} - q\theta} v$$

(iv) $$t_1^{-1} v = \frac{q\theta(k_1+k_1^{-1}) - k_2 - k_2^{-1}}{q\theta - q^{-1}\theta^{-1}} v$$

(v) $$\theta \in \left\{ \frac{k_1}{qk_2}, \frac{k_2}{qk_1}, \frac{k_1k_2}{q}, \frac{1}{qk_1k_2} \right\}$$

pf Sim to pf of Prop 16 □

LEM 22 Let V denote an irred. $\hat{H}_q$-module

Assume $\mp 1, \mp q^{-1}$ are not eigenvalues of X on V and that k_i exists for $i \in \mathbb{I}$.

Assume X has at least one eigenvector v on V

[automatic if $\dim V < \infty$]

Then

(i) V is spanned by

$$\ldots \quad G_0G_2v, \quad G_2v, \quad v, \quad G_0v, \quad G_2G_0v, \quad G_0G_2G_0v, \ldots \qquad *$$

[some might be 0]

(ii) Vectors $*$ are eigenvectors for X on V

(iii) X is diagonalizable on V

pf (i) Span of $*$ is invar under $\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$.

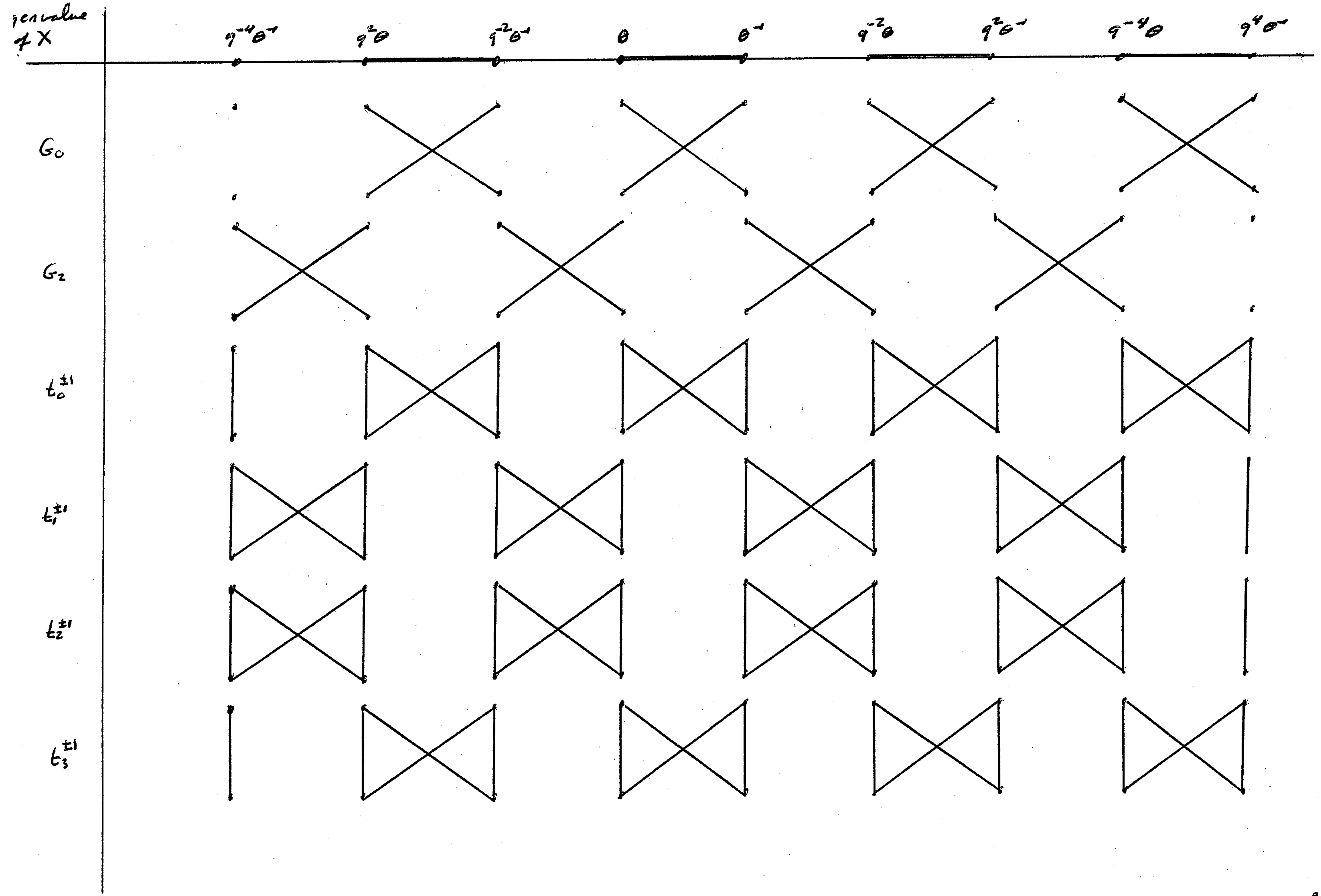
(ii) By Lem 5

(iii) By (i), (ii) $\square$

With ref to L22

$$
\begin{tikzcd}
& & & G_2G_0G_2G_0V \arrow[r, dash] & \cdots \\
& & G_2G_0V \arrow[r, leftrightarrow, "G_0"'] \arrow[d, leftrightarrow, "G_2"] & G_0G_2G_0V \arrow[u, leftrightarrow, "G_2"'] & \\
& V \arrow[r, leftrightarrow, "G_0"'] \arrow[d, leftrightarrow, "G_2"] & G_0V & & \\
G_0G_2V \arrow[r, leftrightarrow, "G_0"'] & G_2V & & & \\
\vdots & & & &
\end{tikzcd}
$$

Actions of $G_0, G_2, \{t_i^{\pm 1}\}_{i \in I\!I}$ on the eigenspaces of X



Cor 23 Let V denote an $\hat{H}_q$-module

Given $0 \neq \theta \in \mathbb{F}$

(i) Assume $\theta \neq \pm 1$ and k_0, k_3 exist for V. Then both

$$t_0^{\pm} V_X(\theta) \subseteq V_X(\theta) + V_X(\theta^{-1})$$

$$t_3^{\pm} V_X(\theta) \subseteq V_X(\theta) + V_X(\theta^{-1})$$

(ii) Assume $\theta \neq \mp q^{-1}$ and k_1, k_2 exist for V. Then both

$$t_1^{\pm 1} V_X(\theta) \subseteq V_X(\theta) + V_X(q^{-2}\theta^{-1})$$

$$t_2^{\pm 1} V_X(\theta) \subseteq V_X(\theta) + V_X(q^{-2}\theta^{-1})$$

pf (i) By P15, 16 and since $\theta \neq \theta^{-1}$, for $v \in V_X(\theta)$

$$\begin{aligned} t_0^{\pm} v &\in \text{Span}(v, G_0 v) \\ &\subseteq V_X(\theta) + V_X(\theta^{-1}) \end{aligned}$$

and

$$\begin{aligned} t_3^{\pm 1} v &\in \text{Span}(v, G_0 v) \\ &\subseteq V_X(\theta) + V_X(\theta^{-1}) \end{aligned}$$

(ii) Similar. $\square$

Recall $A = Y + Y^{-1}$ $\qquad$ $Y = t_0 t_1$

For an $\hat{H}_q$-module V find the action of A on the eigenspaces of X

LEM 24 Let V denote an $\hat{H}_q$-module.
Assume k_i exists for V $(\forall i \in \mathbb{I})$.
Given $0 \neq \theta \in \mathbb{F}$.

(i) Assume $\theta \notin \{\pm 1, \pm q^{-1}, \pm q^{-2}\}$. then

$$Y V_X(\theta) \subseteq V_X(q^2\theta) + V_X(q^{-2}\theta^{-1}) + V_X(\theta) + V_X(\theta^{-1})$$

(ii) Assume $\theta \notin \{\pm q, \pm 1, \pm q^{-1}\}$. then

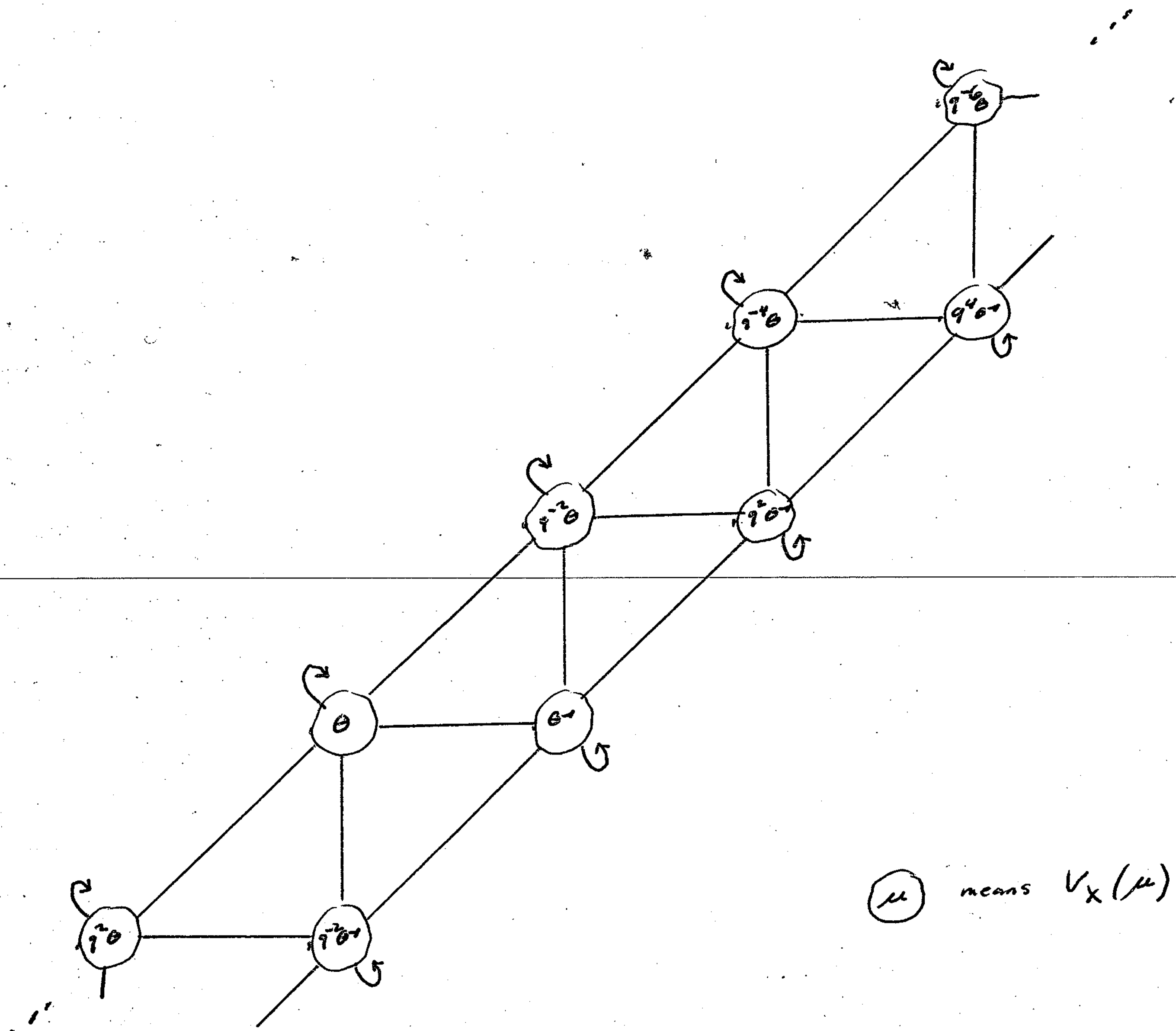
$$Y^{-1} V_X(\theta) \subseteq V_X(q^{-2}\theta^{-1}) + V_X(\theta) + V_X(\theta^{-1}) + V_X(q^{-2}\theta)$$

(iii) Assume $\theta \notin \{\pm q, \pm 1, \pm q^{-1}, \pm q^{-2}\}$ then

$$A Y_X(\theta) \subseteq V_X(q^2\theta) + V_X(q^{-2}\theta^{-1}) + V_X(\theta) + V_X(\theta^{-1}) + V_X(q^{-2}\theta)$$

pf By Cor 23. $\square$

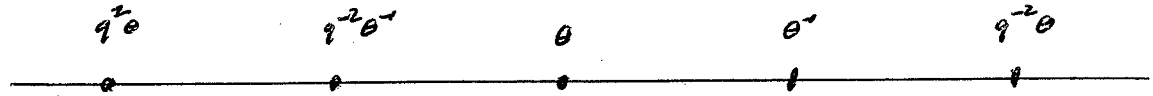
The action of A on the eigenspaces of X

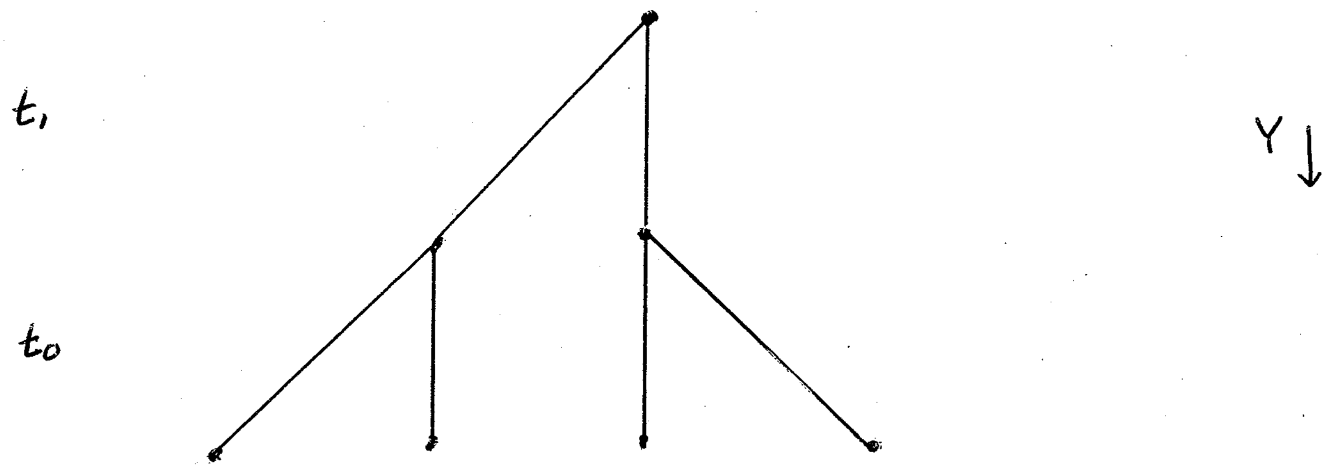


A sends each eigenspace into the sum of adjacent eigenspaces

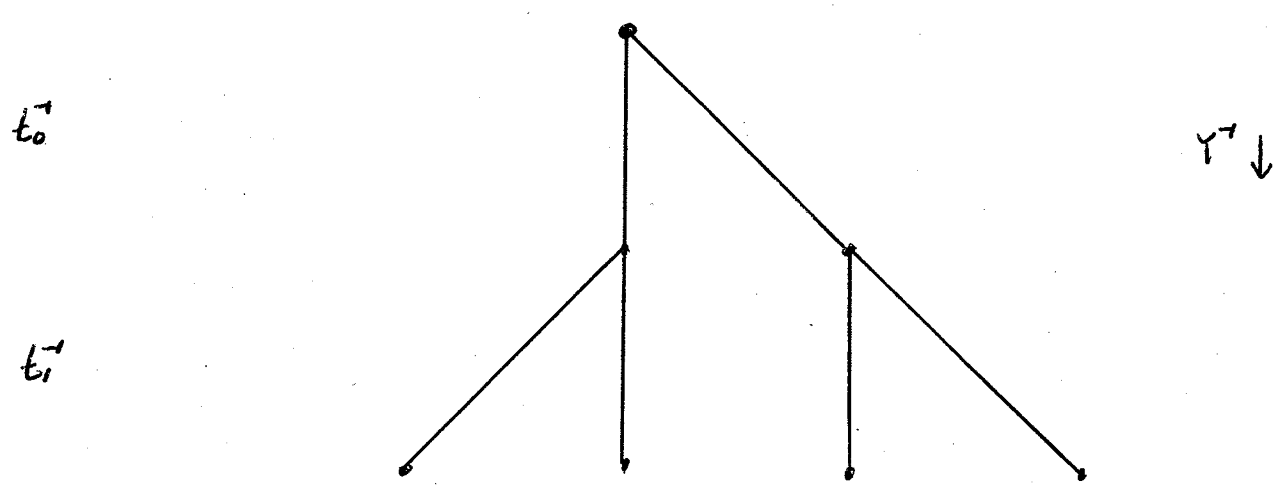
Recall $\quad A = Y + Y^{-1} \qquad Y = t_0 t_1$

Find action of A on the eigenspaces of X





$$Y V_X(\theta) \subseteq V_X(q^2\theta) + V_X(q^{-2}\theta^{-1}) + V_X(\theta) + V_X(\theta^{-1})$$



$$Y^{-1} V_X(\theta) \subseteq V_X(q^{-2}\theta^{-1}) + V_X(\theta) + V_X(\theta^{-1}) + V_X(q^{-2}\theta)$$

$$A V_X(\theta) \subseteq V_X(q^2\theta) + V_X(q^{-2}\theta^{-1}) + V_X(\theta) + V_X(\theta^{-1}) + V_X(q^{-2}\theta)$$

Next goal

Consider $A = Y + Y^{-1}$

$B = X + X^{-1}$

Let V denote an $\hat{H}_q$-module

Find the action of A on the eigenspace of B

LEM 25 Let V denote an $\hat{H}_q$-module

Given $0 \neq \theta \in \mathbb{F}$

$$V_B(\theta + \theta^{-1}) = V_X(\theta) + V_X(\theta^{-1})$$

provided $\theta \neq \mp 1$

pf $\supseteq$: clear

$\subseteq$: Given $v \in V_B(\theta+\theta^{-1})$ show $v \in V_X(\theta) + V_X(\theta^{-1})$

$Bv = (\theta+\theta^{-1})v$

$(X + X^{-1})v = (\theta + \theta^{-1})v$

$(X - \theta)(X - \theta^{-1})v = 0$

$$\frac{X - \theta^{-1}}{\theta - \theta^{-1}} v \in V_X(\theta)$$

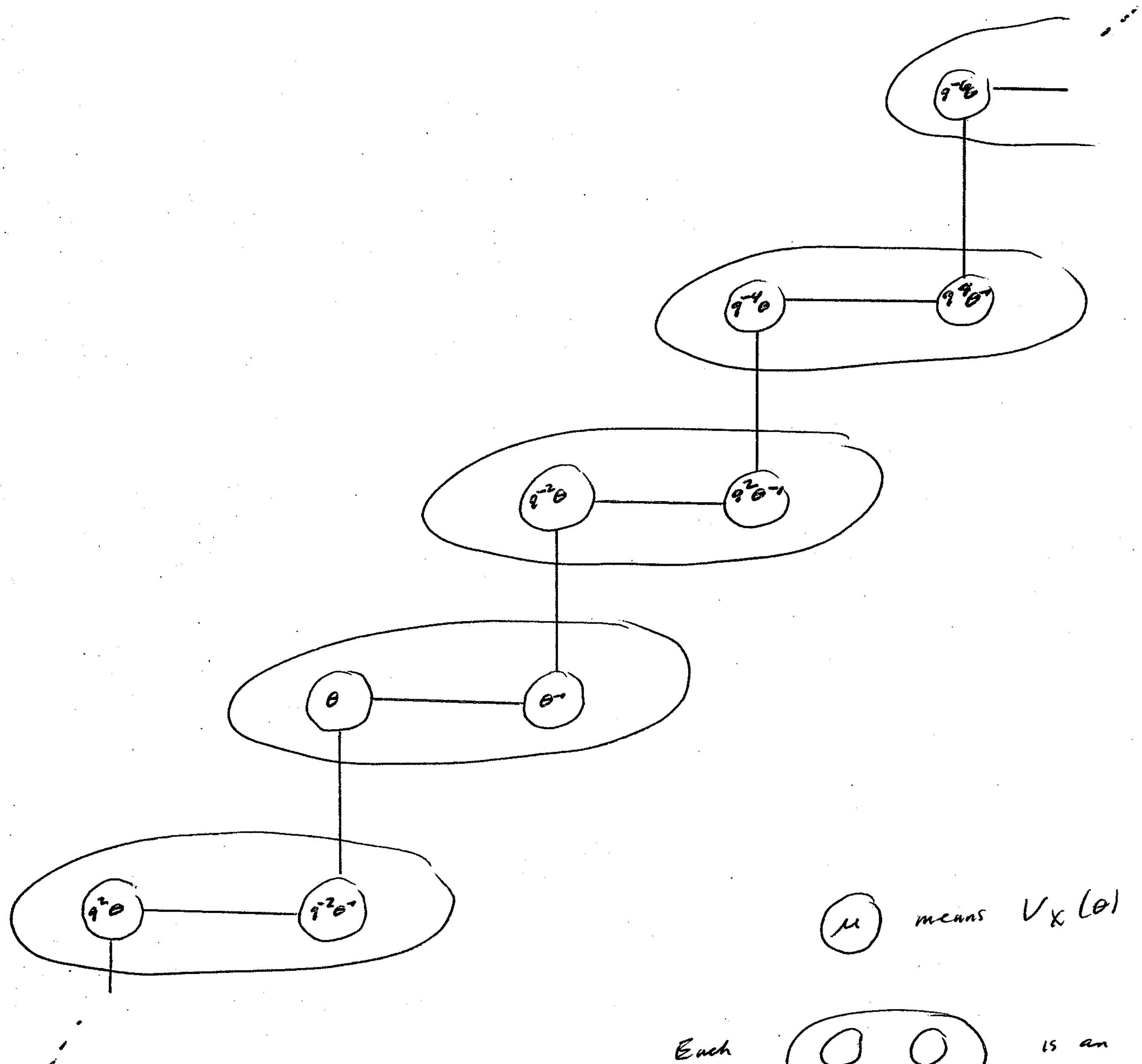
$$\frac{X - \theta}{\theta^{-1} - \theta} v \in V_X(\theta^{-1})$$

$$v = \frac{X - \theta^{-1}}{\theta - \theta^{-1}} v + \frac{X - \theta}{\theta^{-1} - \theta} v$$

$\in V_X(\theta) + V_X(\theta^{-1})$

$\square$

the eigenspaces of $B = X + X^{-1}$



μ means $V_X(\theta)$

Each is an eigenspace of B

Thm 26 Let V denote an $\hat{H}_q$ module

Assume k_i exist for V $\forall i \in \mathbb{I}$.

Given $0 \neq \theta \in \mathbb{F}$. Then

$$A\, V_B(\theta + \theta^{-1}) \subseteq$$

$$V_B(q^2\theta + q^{-2}\theta^{-1}) + V_B(\theta + \theta^{-1}) + V_B(q^{-2}\theta + q^2\theta^{-1})$$

provided

$$\theta \notin \{\mp q^{-2}, \pm q^{-1}, \pm 1, \pm q, \pm q^2\}$$

pf Combine L 24, L 25. $\square$

[B acts in a sim fashion on the eigenspaces of A]

$\mathbb{F}$ alg closed

$0 \neq q \in \mathbb{F}$ $\quad q^4 \neq 1$

Continue to describe the $\hat{H}_q$-modules

Let V denote a $\hat{H}_q$-module

Cases of interest:

X is diagonalizable on V $\quad *$

or

Y is diagonalizable on V

(or both)

WLOG focus on $*$

Under assumption $*$ we can improve some of our earlier results as follows

LEM 27 Let V denote an $\hat{H}_q$-module on which X is diagonalizable. Then for $0 \neq \theta \in \mathbb{F}$

$$V_B(\theta + \theta^{-1}) = V_X(\theta) + V_X(\theta^{-1})$$

[even if $\theta = \pm 1$]

pf Assume $\theta = \pm 1$ else done by L25

So $\theta = \theta^{-1}$

$\supseteq$ ✓

$\subseteq$: Given $v \in V_B(\theta + \theta^{-1})$ show $v \in V_X(\theta)$

$$(X + X^{-1})v = (\theta + \theta^{-1})v$$

$$(X - \theta)(X - \theta^{-1})v = 0$$

$$(X - \theta)^2 v = 0$$

$$(X - \theta)v = 0 \qquad \text{since } X \text{ diagonalizable}$$

$$v \in V_X(\theta)$$

LEM 28 Let V denote an $\hat{H}_q$-module on which X is diagonalizable. Then for $0 \neq \theta \in \mathbb{F}$

(i) $t_0^{\pm 1} V_X(\theta) \subseteq V_X(\theta) + V_X(\theta^{-1})$

(ii) $t_3^{\pm 1} V_X(\theta) \subseteq V_X(\theta) + V_X(\theta^{-1})$

(iii) $t_1^{\pm 1} V_X(\theta) \subseteq V_X(\theta) + V_X(q^{-2}\theta^{-1})$

(iv) $t_2^{\pm 1} V_X(\theta) \subseteq V_X(\theta) + V_X(q^{-2}\theta^{-1})$

pf (i) t_0 commutes with $B = X + X^{-1}$.

So

$$t_0^{\pm 1} V_B(\theta+\theta^{-1}) \subseteq V_B(\theta+\theta^{-1})$$

So

$$\begin{aligned} t_0^{\pm 1} V_X(\theta) &\subseteq t_0^{\pm 1} V_B(\theta+\theta^{-1}) \\ &\subseteq V_B(\theta+\theta^{-1}) \\ &= V_X(\theta) + V_X(\theta^{-1}) \end{aligned}$$

(ii) By (i) and $X = t_3 t_0$

(iii), (iv) Sim

$\square$

LEM 29 Let V denote an $\hat{H}_q$-module on which X is diagonalizable. Then for $0 \neq \theta \in \mathbb{F}$

$$Y V_X(\theta) \subseteq V_X(q^2\theta) + V_X(q^{-2}\theta^{-1}) + V_X(\theta) + V_X(\theta^{-1})$$

$$Y^{-1} V_X(\theta) \subseteq V_X(q^{-2}\theta^{-1}) + V_X(\theta) + V_X(\theta^{-1}) + V_X(q^{-2}\theta)$$

$$A V_X(\theta) \subseteq V_X(q^2\theta) + V_X(q^{-2}\theta^{-1}) + V_X(\theta) + V_X(\theta^{-1}) + V_X(q^{-2}\theta)$$

pf by L 28 and $Y = t_0 t_1$, $A = Y + Y^{-1}$ $\square$

thm 30 Let V denote an $\hat{H}_q$-module on which X is diagonalizable. Then for $0 \neq \theta \in \mathbb{F}$

$$A\, V_B(\theta+\theta^{-1}) \subseteq V_B(q^2\theta + q^{-2}\theta^{-1}) + V_B(\theta+\theta^{-1}) + V_B(q^{-2}\theta + q^2\theta^{-1})$$

pf Combine L 27, 29. $\square$

Caution Ref to m30 assume V is irred.

In gen V is not irred as a module for A, B.

Indeed suppose k_0 exists and $k_0 \neq \pm 1$

We saw earlier

$$V = V_{t_0}(k_0) + V_{t_0}(k_0^{-1}) \qquad \text{ds}$$

with each of

$$V_{t_0}(k_0), \quad V_{t_0}(k_0^{-1})$$

invar under A, B.

Next goal: describe the actions of A, B on $V_{t_0}(k_0^{\pm 1})$

obs:

el of $\hat{H}_q$	action on $V_{t_0}(k_0)$	action on $V_{t_0}(k_0^{-1})$
$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}$	1	0
$\frac{t_0 - k_0}{k_0^{-1} - k_0}$	0	1

So

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}, \qquad \frac{t_0 - k_0}{k_0^{-1} - k_0}$$

are the primitive idempotents for the action of t_0 on V

Note that

$$1 = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} + \frac{t_0 - k_0}{k_0^{-1} - k_0}$$

$$t_0 = k_0 \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} + k_0^{-1} \frac{t_0 - k_0}{k_0^{-1} - k_0}$$

$$t_0^{-1} = k_0^{-1} \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} + k_0 \frac{t_0 - k_0}{k_0^{-1} - k_0}$$

Let W denote an eigenspace for the action of A on V. Then

$$W = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} W + \frac{t_0 - k_0}{k_0^{-1} - k_0} W \qquad (ds)$$

Let U denote an eigenspace for the action of B on V. Then

$$U = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} U + \frac{t_0 - k_0}{k_0^{-1} - k_0} U \qquad (ds)$$

348

Thm 31 Let V denote an $\hat{H}_q$-module on which X is diagonalizable.

Assume k_0 exists for V and $k_0 \neq \pm 1$.

Then for $0 \neq \theta \in \mathbb{F}$ both

$$
\begin{aligned}
A \, \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V_B(\theta + \theta^{-1}) \subseteq \; & \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V_B(q^2\theta + q^{-2}\theta^{-1}) \\
& + \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V_B(\theta + \theta^{-1}) \\
& + \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} V_B(q^{-2}\theta + q^{2}\theta^{-1})
\end{aligned}
$$

$$
\begin{aligned}
A \, \frac{t_0 - k_0}{k_0^{-1} - k_0} V_B(\theta + \theta^{-1}) \subseteq \; & \frac{t_0 - k_0}{k_0^{-1} - k_0} V_B(q^2\theta + q^{-2}\theta^{-1}) \\
& + \frac{t_0 - k_0}{k_0^{-1} - k_0} V_B(\theta + \theta^{-1}) \\
& + \frac{t_0 - k_0}{k_0^{-1} - k_0} V_B(q^{-2}\theta + q^{2}\theta^{-1})
\end{aligned}
$$

Pf By thm 30 and comments below it. $\square$

Next goal: describe terms in above thm.

Until further notice:

V denotes an $\hat{H}_q$-module on which X is diagonalizable

Assume k_0, k_3 exist for V and $k_0 \neq \mp 1$.

Given $0 \neq \theta \in \mathbb{F}$ and

$0 \neq v \in V_X(\theta)$

Assume $\theta \neq \theta^{-1}$ and $G_0 v \neq 0$

By Prop 15

$v, \quad G_0 v$ $\qquad *$

is a basis for a subspace of V that is inv under $t_0^{\pm 1}, t_3^{\pm 1}$

For t_0 the eigenvalues on this space are k_0, k_0^{-1}.

Corresp eigvectors are

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v, \qquad \frac{t_0 - k_0}{k_0^{-1} - k_0} v \qquad **$$

The vectors $**$ form a basis for $\mathrm{Span}(v, G_0 v)$.

Compare the bases $*, **$.

LEM 32 With above notation

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v = \frac{\theta k_0 + \theta^{-1} k_0^{-1} - k_3 - k_3^{-1}}{(\theta - \theta^{-1})(k_0 - k_0^{-1})} v + \frac{\theta}{(\theta - \theta^{-1})(k_0 - k_0^{-1})} G_0 v$$

$$\frac{t_0 - k_0}{k_0^{-1} - k_0} v = \frac{\theta k_0^{-1} + \theta^{-1} k_0 - k_3 - k_3^{-1}}{(\theta - \theta^{-1})(k_0^{-1} - k_0)} v + \frac{\theta}{(\theta - \theta^{-1})(k_0^{-1} - k_0)} G_0 v$$

and

$$V = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v + \frac{t_0 - k_0}{k_0^{-1} - k_0} v$$

$$G_0 v = \frac{k_3 + k_3^{-1} - \theta k_0^{-1} - \theta^{-1} k_0}{\theta} \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v$$

$$+ \frac{k_3 + k_3^{-1} - \theta k_0 - \theta^{-1} k_0^{-1}}{\theta} \frac{t_0 - k_0}{k_0^{-1} - k_0} v$$

pf Prop 15 and lin alg □

LEM 33 with above notation

(i) $$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} G_0 v = \frac{k_3 + k_3^{-1} - \theta k_0^{-1} - \theta^{-1} k_0}{\theta} \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v$$

(ii) $$\frac{t_0 - k_0}{k_0^{-1} - k_0} G_0 v = \frac{k_3 + k_3^{-1} - \theta k_0 - \theta^{-1} k_0^{-1}}{\theta} \frac{t_0 - k_0}{k_0^{-1} - k_0} v$$

$\square$

pf Use Prop 15 and lin alg.

LEM 34 with normalization

nil basis

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}\, v, \qquad \frac{t_0 - k_0}{k_0^{-1} - k_0}\, v$$

the matrices rep t_0, $X^{\pm 1}$, $t_3^{\pm 1}$

are

$$t_0 : \begin{pmatrix} k_0 & 0 \\ 0 & k_0^{-1} \end{pmatrix}$$

$$X : \begin{pmatrix} \dfrac{(\theta+\theta^{-1})k_0 - k_3 - k_3^{-1}}{k_0 - k_0^{-1}} & \dfrac{\theta k_0^{-1} + \theta^{-1} k_0 - k_3 - k_3^{-1}}{k_0^{-1} - k_0} \\[4em] \dfrac{\theta k_0 + \theta^{-1} k_0^{-1} - k_3 - k_3^{-1}}{k_0 - k_0^{-1}} & \dfrac{(\theta+\theta^{-1})k_0^{-1} - k_3 - k_3^{-1}}{k_0^{-1} - k_0} \end{pmatrix}$$

$$X^{-1}: \begin{pmatrix} \dfrac{(\theta+\theta^{-1})k_0^{-1}-k_3-k_3^{-1}}{k_0^{-1}-k_0} & \dfrac{\theta k_0^{-1}+\theta^{-1}k_0-k_3-k_3^{-1}}{k_0-k_0^{-1}} \\ & \\ \dfrac{\theta k_0+\theta^{-1}k_0^{-1}-k_3-k_3^{-1}}{k_0^{-1}-k_0} & \dfrac{(\theta+\theta^{-1})k_0-k_3-k_3^{-1}}{k_0-k_0^{-1}} \end{pmatrix}$$

$$t_3:\begin{pmatrix} \dfrac{\theta+\theta^{-1}-(k_3+k_3^{-1})k_0^{-1}}{k_0-k_0^{-1}} & \dfrac{(\theta k_0^{-1}+\theta^{-1}k_0-k_3-k_3^{-1})k_0}{k_0^{-1}-k_0} \\ \dfrac{(\theta k_0+\theta^{-1}k_0^{-1}-k_3-k_3^{-1})k_0^{-1}}{k_0-k_0^{-1}} & \dfrac{\theta+\theta^{-1}-(k_3+k_3^{-1})k_0}{k_0^{-1}-k_0} \end{pmatrix}$$

$$t_3^{-1}:\begin{pmatrix} \dfrac{\theta+\theta^{-1}-(k_3+k_3^{-1})k_0}{k_0^{-1}-k_0} & \dfrac{(\theta k_0^{-1}+\theta^{-1}k_0-k_3-k_3^{-1})k_0}{k_0-k_0^{-1}} \\ \dfrac{(\theta k_0+\theta^{-1}k_0^{-1}-k_3-k_3^{-1})k_0^{-1}}{k_0^{-1}-k_0} & \dfrac{\theta+\theta^{-1}-(k_3+k_3^{-1})k_0^{-1}}{k_0-k_0^{-1}} \end{pmatrix}$$

pf use Prop 15 and lin alg. $\square$

$$T = \begin{pmatrix} 1 & \dfrac{k_3 + k_3^{-1} - \theta k_0^{-1} - \theta^{-1} k_0}{\theta} \\ 1 & \dfrac{k_3 + k_3^{-1} - \theta k_0 - \theta^{-1} k_0^{-1}}{\theta} \end{pmatrix}$$

$$X^{new} = T \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix} T^{-1}$$

X^{new} $tr = \theta + \theta^{-1}$, $det = 1$

$\mathbb{F}$ alg closed

$0 \neq q \in \mathbb{F}$ $q^4 \neq 1$

Continue to describe the $\hat{H}_q$-modules

Until further notice:

- V denotes an $\hat{H}_q$-module on which X is diagonalizable
- Assume k_i exists for V $\forall i \in \mathbb{I}$
- $k_0 \neq \mp 1$

Given $0 \neq \theta \in \mathbb{F}$

$0 \neq v \in V_X(\theta)$

Find action of $A = Y + Y^{-1}$ on

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v, \qquad \frac{t_0 - k_0}{k_0^{-1} - k_0} v$$

Since k_0 only defined up to reciprocal, we are free to replace k_0 by k_0^{-1}. So WLOG focus on

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v$$

with above notation, $A \quad \overline{\quad\quad} \atop k_0 - k_0^{-1}$

term	coef
$\dfrac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} G_2 v$	$\dfrac{\theta k_0 + \theta^{-1} k_0^{-1} - k_3 - k_3^{-1}}{(\theta - \theta^{-1})(q\theta - q^{-1}\theta^{-1})}$
$\dfrac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v$	$\dfrac{\theta k_0 + \theta^{-1} k_0^{-1} - k_3 - k_3^{-1}}{\theta - \theta^{-1}} \dfrac{q^{-1}\theta^{-1}(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{q^{-1}\theta^{-1} - q\theta} + \dfrac{q^{-1}\theta(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{\theta - \theta^{-1}} \dfrac{k_3 + k_3^{-1} - \theta k_0^{-1} - \theta^{-1} k_0}{q^{-1}\theta - q\theta^{-1}} + \dfrac{k_1 + k_1^{-1}}{k_0}$
$\dfrac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} G_2 G_0 v$	$\dfrac{\theta}{(\theta - \theta^{-1})(q\theta^{-1} - q^{-1}\theta)}$

provided

$$\theta \neq \theta^{-1}, \quad q\theta \neq q^{-1}\theta^{-1}, \quad q^{-1}\theta \neq q\theta^{-1}$$

pf

$$A \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v = \Big(t_0 t_1 + \underbrace{t_1^{-1} t_0^{-1}}_{T_1 t_1} \Big) \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v$$

eigenvector for t_0 with eigenval k_0

$$= \Big(t_0 t_1 + (k_1 + k_1^{-1} - t_1) k_0^{-1} \Big) \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v$$

$$= (t_0 - k_0^{-1}) \; t_1 \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v$$

$$+ \frac{k_1 + k_1^{-1}}{k_0} \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v$$

$$\left.\begin{aligned} &= (k_0 - k_0^{-1}) \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} t_1 \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v \\ &\qquad + \frac{k_1 + k_1^{-1}}{k_0} \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v \end{aligned}\right\} \ast$$

Find

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} t_1 \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v$$

$$t_1 \frac{k_0 - k_0^{-1}}{k_0 - k_0^{-1}} v =$$

$$t_1 \left(\frac{\theta k_0 + \theta^{-1} k_0^{-1} - k_3 - k_3^{-1}}{(\theta - \theta^{-1})(k_0 - k_0^{-1})} v + \frac{\theta}{(\theta - \theta^{-1})(k_0 - k_0^{-1})} G_0 v \right)$$

(by L19)

$$t_1 v = \frac{q^{-1}\theta^{-1}(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{q^{-1}\theta^{-1} - q\theta} v + \frac{1}{q\theta - q^{-1}\theta^{-1}} G_2 v$$

(by L19)

$$t_1 G_0 v = \frac{q^{-1}\theta(k_1 + k_1^{-1}) - k_2 - k_2^{-1}}{q^{-1}\theta - q\theta^{-1}} G_0 v + \frac{1}{q\theta^{-1} - q^{-1}\theta} G_2 G_0 v$$

(by L19, with v replaced by $G_0 v$ and θ replaced by θ^{-1})

Now eval

v, $G_2 v$, $G_0 v$, $G_2 G_0 v$

$$v = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v + \frac{t_0 - k_0}{k_0^{-1} - k_0} v \qquad \text{by } L32$$

$$G_2 v = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} G_2 v + \frac{t_0 - k_0}{k_0^{-1} - k_0} G_2 v \qquad L32$$

$$G_0 v = \frac{k_3 + k_3^{-1} - \theta k_0^{-1} - \theta^{-1} k_0}{\theta} \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} v$$

$$+ \frac{k_3 + k_3^{-1} - \theta k_0 - \theta^{-1} k_0^{-1}}{\theta} \frac{t_0 - k_0}{k_0^{-1} - k_0} v$$

L32 if $G_0 v \neq 0$ L14(i) if $G_0 v = 0$

$$G_2 G_0 v = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} G_2 G_0 v + \frac{t_0 - k_0}{k_0^{-1} - k_0} G_2 G_0 v \qquad L32$$

Observe

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} = \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}$$

$$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} \frac{t_0 - k_0}{k_0^{-1} - k_0} = 0$$

Combining the above info we get

$$\frac{t_0-k_0^{-1}}{k_0-k_0^{-1}}\, t_1\, \frac{t_0-k_0^{-1}}{k_0-k_0^{-1}}\, v =$$

term	coef
$\frac{t_0-k_0^{-1}}{k_0-k_0^{-1}} G_2 v$	$\frac{\theta k_0+\theta^{-1}k_0^{-1}-k_3-k_3^{-1}}{(\theta-\theta^{-1})(k_0-k_0^{-1})} \quad \frac{1}{q\theta-q^{-1}\theta^{-1}}$
$\frac{t_0-k_0^{-1}}{k_0-k_0^{-1}} v$	$\frac{\theta k_0+\theta^{-1}k_0^{-1}-k_3-k_3^{-1}}{(\theta-\theta^{-1})(k_0-k_0^{-1})} \quad \frac{q^{-1}\theta^{-1}(k_1+k_1^{-1})-k_2-k_2^{-1}}{q^{-1}\theta^{-1}-q\theta}$ $+$ $\frac{q^{-1}\theta(k_1+k_1^{-1})-k_2-k_2^{-1}}{(\theta-\theta^{-1})(k_0-k_0^{-1})} \quad \frac{k_3+k_3^{-1}-\theta k_0^{-1}-\theta^{-1}k_0}{q^{-1}\theta-q\theta^{-1}}$
$\frac{t_0-k_0^{-1}}{k_0-k_0^{-1}} G_2 G_0 v$	$\frac{\theta}{(\theta-\theta^{-1})(k_0-k_0^{-1})} \quad \frac{1}{q\theta^{-1}-q^{-1}\theta}$

Result follows using $*$.

$36e^2$ □

Shortly we will return to the 3-term rec of Th 35

First we need to describe the irred $\hat{H}_q$-modules

<u>Notation</u> For indets $\lambda_1, \lambda_2, \lambda_3$

$$G(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1 + \lambda_1^{-1})^2 + (\lambda_2 + \lambda_2^{-1})^2 + (\lambda_3 + \lambda_3^{-1})^2$$

$$- (\lambda_1 + \lambda_1^{-1})(\lambda_2 + \lambda_2^{-1})(\lambda_3 + \lambda_3^{-1}) \quad - 4$$

$$= \lambda_1^{-2} (\lambda_1 - \lambda_2 \lambda_3^{-1})(\lambda_1 - \lambda_2^{-1} \lambda_3)(\lambda_1 - \lambda_2 \lambda_3)(\lambda_1 - \lambda_2^{-1} \lambda_3^{-1})$$

With ref to Cor 9, for $0 \neq \theta \in \mathbb{F}$

G_0^2 acts on $V_X(\theta)$ as $G(\theta, k_0, k_3)\, I$

G_1^2 .. $V_Y(\theta)$ .. $G(\theta, k_0, k_1)\, I$

G_2^2 .. $V_X(\theta)$ $G(q^{-1}\theta^{-1}, k_1, k_2)\, I$

G_3^2 .. $V_Y(\theta)$ $G(q^{-1}\theta^{-1}, k_2, k_3)\, I$