

LEM 150 We have

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$$\frac{qXA - q^{-1}AX}{q^2 - q^{-2}} = qYX + \frac{A\epsilon_0 T_3 - Y\epsilon_0 T_3 - q\epsilon_0 T_2}{q + q^{-1}}$$

$$\frac{qAYX - q^{-1}YXA}{q^2 - q^{-2}} = q^2 X + \frac{q^{-1}Y\epsilon_0 T_2 - q^{-2}\epsilon_0 T_3}{q + q^{-1}}$$

$$\frac{qBY - q^{-1}YB}{q^2 - q^{-2}} = qYX + \frac{T_1 T_3 - q\epsilon_0 T_2 - X\epsilon_0^{-1} T_1}{q + q^{-1}}$$

$$\frac{qYXB - q^{-1}BYX}{q^2 - q^{-2}} = q^{-1}Y + \frac{q^{-1}B\epsilon_0 T_2 - q^{-2}\epsilon_0 T_1 + q^{-1}X\epsilon_0^{-1} T_2 - q^{-1}T_2 T_3}{q + q^{-1}}$$

pf Use reduction rules to verify

□

\mathbb{F} arb $0 \neq q \in \mathbb{F} \quad q^2 \neq 1$

Recall

$$\hat{H}_q^+ = \{ h \in \hat{H}_q \mid t_0 h = h t_0 \}$$

$$\hat{H}_q^- = \{ h \in \hat{H}_q \mid t_0 h = h t_0^{-1} \}$$

Next goal describe \hat{H}_q^-

Recall our decomp

$$\begin{aligned} \hat{H}_q &= \langle A \rangle \langle B \rangle \pi + \langle A \rangle x \langle B \rangle \pi + \langle A \rangle y \langle B \rangle \pi \\ &\quad + \langle A \rangle yx \langle B \rangle \pi \\ &= \langle A \rangle \langle B \rangle \pi + \langle A \rangle t_1 \langle B \rangle \pi + \langle A \rangle t_2 \langle B \rangle \pi + \\ &\quad \langle A \rangle t_3 \langle B \rangle \pi \end{aligned}$$

LEM 15)

$$\begin{aligned} \hat{H}_q^- &\subseteq \langle A \rangle \langle B \rangle \pi + \langle A \rangle x \langle B \rangle \pi + \langle A \rangle y \langle B \rangle \pi \quad * \\ &= \langle A \rangle \langle B \rangle \pi + \langle A \rangle t_1 \langle B \rangle \pi + \langle A \rangle t_3 \langle B \rangle \pi \end{aligned}$$

pf Let $W_1 = \text{RHS} \neq *$

$$W_2 = \langle A \rangle yx \langle B \rangle \pi$$

so $\hat{H}_q^- = W_1 + W_2$ (ds qus)

show $\hat{H}_q^- \subseteq W_1$

Given $h \in \hat{H}_q^-$ show $h \in W_1$

Consider map

$$\begin{aligned} \hat{H}_g &\longrightarrow \hat{H}_g \\ \theta: h &\longrightarrow \cancel{h} - h\vec{t}_0 \end{aligned}$$

$$\hat{H}_g^- = \ker(\theta)$$

Write

$$h = \overset{\wedge}{h_1} + \overset{\wedge}{h_2}$$

$W_1 \qquad W_2$

**

show $h_2 = 0$

Recall W_1 is θ -inv

Also θ acts on \hat{H}_g/W_1 as $z \rightarrow z(t_0 - \vec{t}_0)$

Apply θ to **

$$0 = \underset{W_1}{\overset{\wedge}{\theta(h_1)}} + \underbrace{\overset{\wedge}{\theta(h_2)} - h_2(t_0 - \vec{t}_0)}_{\overset{\wedge}{W_1}} + \underbrace{h_2(t_0 - \vec{t}_0)}_{\overset{\wedge}{W_2}}$$

So $0 = h_2(t_0 - \vec{t}_0)$

So $0 = h_2$

□

Thm 152 $\hat{H}_q^- = \underbrace{\{t_0 h - h t_0 \mid h \in \hat{H}_2\}}_{H^-}$

pf \supseteq : By L 118

\subseteq :

obs $H^- = \langle A \rangle (t_0 t_1 - t_1 t_0) \langle B \rangle \pi + \langle A \rangle (t_0 t_2 - t_2 t_0) \langle B \rangle \pi$

+ $\langle A \rangle (t_0 t_3 - t_3 t_0) \langle B \rangle \pi$

Recall

$$t_0 t_1 - t_1 t_0 = \underbrace{A - t_0^{-1} T_1}_{\pi} - t_1 (t_0 - t_0^{-1}) \quad (1)$$

$$\langle A \rangle \langle B \rangle \pi$$

$$t_0 t_3 - t_3 t_0 = \underbrace{B - t_0^{-1} T_3}_{\pi} - t_3 (t_0 - t_0^{-1}) \quad (2)$$

$$\langle A \rangle \langle B \rangle \pi$$

$$t_0 t_2 - t_2 t_0 = q^{-1} B t_0^{-1} T_1 - q^{-1} A B \quad \in \langle A \rangle \langle B \rangle \pi$$

$$+ q^{-1} t_1 T_3 - q^{-1} t_1 B t_0^{-1} \quad \in \langle A \rangle t_1 \langle B \rangle \pi$$

$$+ q^{-1} A t_3 t_0 - q^{-1} t_3 T_1 \quad \in \langle A \rangle t_3 \langle B \rangle \pi \quad (3)$$

By (1)

$$t_1(t_0 - t_0^{-1}) \in H^- + \langle A \rangle \langle B \rangle \Pi \quad (1')$$

By (2)

$$t_3(t_0 - t_0^{-1}) \in H^- + \langle A \rangle \langle B \rangle \Pi \quad (2')$$

By (3) and (1')

$$A t_3 t_0 - t_3 T_1 - t_1 B t_0 + t_1 T_3 \in H^- + \langle A \rangle \langle B \rangle \Pi \quad (3')$$

mult each term in (3') by t_0^{-1} and use (1'), (2')

to get

$$A t_3 - t_3 t_0 T_1 - t_1 B + t_1 t_0 T_3 \in H^- + \langle A \rangle \langle B \rangle \Pi \quad (3'')$$

We assume

$$\exists h \in \hat{H}_g^- \setminus H^-$$

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and get a contradiction.

By L151

$$h = \sum_{i_1 \in \mathbb{N}} A^{i_1} B^2 t_{i_1} + \sum_{i_2 \in \mathbb{N}} A^{i_2} t_{i_2} B^2 t_{i_2}' + \sum_{i_3 \in \mathbb{N}} A^{i_3} t_{i_3} B^2 t_{i_3}''$$

$$t_{i_1}, t_{i_2}', t_{i_3}'' \in \mathbb{T} = \langle t_0^{\neq 1}, T_1, T_2, T_3 \rangle$$

By (1) and since

1. t_0 is a basis for a complement of $\langle t_0^{\neq 1} \rangle$ (to t_0)
in $\langle t_0^{\neq 1} \rangle$

WLOG

$$t_{i_1}' \in \underbrace{\langle T_1, T_2, T_3 \rangle}_{\mathbb{Z}} + t_0 \langle T_1, T_2, T_3 \rangle \quad i_1 \in \mathbb{N}$$

Similarly using (2). WLOG

$$t_{i_2}'' \in \mathbb{Z} + t_0 \mathbb{Z} \quad i_2 \in \mathbb{N}$$

Write

$$h = \sum_{i_1 \in \mathbb{N}} A^{i_1} B^2 t_{i_1} + \sum_{i_2 \in \mathbb{N}} A^{i_2} t_{i_2} B^2 (\underbrace{z_{i_2}}_{\mathbb{Z}} + t_0 \underbrace{z_{i_2}'}_{\mathbb{Z}}) + \sum_{i_3 \in \mathbb{N}} A^{i_3} t_{i_3} B^2 (\underbrace{z_{i_3}''}_{\mathbb{Z}} + t_0 \underbrace{z_{i_3}''' }_{\mathbb{Z}})$$

Using (3') WLOG

$$z_{i_2}' = 0 \quad \text{if } i_2 > 0 \quad i_2 \in \mathbb{N}$$

Using (3'') WLOG

$$z_{i_3}''' = 0 \quad \text{if } i_3 > 0 \quad i_3 \in \mathbb{N}$$

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Now

$$\begin{aligned}
 0 &= t_0 h - h t_0^\dagger \\
 &= \sum_{i_1 \in \mathbb{N}} A^{i_1} B^\dagger t_{i_1} (t_0 - t_0^\dagger) \\
 &+ \sum_{i_1 \in \mathbb{N}} A^{i_1} \underbrace{(A - t_0^\dagger T_1)}_{\text{" } t_0 t_1 - t_1 t_0^\dagger} B^\dagger (z_{i_1} + t_0 z_{i_1}^{\prime}) \\
 &+ \sum_{i_1 \in \mathbb{N}} A^{i_1} \underbrace{(B - t_0^\dagger T_3)}_{\text{" } t_0 t_3 - t_3 t_0^\dagger} B^\dagger (z_{i_1}^{\prime\prime} + t_0 z_{i_1}^{\prime\prime\prime})
 \end{aligned}
 \quad \left. \vphantom{\sum_{i_1 \in \mathbb{N}}} \right\} \star$$

Each term above is in $\langle A \rangle \langle B \rangle \Pi$

obs

$$\begin{aligned}
 \langle A \rangle \langle B \rangle \Pi &= \sum_{i_1 \in \mathbb{N}} A^{i_1} B^\dagger z + \sum_{i_1 \in \mathbb{N}} A^{i_1} B^\dagger z t_0 \\
 &+ \langle A \rangle \langle B \rangle \Pi (t_0 - t_0^\dagger)
 \end{aligned}
 \quad (ds 105)$$

For $i_1 \in \mathbb{N}$ consider contribution to \star from $A^{i_1} B^\dagger z =$

$$0 = z_{i_1, j} - T_1 z_{i_1}^{\prime} + z_{i_1, j}^{\prime\prime} - T_3 z_{i_1}^{\prime\prime\prime}$$

\uparrow
 0 if $i=0$

For $i_1 \in \mathbb{N}$ consider contrib to \star from $A^{i_1} B^\dagger z t_0 =$

$$0 = -T_1 z_{i_1} + z_{i_1, j}^{\prime} - T_3 z_{i_1}^{\prime\prime} + z_{i_1, j}^{\prime\prime\prime}$$

\uparrow
 0 if $i=0$

Show

$$z_{i,j}, z'_{i,j}, z''_{i,j}, z'''_{i,j} \text{ all } 0$$

$$\forall i, j \in \mathbb{N}$$

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Suppose not.

Define

$$n = \min \left\{ i \mid \begin{array}{l} i \in \mathbb{N}, \text{ at least one of } \{z_{i,j}\}_{j \in \mathbb{N}} \\ \{z'_{i,j}\}_{j \in \mathbb{N}}, \{z''_{i,j}\}_{j \in \mathbb{N}}, \{z'''_{i,j}\}_{j \in \mathbb{N}} \text{ is non } 0 \end{array} \right\}$$

For $j \in \mathbb{N}$

$$0 = -T_1 z'_{n,j} + z''_{n,j} - T_3 z'''_{n,j}$$

$$0 = -T_1 z_{n,j} - T_3 z''_{n,j} + z'''_{n,j}$$

Case $j=0$

$$0 = -T_1 z'_{n,0} - T_3 z'''_{n,0}$$

$$0 = -T_1 z_{n,0} - T_3 z''_{n,0}$$

Case $j \geq 1$

$$0 = z'''_{n,j} - T_3 z''_{n,j}$$

$$0 = -T_3 z''_{n,j} + z'''_{n,j}$$

$$\begin{aligned}
 z_{n0}^I = 0 &\leftrightarrow z_{n0}^{III} = 0 \\
 &\leftrightarrow z_{n1}^{II} = 0 \\
 &\leftrightarrow z_{n2}^{III} = 0 \\
 &\leftrightarrow z_{n3}^{II} = 0 \\
 &\vdots
 \end{aligned}$$

finitely many of these
are non zero
so all 0

$$\begin{aligned}
 z_{n0} = 0 &\leftrightarrow z_{n0}^{II} = 0 \\
 &\leftrightarrow z_{n1}^{III} = 0 \\
 &\leftrightarrow z_{n2}^{II} = 0 \\
 &\leftrightarrow z_{n3}^{III} = 0 \\
 &\vdots
 \end{aligned}$$

finitely many of these
are non 0
so all are 0.

this contradicts the def of n .

We have shown

$$z_{ij}, z_{ij}^I, z_{ij}^{II}, z_{ij}^{III} \text{ all } 0 \quad i, j \in \mathbb{N}$$

Now

$$\begin{aligned}
 h &= \sum_{i, j \in \mathbb{N}} A^i B^j t_{ij} \\
 &\in \hat{H}_g^+
 \end{aligned}$$

But now

$$h \in \hat{H}_g^+ \cap \hat{H}_g^- = 0 \in \hat{H}^- \text{ cont}$$

□

Cor 153 the \mathbb{F} -algebra $H_9^+ + H_9^-$

is generated by

A, B, C

$t_0^{\pm 1}, T_1, T_2, T_3$

$t_0 t_1 - t_1 t_0, t_0 t_2 - t_2 t_0, t_0 t_3 - t_3 t_0$

pf By M152 and since H_9^+ is gen by

$A, B, C, t_0^{\pm 1}, T_1, T_2, T_3$

□

\mathbb{F} arb

$$0 \neq q \in \mathbb{F} \quad q^2 \neq 1$$

Continue to describe \hat{H}_q^\pm

Cor 15.4 For the \mathbb{F} -alg $\hat{H}_q^+ + \hat{H}_q^-$

each of the following is a gen set:

(i) $A, B, C, t_0^{\pm 1}, T_1, T_2, T_3, t_1(t_0 - t_0^{-1}), t_2(t_0 - t_0^{-1}), t_3(t_0 - t_0^{-1})$

(ii) $A, B, C, t_0, T_1, T_2, T_3, (t_0 - t_0^{-1})t_1, (t_0 - t_0^{-1})t_2, (t_0 - t_0^{-1})t_3$

pf (i) Recall

$$\begin{aligned} t_0 t_1 - t_1 t_0 &= A - t_0^{-1} T_1 - t_1(t_0 - t_0^{-1}) \\ t_0 t_2 - t_2 t_0 &= C - t_0^{-1} T_2 - t_2(t_0 - t_0^{-1}) \\ t_0 t_3 - t_3 t_0 &= B - t_0^{-1} T_3 - t_3(t_0 - t_0^{-1}) \end{aligned}$$

Now use Ca 15.3

(ii) One checks

$$\begin{aligned} t_0 t_1 - t_1 t_0 &= -A + t_0^{-1} T_1 + (t_0 - t_0^{-1}) t_1 \\ t_0 t_2 - t_2 t_0 &= -C + t_0^{-1} T_2 + (t_0 - t_0^{-1}) t_2 \\ t_0 t_3 - t_3 t_0 &= -B + t_0^{-1} T_3 + (t_0 - t_0^{-1}) t_3 \end{aligned}$$

Now use Ca 15.3. □

Next goal: Find a basis for \hat{H}_q^-

Recall

$$\begin{aligned} \tilde{H}_g &= \langle A \rangle \langle B \rangle \pi + \langle A \rangle x \langle B \rangle \pi + \langle A \rangle y \langle B \rangle \pi + \langle A \rangle yx \langle B \rangle \pi (t_0 - t_0^{-1}) \\ &= \langle A \rangle \langle B \rangle \pi + \langle A \rangle x \langle B \rangle \pi + \langle A \rangle y \langle B \rangle \pi + \langle A \rangle c \langle B \rangle \pi \\ &= \langle A \rangle \langle B \rangle \pi + \langle A \rangle t_1 \langle B \rangle \pi + \langle A \rangle t_3 \langle B \rangle \pi + \langle A \rangle t_2 \langle B \rangle \pi (t_0 - t_0^{-1}) \end{aligned}$$

obs

$$\hat{H}_g^+ + \hat{H}_g^- \subseteq \tilde{H}_g$$

Since

$$\hat{H}_g^+ = \langle A \rangle \langle B \rangle \pi + \langle A \rangle c \langle B \rangle \pi$$

$$\hat{H}_g^- \subseteq \langle A \rangle \langle B \rangle \pi + \langle A \rangle x \langle B \rangle \pi + \langle A \rangle y \langle B \rangle \pi$$

LEM 155 the following is a basis for a complement of \tilde{H}_g in \hat{H}_g

$$A^i t_2 B^r T_1^r T_2^s T_3^t$$

$$A^i t_2 B^r t_0 T_0^r T_2^s T_3^t$$

$$i, r, s, t \in \mathbb{N}$$

pf By def of \tilde{H}_g and recall

$$\langle t_0^{\pm 1} \rangle = \mathbb{F}1 + \mathbb{F}t_0 + \langle t_0^{\pm 1} \rangle (t_0 - t_0^{-1}) \quad ds \quad \square$$

let $Z = \langle T_1, T_2, T_3 \rangle$

Cor 156

$$\hat{H}_g = \tilde{H}_g + \langle A \rangle t_2 \langle B \rangle Z + \langle A \rangle t_2 \langle B \rangle t_0 Z \quad (ds \text{ vs})$$

pf By L155.

Thm 157 the following is a basis for the \mathbb{F} -vector space \hat{H}_g^-

$$\left\{ \begin{array}{l} A^i (t_0 t_1 - t_1 t_0) B^j t_0^k T_1^r T_2^s T_3^t \quad k \in \mathbb{Z} \\ A^i (t_0 t_3 - t_3 t_0) B^j t_0^k T_1^r T_2^s T_3^t \quad i, j, r, s, t \in \mathbb{N} \\ A^i (t_0 t_2 - t_2 t_0) B^j T_1^r T_2^s T_3^t \\ A^i (t_0 t_2 - t_2 t_0) B^j t_0^k T_1^r T_2^s T_3^t \end{array} \right.$$

pf the following is a basis for a complement of \hat{H}_g^- in \hat{H}_g :

$$\hat{H}_g^+ \text{ in } \hat{H}_g :$$

$$\begin{array}{l} A^i t_1 B^j t_0^k T_1^r T_2^s T_3^t \quad k \in \mathbb{Z}, \quad i, j, r, s, t \in \mathbb{N} \\ A^i t_3 B^j t_0^k T_1^r T_2^s T_3^t \end{array} \quad **$$

So $**U**$ is a basis for a complement of \hat{H}_g^+ in \hat{H}_g .

Apply the map $h \rightarrow t_0 h - h t_0$ to each basis vector in $**U**$ and recall

$$\begin{array}{l} \hat{H}_g^+ = \text{ker of map} \quad \text{by def of } \hat{H}_g^+ \\ \hat{H}_g^- = \text{image of map} \quad \text{by Th 152} \end{array}$$

□

Cor 158 the following sum is direct:

$$\begin{aligned} \hat{H}_g^- &= \langle A \rangle (t_0 t_1 - t_1 t_0) \langle B \rangle \mathbb{T} \\ &+ \langle A \rangle (t_0 t_3 - t_3 t_0) \langle B \rangle \mathbb{T} \\ &+ \langle A \rangle (t_0 t_2 - t_2 t_0) \langle B \rangle \mathbb{Z} \\ &+ \langle A \rangle (t_0 t_2 - t_2 t_0) \langle B \rangle t_0 \mathbb{Z} \end{aligned}$$

it by Th 157

□
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Th 159 The following is a basis for a complement

of \hat{H}_g^- in

$$\langle A \rangle \langle B \rangle \Pi + \langle A \rangle \times \langle B \rangle \Pi + \langle A \rangle \vee \langle B \rangle \Pi$$

$$(\langle A \rangle \langle B \rangle \Pi + \langle A \rangle \wedge \langle B \rangle \Pi + \langle A \rangle \vee \langle B \rangle \Pi) =$$

$$A^i B^j t_0^k T_1^r T_2^s T_3^t \quad k \in \mathbb{Z}, \text{ i, r, s, t} \in \mathbb{N}$$

$$A^i t_1 T_1^r T_2^s T_3^t \quad \text{i, r, s, t} \in \mathbb{N}$$

$$A^i t_2 t_0 T_1^r T_2^s T_3^t$$

$$A^i t_3 B^j T_1^r T_2^s T_3^t$$

$$A^i t_3 B^j t_0 T_1^r T_2^s T_3^t$$

i, r, s, t $\in \mathbb{N}$

pf

We invoke Th 157

Recall

$$t_0 t_1 - t_1 t_0 = A - t_0 T_1 - t_1 (t_0 - t_0 T_1)$$

$$t_0 t_2 - t_2 t_0 = 0 - t_0 T_2 - t_2 (t_0 - t_0 T_2)$$

$$t_0 t_3 - t_3 t_0 = C - t_0 T_3 - t_3 (t_0 - t_0 T_3)$$

$$= q^{-1} B t_0 T_1 - q^{-1} A B$$

$$+ q^{-1} t_1 T_2 - q^{-1} t_1 B t_0$$

$$+ q^{-1} A t_3 t_0 - q^{-1} t_3 T_1$$

show

$$\hat{H}_g^- + \text{Span}(\star) = \langle A \rangle \langle B \rangle \Pi + \langle A \rangle t_3 \langle B \rangle \Pi + \langle A \rangle t_1 \langle B \rangle \Pi$$

As we saw in pt of #152

$$t_1(t_0 - t_0^{-1}) \in \hat{H}_g^- + \langle A \rangle \langle B \rangle \Pi \quad (1)$$

$$t_3(t_0 - t_0^{-1}) \in \hat{H}_g^- + \langle A \rangle \langle B \rangle \Pi \quad (2)$$

$$At_3t_0 - t_3T_1 - t_1Bt_0 + t_1T_3 \in \hat{H}_g^- + \langle A \rangle \langle B \rangle \Pi \quad (3)$$

$$At_3 - t_3t_0T_1 - t_1B + t_1t_0T_3 \in \hat{H}_g^- + \langle A \rangle \langle B \rangle \Pi \quad (4)$$

Note $\langle A \rangle \langle B \rangle \Pi \subseteq \text{Span}(\star)$

show $\langle A \rangle t_3 \langle B \rangle \Pi \subseteq \hat{H}_g^- + \text{Span}(\star)$

$\langle A \rangle t_3 \langle B \rangle \Pi$ has basis

$$\begin{array}{l} \underbrace{A^i t_3 B^j T_1^r T_2^s T_3^t}_{\wedge \star} \quad \text{with } r, s, t \in \mathbb{N} \\ \underbrace{A^i t_3 B^j t_0 T_1^r T_2^s T_3^t}_{\wedge \star} \quad \text{---} \\ \underbrace{A^i t_3 B^j t_0^k (t_0 - t_0^{-1}) T_1^r T_2^s T_3^t}_{\wedge \star} \quad \text{---} \end{array}$$

$\wedge (2)$

$$\hat{H}_g^- + \langle A \rangle \langle B \rangle \Pi$$

$$\subseteq \hat{H}_g^- + \text{Span}(\star)$$

$\langle A \rangle t_1, \langle B \rangle \pi$ has basis

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$$A^i t_1, B^j T_1^r T_2^s T_3^t$$

disjoint $\in \mathbb{N}$ (5)

$$A^i t_1, B^j t_0, T_1^r T_2^s T_3^t$$

... (6)

$$A^i t_1, B^j t_0^k (t_0 - t_0^k) T_1^r T_2^s T_3^t$$

... $k \in \mathbb{Z}$

$$\hat{H}_g^- + \langle A \rangle \langle B \rangle \pi$$

$$\hat{H}_g^- + \text{Span}(\star)$$

show (5), (6) in $\hat{H}_g^- + \text{Span}(\star)$

By (3)

$$A^{i_3} t_3 B^j t_0 T_1^r T_2^s T_3^t - A^{i_3} t_3 B^j T_1^r T_2^s T_3^t$$

$$- A^{i_1} t_1 B^{j_1} t_0 T_1^r T_2^s T_3^t + A^{i_1} t_1 B^{j_1} T_1^r T_2^s T_3^{t_1}$$

$$\in \hat{H}_g^- + \langle A \rangle \langle B \rangle \pi$$

$$\subseteq \hat{H}_g^- + \text{Span}(\star)$$

disjoint $\in \mathbb{N}$

So

$$A^i t_1 B^{j_1} t_0 T_1^r T_2^s T_3^t - A^i t_1 B^{j_1} T_1^r T_2^s T_3^{t_1}$$

(7)

$$\in \hat{H}_g^- + \text{Span}(\star)$$

disjoint $\in \mathbb{N}$

By (4)

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$$A^{i_1} t_3 B^{\beta} T_1^r T_2^s T_3^t - A^{i_2} t_3 B^{\beta} t_0 T_1^{r_0} T_2^s T_3^t$$

$$- A^{i_1} t_1 B^{\beta r_0} T_1^r T_2^s T_3^t + A^{i_2} t_1 B^{\beta} t_0 T_1^r T_2^s T_3^t$$

$$\in H_9^{\wedge} + \langle A \rangle \langle B \rangle \pi$$

$$\in H_9^{\wedge} + \text{Span}(\star)$$

$i_1, r, s, t \in \mathbb{N}$

Σ_0

$$A^{i_1} t_1 B^{\beta r_0} T_1^r T_2^s T_3^t - A^{i_2} t_1 B^{\beta} t_0 T_1^r T_2^s T_3^t \quad (8)$$

$$\in H_9^{\wedge} + \text{Span}(\star)$$

$i_1, r, s, t \in \mathbb{N}$

By (7), (8) and end in 7

$$A^{i_1} t_1 B^{\beta} T_1^r T_2^s T_3^t \in H_9^{\wedge} + \text{Span}(\star)$$

$$A^{i_2} t_1 B^{\beta} t_0 T_1^r T_2^s T_3^t$$

$i_1, r, s, t \in \mathbb{N} \quad l \geq 1$

By this and def of \star we find

$$(5), (6) \in H_9^{\wedge} + \text{Span}(\star)$$

One checks

Show vector \star together with basis for H_9 in $\mathbb{P}^n(52)$ 8

form lin indep set:

Apply map $h \rightarrow t \circ h - h \circ t^{-1}$ to above vectors

recall \hat{H}_9^- is ker of map in \hat{H}_9

Apply map to \star

$$\left\{ \begin{array}{l} A^i B_j^{\rightarrow} t_0^k (t_0 - t_0^{-1}) T_1^r T_2^s T_3^t \\ A^i (A - t_0^{-1} T_1) T_1^r T_2^s T_3^t \\ A^i (A - t_0^{-1} T_1) t_0 T_1^r T_2^s T_3^t \\ A^i (B - t_0^{-1} T_3) B^{\rightarrow} T_1^r T_2^s T_3^t \\ A^i (B - t_0^{-1} T_3) B^{\rightarrow} t_0 T_1^r T_2^s T_3^t \end{array} \right.$$

$$i, j, r, s, t \in \mathbb{N} \quad k \in \mathbb{Z}$$

$$i, r, s, t \in \mathbb{N}$$

$$i, r, s, t \in \mathbb{N}$$

One checks $\{ \}$ are lin indep and result follows. □

COR 160 The following sum is direct

9

$$\begin{aligned} & \langle A \rangle \langle B \rangle \Pi + \langle A \rangle \times \langle B \rangle \Pi + \langle A \rangle \vee \langle B \rangle \Pi \\ = & \quad \overset{\sim}{H_9} - \quad + \quad \langle A \rangle \langle B \rangle \Pi \\ & + \quad \langle A \rangle t_1 \mathbb{Z} \quad + \quad \langle A \rangle t_0 \mathbb{Z} \\ & + \quad \langle A \rangle t_3 \langle B \rangle \mathbb{Z} \\ & \quad \quad + \quad \langle A \rangle t_3 \langle B \rangle t_0 \mathbb{Z} \end{aligned}$$

pf By th 159.

□

th 161 the following is a basis for a complement of $\hat{H}_9^+ + \hat{H}_9^-$ in \tilde{H}_9

$$\begin{aligned}
 & A^i t_1 T_1^r T_2^s T_3^t && i, r, s, t \in \mathbb{N} \\
 & A^i t_1 t_0 T_1^r T_2^s T_3^t && \\
 & A^i t_3 B^j T_1^r T_2^s T_3^t && i, j, r, s, t \in \mathbb{N} \\
 & A^i t_3 B^j t_0 T_1^r T_2^s T_3^t &&
 \end{aligned}$$

★

pt By th 159 the sum

$$\hat{H}_9^- + \langle A \rangle \langle B \rangle \Pi$$

is direct.

By th 159.

★

is a basis for a complement of $\hat{H}_9^- + \langle A \rangle \langle B \rangle \Pi$

in

$$\langle A \rangle \langle B \rangle \Pi + \langle A \rangle t_1 \langle B \rangle \Pi + \langle A \rangle t_3 \langle B \rangle \Pi$$

So ★ is a basis for a complement of

$$\hat{H}_9^- + \underbrace{\langle A \rangle \langle B \rangle \Pi + \langle A \rangle C \langle B \rangle \Pi}_{\hat{H}_9^+}$$

in

$$\langle A \rangle \langle B \rangle \Pi + \langle A \rangle t_1 \langle B \rangle \Pi + \langle A \rangle t_3 \langle B \rangle \Pi + \langle A \rangle C \langle B \rangle \Pi$$

$$(\cong \tilde{H}_9)$$

□

COR 162 The following sum is direct

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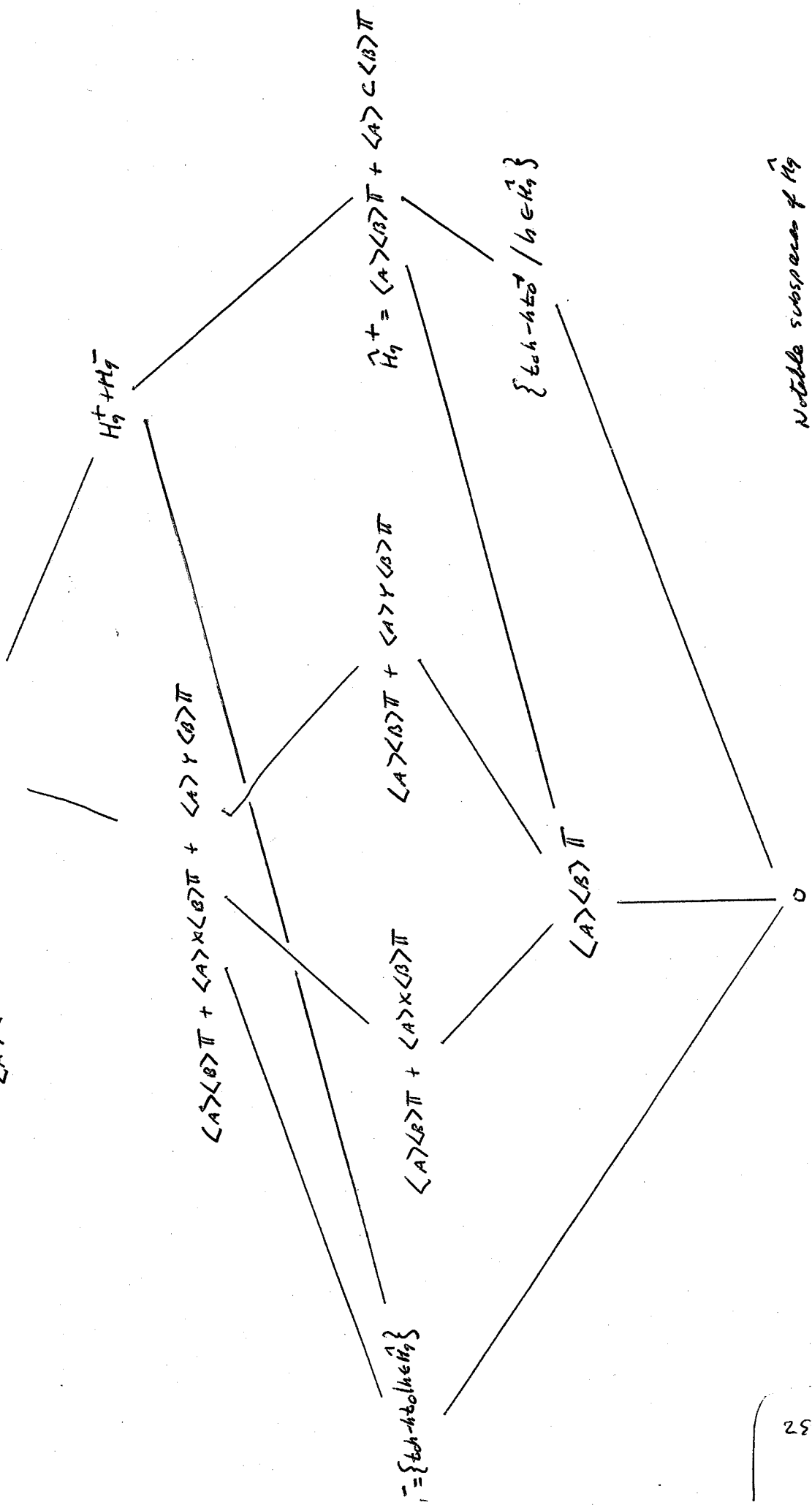
$$\begin{aligned} \tilde{H}_g = & \hat{H}_g^+ + \hat{H}_g^- + \langle A \rangle t_1 \mathbb{Z} \\ & + \langle A \rangle t_1 t_0 \mathbb{Z} + \langle A \rangle t_3 \langle B \rangle \mathbb{Z} \\ & + \langle A \rangle t_3 \langle B \rangle t_0 \mathbb{Z} \end{aligned}$$

pf by th 16.1

□

\hat{H}_9

$$\begin{aligned} & \langle A \rangle \langle B \rangle \pi + \langle A \rangle \langle X \rangle \langle B \rangle \pi + \langle A \rangle \langle Y \rangle \langle B \rangle \pi + \langle A \rangle \langle X \rangle \langle B \rangle \pi + \langle A \rangle \langle Y \rangle \langle B \rangle \pi + \langle A \rangle \langle X \rangle \langle B \rangle \pi + \langle A \rangle \langle Y \rangle \langle B \rangle \pi \\ & = \langle A \rangle \langle B \rangle \pi + \langle A \rangle \langle X \rangle \langle B \rangle \pi + \langle A \rangle \langle Y \rangle \langle B \rangle \pi \end{aligned}$$



Notable subspace of \hat{H}_9

\mathbb{F} arb

$$0 \neq q \in \mathbb{F} \quad q^4 \neq 1$$

Next general goal:

describe the center $Z(\hat{H}_q)$

We will show

- the \mathbb{F} -algebra $Z(\hat{H}_q)$ is gen by T_0, T_1, T_2, T_3 provided q is not a root of 1.

Recall that the following is a basis for \hat{H}_q :

$$q^i x^k t_0^k T_0^l T_1^r T_2^a T_3^b$$

$$i, j \in \mathbb{Z}$$

$$k \in \{0, 1\}$$

$$l, r, a, b \in \mathbb{N}$$

So the following sum is direct:

$$\hat{H}_q = \sum_{l, r, a, b \in \mathbb{N}} \langle y \rangle \langle x \rangle T_0^l T_1^r T_2^a T_3^b$$

$$+ \sum_{l, r, a, b \in \mathbb{N}} \langle y \rangle \langle x \rangle t_0 T_0^l T_1^r T_2^a T_3^b$$

DEF 163 let K denote the
2-sided ideal of \hat{H}_g generated by $\{T_i\}_{i \in \mathbb{I}}$

So

$$K = \sum_{i \in \mathbb{I}} \hat{H}_g T_i$$

LEM 164 the following is a basis for
the \mathbb{F} -vector space K :

$$y^i x^j t_0^k T_0^l T_1^r T_2^s T_3^t$$

$$\begin{aligned} i, j &\in \mathbb{Z} \\ k \in \{0, 1\} \\ l, r, s, t &\in \mathbb{N} \\ (l, r, s, t) &\neq (0, 0, 0, 0) \end{aligned}$$

**

pt clear using *

□

COR 165 The following is a basis for
a complement of K in \hat{H}_g

$$y_i x^j t^k$$

$$i \in \mathbb{Z}$$

$$k \in \{0, 1\}$$

pf Compare x and $x^{\#}$



DEF 166 let \bar{H}_q denote the quotient
 \mathbb{F} -algebra

$$\bar{H}_q = \hat{H}_q / K$$

For $h \in \hat{H}_q$ let \bar{h} denote the image of h
under the canonical map

$$\hat{H}_q \rightarrow \bar{H}_q$$

By constr

$$\bar{T}_i = 0$$

$$i \in \mathbb{I}$$

LEM 167 The following is a basis
for the \mathbb{F} -vector space $\overline{\mathbb{F}}_q$:

$$\overline{\gamma}^i \overline{x}^j \overline{E}_0^k$$

$$i, j \in \mathbb{Z} \\ k \in \{0, 1\}$$

pf By Cor 165

□

LEM 168

The following relations hold

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in \overline{H}_q

$$(i) \quad \overline{X} \overline{Y} = q^2 \overline{Y} \overline{X}$$

$$(ii) \quad \overline{E}_0 \overline{X} = \overline{X}^{-1} \overline{E}_0$$

$$(iii) \quad \overline{E}_0 \overline{Y} = \overline{Y}^{-1} \overline{E}_0$$

$$(iv) \quad \overline{E}_0^2 = -1$$

pf (i) Recall

$$C_0 = q(YX - q^{-1}XY)$$

show $\overline{C}_0 = 0$

By Prop 30

$$C_0 = q t_0 T_2 + T_3 t_1 + q^{-1} T_0 t_2 + T_1 t_3 - q^{-1} T_0 T_2 - T_1 T_3$$

 $\in K$ so $\overline{C}_0 = 0$

(ii) By L25

$$t_0 X = X^{-1} t_0 + X T_0 - T_3$$

$$\text{so } \overline{E}_0 \overline{X} = \overline{X}^{-1} \overline{E}_0$$

(iii) Sim to (ii)

$$L^2 + 1 = T_0 t_0 \in K$$

$$\text{so } \overline{E}_0^2 + 1 = 0$$

□

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LEM 169

We have

$$\bar{E}_1 = -\bar{Y}^{-1} \bar{E}_0$$

$$\bar{E}_2 = q^{-1} \bar{Y}^{-1} \bar{X} \bar{E}_0$$

$$\bar{E}_3 = -\bar{X} \bar{E}_0$$

pf

Use L 39

□

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Notation We endow \mathbb{N}^4 with
 partial order \leq as follows

Given $(l, r, s, t) \in \mathbb{N}^4$
 -- $(l', r', s', t') \in \mathbb{N}^4$

$(l, r, s, t) \leq (l', r', s', t')$ whenever

$l \leq l'$ and $r \leq r'$ and $s \leq s'$ and $t \leq t'$.

— 0 —

Fix $(l, r, s, t) \in \mathbb{N}^4$ write

$$L = \hat{H}_q T_0^l T_1^r T_2^s T_3^t$$

L is 2-sided ideal of \hat{H}_q with basis

$$y^i x^j t_0^k T_0^{l'} T_1^{r'} T_2^{s'} T_3^{t'} \quad \begin{matrix} i, j \in \mathbb{Z} \\ k \in \{0, 1\} \end{matrix}$$

$(l', r', s', t') \in \mathbb{N}^4$

$$(l', r', s', t') \geq (l, r, s, t)$$

Obs

LK is 2-sided ideal of \hat{H}_g
that is contained in L

Describe

L/LK

$$\begin{aligned} LK &= L(\hat{H}_g T_0 + \hat{H}_g T_1 + \hat{H}_g T_2 + \hat{H}_g T_3) \\ &= LT_0 + LT_1 + LT_2 + LT_3 \\ &= \hat{H}_g T_0^{2m} T_1^r T_2^s T_3^t \\ &\quad + \hat{H}_g T_0^l T_1^{2m} T_2^r T_3^t \\ &\quad + \hat{H}_g T_0^l T_1^r T_2^{2m} T_3^t \\ &\quad + \hat{H}_g T_0^l T_1^r T_2^r T_3^{2m} \end{aligned}$$

So the following is a basis for a complement of LK
in L :

$$y^i x^j t_0^k T_0^l T_1^r T_2^s T_3^t \quad \begin{array}{l} i, j \in \mathbb{Z} \\ k \in \{0, 1\} \end{array}$$

So \mathbb{F} -vector space L/LK has basis

$$y^i x^j t_0^k + LK \quad \begin{array}{l} i, j \in \mathbb{Z} \\ k \in \{0, 1\} \end{array}$$

\hat{H}_g acts on $V = H_g$ by

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$$\hat{H}_g \times V \rightarrow V$$

$$h \quad u \rightarrow hu$$

So V is \hat{H}_g -module

Each of L, LK is an \hat{H}_g -submodule since they are ideals

Consider quotient module

$$L/LK$$

Each element of K acts on this module as 0

Action of \hat{H}_g on L/LK induces an action of

$$\hat{H}_g/K = \bar{H}_g \text{ on } L/LK.$$

DEF 170

Given $h \in \hat{H}_g$ write

$$h = \sum_{(l, r, s, t) \in \mathbb{N}^4} h_{l, r, s, t} T_0^l T_1^r T_2^s T_3^t$$

$$h_{l, r, s, t} \in \langle y \rangle \langle x \rangle + \langle y \rangle \langle x \rangle b_0$$

$\forall (l, r, s, t) \in \mathbb{N}^4$ call it primary ^(for h) whenever

(i) $h_{l, r, s, t} \neq 0$

(ii) $h_{l, r, s, t} = 0 \quad \forall (l', r', s', t') \in \mathbb{N}^4$ s.t.
 $(l', r', s', t') < (l, r, s, t)$

let $P(h) = \left\{ (l, r, s, t) \in \mathbb{N}^4 \mid (l, r, s, t) \text{ is primary rel } h \right\}$

By const. for $h \in \hat{H}_g$

$$h \in \sum_{(l, r, s, t) \in P(h)} \hat{H}_g T_0^l T_1^r T_2^s T_3^t$$

Thm 171 Assume g is not a root
 $\neq 1$. Then the \mathbb{F} -alg $Z(\hat{H}_g)$ is gen by

$$T_0, T_1, T_2, T_3$$

pf. Given $h \in Z(\hat{H}_g)$

show

$$h \in \langle T_0, T_1, T_2, T_3 \rangle$$

We suppose not and get contradiction.

Write

$$h = \sum_{(l, r, s, t) \in \mathbb{N}^4} h_{l,r,s,t} T_0^l T_1^r T_2^s T_3^t$$

$$h_{l,r,s,t} \in \langle Y \rangle \langle X \rangle + \langle Y \rangle \langle X \rangle t_0$$

Define

$$N_h = \left\{ (l, r, s, t) \in \mathbb{N}^4 \mid h_{l,r,s,t} \neq 0 \right\}$$

wlog choose h s.t. N_h is minimal among all

$$h \in Z(\hat{H}_g) \setminus \langle T_0, T_1, T_2, T_3 \rangle$$

$h \neq 0$ so

$$P(h) \neq \emptyset$$

Fix $(l, r, s, t) \in P(h)$

Define

$$G = \begin{aligned} & \hat{H}_g T_0^{2m} + \hat{H}_g T_1^{2m} \\ & + \hat{H}_g T_2^{2m} + \hat{H}_g T_3^{2m} \end{aligned}$$

(2-sided ideal of \hat{A}_g)

G has basis

$$y^i x^{2k} t_0, T_0^{l'} T_1^{r'} T_2^{s'} T_3^{t'}$$

$$i, j \in \mathbb{Z}$$

$$k \in \{0, 1\}$$

$$(l', r', s', t') \in \mathbb{N}^4$$

$$(l', r', s', t') \notin (\text{l.i.a.t.})$$

By constr.

$$h = h_{\text{e.r.a.t.}} T_0^{l'} T_1^{r'} T_2^{s'} T_3^{t'} \in G \quad \star$$

Write

$$h_{\text{e.r.a.t.}} = \sum_{i \in \mathbb{Z}} \alpha_i y^i x^2$$

$$\alpha_i \in \mathbb{F}$$

$$+ \sum_{i \in \mathbb{Z}} \beta_i y^i x^{2k} t_0$$

$$\beta_i \in \mathbb{F}$$

take commutator of \star with x, y, t_0

x :

$$\underbrace{xh - hx}_0 = \underbrace{(x h_{\text{e.r.a.t.}} - h_{\text{e.r.a.t.}} x)}_0 T_0^{l'} T_1^{r'} T_2^{s'} T_3^{t'} \in G$$

$$\sum_{i \in \mathbb{Z}} \alpha_i \left(\underbrace{x y^i x^2}_{q^{2i} y^i x} - y^i x^2 x \right) + \sum_{i \in \mathbb{Z}} \beta_i \left(\underbrace{x y^i x^{2k} t_0}_{q^{2i} y^i x} - y^i x^{2k} t_0 x \right)$$

$$\left(\sum_{i \in \mathbb{Z}} \alpha_i y^i x^{2k} (q^{2i} - 1) + \sum_{i \in \mathbb{Z}} \beta_i (y^i x^{2k+1} t_0 q^{2i} - y^i x^{2k+1} t_0) \right) T_0^{l'} T_1^{r'} T_2^{s'} T_3^{t'}$$

$$\in G$$

Adjusting indices ii)

$$\left[\begin{aligned} & \sum_{i, j \in \mathbb{Z}} y^i x^j \Delta_{i, j} (q^{2i} - 1) \\ & + \sum_{i, j \in \mathbb{Z}} y^i x^j t_0 (\beta_{i, j} q^{2i} - \beta_{i, j}) \end{aligned} \right]_{T_0^l T_1^r T_2^s T_3^t} \in G$$

But $y^i x^j t_0^k T_0^l T_1^r T_2^s T_3^t$ $i, j \in \mathbb{Z}$
 $k \in \{0, 1, 3\}$

are linear independ mod G

so $\Delta_{i, j} (q^{2i} - 1) = 0 \quad \forall i, j \in \mathbb{Z} \quad (1)$

$\beta_{i, j} q^{2i} - \beta_{i, j} = 0 \quad (2)$

taking the commutator of \star with y we see that

$\Delta_{i, j} (q^{2i} - 1) = 0 \quad \forall i, j \in \mathbb{Z} \quad (3)$

$\beta_{i, j} - \beta_{i, j} q^{-2j} = 0 \quad (4)$

taking the com of \star with t_0 we see that

$\Delta_{i, j} = \Delta_{i, j} q \quad \forall i, j \in \mathbb{Z} \quad (5)$

$\beta_{i, j} = \beta_{i, j} q \quad (6)$

By (1) and (3) and since q not a root of 1
 $\Delta_{i, j} = 0$ if $(i, j) \neq (0, 0)$

By (2) and some finitely many β_i are non 0

$$\beta_i = 0 \quad \forall i \in \mathbb{Z}$$

Now $h_{e,r,s,t} = \alpha_{00} 1$

Define $h' = h - \alpha_{00} t_0^l t_1^r t_2^s t_3^t$

then $h' \in \mathbb{Z}(\mathbb{H}_9) \setminus \langle t_0, t_1, t_2, t_3 \rangle$

and $N(h') = N_h - 1$ ent.

□

Problem 172. Is thm 171 true if we drop the assumption that z is a root of 1?

Cor 172 Assume q not a root of 1.

\exists iso of F -algebras

$$\mathbb{Z}(\mu_q) \rightarrow F[\lambda_0, \lambda_1, \lambda_2, \lambda_3]$$

that sends

$$T_i \rightarrow \lambda_i \quad i \in \mathbb{I}$$

pt. We saw $\{T_i\}_{i \in \mathbb{I}}$ are alg indep.

□

\mathbb{F} arb $0 \neq q \in \mathbb{F}$ $q^4 \neq 1$

View \hat{H}_q as left \hat{H}_q^+ -module with action

$$\begin{array}{ccc} \hat{H}_q^+ & \times & \hat{H}_q \longrightarrow \hat{H}_q \\ h & & u \longrightarrow hu \end{array}$$

or right \hat{H}_q^+ -module with action

$$\begin{array}{ccc} \hat{H}_q & \times & \hat{H}_q^+ \longrightarrow \hat{H}_q \\ u & & h \longrightarrow uh \end{array}$$

LEM 173 As a left or right \hat{H}_q^+ -module

\hat{H}_q is gen by $\{t_i : i \in \mathbb{I}\}$. Mat is

$$(i) \quad \hat{H}_q = \sum_{i \in \mathbb{I}} \hat{H}_q^+ t_i$$

$$(ii) \quad \hat{H}_q = \sum_{i \in \mathbb{I}} t_i \hat{H}_q^+$$

(Cautions: above sums not direct)

(ii) We saw earlier

$$\hat{H}_g = \langle A \rangle \langle B \rangle \Pi + \langle A \rangle t_1 \langle B \rangle \Pi \\ + \langle A \rangle t_2 \langle B \rangle \Pi + \langle A \rangle t_3 \langle B \rangle \Pi$$

$$\left[\langle A \rangle \langle B \rangle \Pi = \langle A \rangle t_0 \langle B \rangle \Pi \text{ since} \right. \\ \left. t_0 \in \Pi \text{ is invertible and} \right. \\ \left. \text{commutes with } B \right]$$

$$= \sum_{i \in \Pi} \langle A \rangle t_i \langle B \rangle \Pi$$

Therefore \hat{H}_g is characterized by the following features:

- (a) contains each of $\{t_i\}_{i \in \Pi}$
- (b) closed under left mult by A
- (c) closed under right mult by B
- (d) closed under right mult by Π

Define

$$H = \sum_{i \in \Pi} t_i \hat{H}_g^{n+}$$

Show H satisfies (a) - (d) above

(a) ✓

(b) ✓

(c) ✓

(d): show $A t_i \in H \quad i \in \Pi$

$$At_0 = t_0 A \in H \quad \checkmark$$

$$At_1 = t_1 A \in H \quad \checkmark$$

$At_2 =$

By L146

$$t_3 + \frac{q t_2 A - q^{-1} A t_2}{q^2 - q^{-2}}$$

$$= \frac{t_0 t_0^{-2} T_1 T_2 - t_0 t_0^{-1} T_2 + q^{-1} T_3}{q + q^{-1}}$$

so $At_2 \in H$

$At_3 =$

By L145

$$t_2 + \frac{q A t_3 - q^{-1} t_3 A}{q^2 - q^{-2}}$$

$$= \frac{t_0^{-1} T_1 T_3 - t_0 t_0^{-1} T_3 + q^{-1} T_2}{q + q^{-1}}$$

so $At_3 \in H \quad \checkmark$

(i) Similar

□

Recall

$$\hat{H}_g^+ \hat{H}_g^- \subseteq \hat{H}_g^-$$

$$\hat{H}_g^- \hat{H}_g^+ \subseteq \hat{H}_g^-$$

so \hat{H}_g^- is a \hat{H}_g^+ -submodule of \hat{H}_g , w.r.t either left or right action.

LEM 174 As a left or right \hat{H}_g^+ -module

\hat{H}_g^- is gen by $\{t_0 t_i - t_i t_0\}_{i=1}^3$. That is

(i)
$$\hat{H}_g^- = \sum_{i=1}^3 \hat{H}_g^+ (t_0 t_i - t_i t_0)$$

(ii)
$$\hat{H}_g^- = \sum_{i=1}^3 (t_0 t_i - t_i t_0) \hat{H}_g^+$$

[above sums not direct]

pf (i) Recall by M152

$$\hat{H}_g^- = \left\{ t_0 h - h t_0 \mid h \in \hat{H}_g^+ \right\}$$

\supseteq : For $i \in \{1, 2, 3\}$
 $t_0 t_i - t_i t_0 \in \hat{H}_g^-$

and $\hat{H}_g^+ \hat{H}_g^- \subseteq \hat{H}_g^-$

\subseteq : Given $u \in \hat{H}_g^-$
 $\exists h \in \hat{H}_g^+$ s.t $u = t_0 h - h t_0$

By L173 (d)

$$h = \sum_{i \in I} h_i t_i$$

$$h_i \in H_g^+$$

$$a = t_0 h - h t_0$$

$$= \sum_{i \in I} h_i (t_0 t_i - t_i t_0)$$

$$= \sum_{i=1}^3 h_i (t_0 t_i - t_i t_0)$$

$$\in \sum_{i=1}^3 H_g^+ (t_0 t_i - t_i t_0) \quad \cup$$

(c) Sim.

□

F arb $0 \neq 1 \in F$

Some 2-sided ideals of \hat{H}_g

Consider the subspace

$$[\hat{H}_g, \hat{H}_g] = \text{Span} \{ uv - vu \mid u, v \in \hat{H}_g \}$$

$\hat{H}_g [\hat{H}_g, \hat{H}_g] \hat{H}_g$ is 2-sided ideal of \hat{H}_g gen by $[\hat{H}_g, \hat{H}_g]$

This ideal is characterized as follows.

LEM 175 let L denote any 2-sided ideal of \hat{H}_g

TFAE

(i) $L \supseteq \hat{H}_g [\hat{H}_g, \hat{H}_g] \hat{H}_g$

(ii) The quotient F -algebra \hat{H}_g / L is commutative.

pt on

LEM 176. The ideal $\hat{H}_g [\hat{H}_1, \hat{H}_2] \hat{H}_g$

is gen by

$$t_i t_j - t_j t_i \quad 0 \leq i < j \leq 3$$

*

pf Let $L =$ 2-sided ideal of \hat{H}_g gen by *

obs $L \subseteq \hat{H}_g [\hat{H}_1, \hat{H}_2] \hat{H}_g$

\hat{H}_g/L is commutative since $\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$ gen \hat{H}_g

and

$$[t_i^{\pm 1}, t_j^{\pm 1}] = \pm [t_i, t_j] \in L \quad i, j \in \mathbb{I}$$

Now $L \supseteq \hat{H}_g [\hat{H}_1, \hat{H}_2] \hat{H}_g$

by L175

□

LEM 177 \exists surj \mathbb{F} -alg hom

$$\hat{H}_g \rightarrow \mathbb{F} \left[\begin{array}{ccc} \pm 1 & \pm 1 & \pm 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{array} \right]$$

\uparrow mod commut

that sends

$$t_i^{\pm 1} \rightarrow \lambda_i^{\pm 1} \quad i=1,2,3$$

$$t_0 \rightarrow q^{-1} \lambda_1^{-1} \lambda_2^{-1} \lambda_3^{-1}$$

$$t_0^{-1} \rightarrow q \lambda_1 \lambda_2 \lambda_3$$

pf The map respects the defining rels for \hat{H}_g □

LEM 178 For the hom in L177
the kernel is $\hat{H}_9[\hat{H}_7, \hat{H}_7] \hat{H}_9$

pf Let $L = \ker$

$$\hat{H}_9/L \cong \mathbb{F}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}, \lambda_3^{\pm 1}] \text{ is com}$$

$$\text{so } \hat{H}_9[\hat{H}_7, \hat{H}_7] \hat{H}_9 \subseteq L$$

Also since

$$\hat{H}_9 / \hat{H}_9[\hat{H}_7, \hat{H}_7] \hat{H}_9$$

is com \exists \mathbb{F} -alg hom

$$\begin{aligned} \mathbb{F}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}, \lambda_3^{\pm 1}] &\rightarrow \hat{H}_9 / \hat{H}_9[\hat{H}_7, \hat{H}_7] \hat{H}_9 \\ \lambda_i^{\pm 1} &\rightarrow t_i^{\pm 1} + \hat{H}_9[\hat{H}_7, \hat{H}_7] \hat{H}_9 \end{aligned}$$

Composition

$$\hat{H}_9 \rightarrow \mathbb{F}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}, \lambda_3^{\pm 1}] \rightarrow \hat{H}_9 / \hat{H}_9[\hat{H}_7, \hat{H}_7] \hat{H}_9$$

has ker $\hat{H}_9[\hat{H}_7, \hat{H}_7] \hat{H}_9$ so

$$L \subseteq \hat{H}_9[\hat{H}_7, \hat{H}_7] \hat{H}_9$$

So

$$L = \hat{H}_9[\hat{H}_7, \hat{H}_7] \hat{H}_9$$

□

COR 179 the following is a basis for

a complement of

$$\hat{H}_9 \left[\hat{H}_9, \hat{H}_9 \right] \hat{H}_9$$

in \hat{H}_9 :

$$t_1^i, t_2^j, t_3^k \quad i, j, k \in \mathbb{Z}$$

*

pf For the hom $\hat{H}_9 \rightarrow \mathbb{F}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}, \lambda_3^{\pm 1}]$ in L177

the images of * form basis for $\mathbb{F}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}, \lambda_3^{\pm 1}]$

Result follows in view of L178.

□

Now consider

$$\hat{H}_g \hat{H}_g^- \hat{H}_g = \text{2-sided ideal of } \hat{H}_g$$

gen by \hat{H}_g^-

Recall

$$\hat{H}_g = \{ b_0 h - h b_0 \mid h \in \hat{H}_g \}$$

$$\subseteq [\hat{H}_g, \hat{H}_g]$$

So

$$\hat{H}_g \hat{H}_g^- \hat{H}_g \subseteq \hat{H}_g [\hat{H}_g, \hat{H}_g] \hat{H}_g$$

[we will show \subseteq]

LEM 180 The ideal $\hat{H}_g \hat{H}_g^- \hat{H}_g$ is gen by

$$b_0 t_i - t_i b_0 \quad i=1,2,3$$

pf.

$$\hat{H}_g \hat{H}_g^- \hat{H}_g \stackrel{L179}{=} \sum_{i=1}^3 \hat{H}_g \hat{H}_g^+ (b_0 t_i - t_i b_0) \hat{H}_g$$

$$= \sum_{i=1}^3 \hat{H}_g (b_0 t_i - t_i b_0) \hat{H}_g$$

□

L181 The quotient F -algebra

$$\hat{H}_2 / \hat{H}_2 \hat{H}_1^{-1} \hat{H}_2$$

has a pres by gens $\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$ and rels

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad i \in \mathbb{I}$$

$$t_0^{\pm 1} \text{ central} \quad i=1,2,3$$

$$t_i t_i^{-1} \text{ central}$$

$$t_0 t_1 t_2 t_3 = q^{-1}$$

pf

By L180 and since the assertion

that $t_0^{\pm 1}$ is central is equiv to

the assertion $t_0 t_i = t_i t_0 \quad i=1,2,3$

□

LEM 182

For $\varepsilon \in \{1, -1\}$ \exists \mathbb{F} -alg

hom

$${}^n H_9 \longrightarrow \text{Mat}_{2 \times 2}(\mathbb{F}) \otimes \mathbb{F}[\lambda^{\pm 1}, \lambda_1^{\pm 1}, \lambda_2^{\pm 1}]$$

that sends

$$t_0 \longrightarrow \varepsilon \begin{pmatrix} q^{-1} & 0 \\ 0 & q^{-1} \end{pmatrix}$$

$$t_1 \longrightarrow \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1^{-1} \end{pmatrix}$$

$$t_2 \longrightarrow \begin{pmatrix} \lambda_2 & \lambda \\ 0 & \lambda_2^{-1} \end{pmatrix}$$

$$t_1, t_2, t_3 \longrightarrow \varepsilon I$$

pf

One checks the def rules of ${}^n H_9$ are respected \square

L183 For the map in L182

$$(i) \quad \hat{H}_9 - \hat{H}_9 \subseteq \ker$$

$$(ii) \quad \hat{H}_9 [\hat{H}_9, \hat{H}_9] \not\subseteq \ker$$

pf (i) For $i=1,2,3$ the map sends

$$b_i b_i - b_i b_i \rightarrow 0$$

$$\text{so } b_i b_i - b_i b_i \in \ker$$

Result follows by L180.

(ii) \hat{H}_9 / \ker is not commutative. □

Cor 184

$$\hat{H}_9 - \hat{H}_9 \subseteq \hat{H}_9 [\hat{H}_9, \hat{H}_9] \hat{H}_9$$

pf by L183 □

LEM 185 In the F -alg

$$\hat{H}_g / \hat{H}_g \hat{H}_g^{-1} \hat{H}_g$$

10

*

$t_i t_j + (t_i t_j)^{-1}$ is central

$1 \leq i < j \leq 3$

pf Take $i=1, j=2$ (other case sim)

$t_1 t_2 + (t_1 t_2)^{-1}$ commutes with t_1, t_2

in \hat{H}_g and hence *

$$t_1 t_2 + (t_1 t_2)^{-1}$$

...

t_3 in *

$$t_1 t_2 + (t_1 t_2)^{-1}$$

...

t_3 in *

since

$$t_1 t_2 t_3 = t_3^{-1}$$

in \hat{H}_g

and hence *

Now in *

$$t_1 t_2 + (t_1 t_2)^{-1}$$

commutes with each $\{t_i\}_{i \in \mathbb{Z}}$

□

LEM 186

In the \mathbb{F} -alg

$$\hat{H}_q / \langle \hat{H}_q, \hat{H}_q^{-1} - \hat{H}_q \rangle$$

$$t_i (q b_0 - q^{-1} b_0^{-1}) \text{ is central } i=1,2,3$$

pf Take $i=3$ other cases similar

By L185 $t_1 t_2 + (t_1 t_2)^{-1}$ central

$$t_1 t_2 = q b_0^{-1} b_3$$

$$q b_0 t_3 + q^{-1} t_0^{-1} t_3^{-1} \text{ central}$$

$$t_3 + t_3^{-1} \text{ central}$$

Also

$$\underbrace{q b_0 t_3 + q^{-1} t_0^{-1} t_3^{-1}}_{t_3 (q b_0 - q^{-1} b_0^{-1})} = q^{-1} t_0^{-1} (t_3 + t_3^{-1}) \text{ central}$$

□

Problem (ex). Find a basis for the \mathbb{F} -vector space

$$\hat{H}_q / \langle \hat{H}_q, \hat{H}_q^{-1} - \hat{H}_q \rangle$$

Find a basis for a complement of $\langle \hat{H}_q, \hat{H}_q^{-1} - \hat{H}_q \rangle$

in \hat{H}_q .

Fix $0 \neq k_i \in \mathbb{F}$

$i \in \mathbb{I}$

Recall (ordinary) DAHA $H(k_0, k_1, k_2, k_3; q)$

of type (C_1, \vee, C_1) is defined by gens $\{t_i\}_{i \in \mathbb{I}}$

and rels

$$(t_i - k_i)(t_i - k_i^{-1}) = 0 \quad i \in \mathbb{I}$$

$$t_0 t_1 t_2 t_3 = q^{-1}$$

Recall by Lem 3 \exists surj \mathbb{F} -algebra hom

$$\hat{H}_q \rightarrow H(k_0, k_1, k_2, k_3; q)$$

that sends $t_i \rightarrow t_i \quad i \in \mathbb{I}$

$$T_i \rightarrow k_i + k_i^{-1}$$

The kernel is

$$\sum_{i \in \mathbb{I}} \hat{H}_q (T_i - (k_i + k_i^{-1}) \mathbb{1})$$

Call this ideal \mathfrak{K} (the ideal \mathfrak{K} from L163 is sp case)

By const

\mathbb{F} -algebra

$$\hat{H}_q / \mathfrak{K} \cong H(k_0, k_1, k_2, k_3; q)$$

Next goal: Find a basis for

\mathfrak{K} and \hat{H}_q / \mathfrak{K}

Define

$$\hat{T}_i = T_i - (k_i + k_i^{-1}) \mathbb{1} \quad i \in \mathbb{I}$$

LEM 187 The following is a basis
 for \mathbb{F} -vector space \hat{H}_9 :

$$y^i x^j t_0^k \quad T_0^l \quad T_1^r \quad T_2^s \quad T_3^t$$

$i, j \in \mathbb{Z}, \quad k \in \{0, 1\}$

pf Recall the following is a basis for \hat{H}_9 :

$$y^i x^j t_0^k \quad T_0^l \quad T_1^r \quad T_2^s \quad T_3^t$$

$i, j \in \mathbb{Z}, \quad k \in \{0, 1\}$

$l, r, s, t \in \mathbb{N}$

LEM 188 The following is a basis
for the F -vector space K :

$$y_i x^k \text{ to } T_0^k \quad T_1^k \quad T_2^k \quad T_3^k$$

$$i, k \in \mathbb{Z} \quad k \in \{0, 1\}$$

$$l, r, s, t \in \mathbb{N}$$

$$(l, r, s, t) \neq (0, 0, 0, 0)$$

pf use L187.

Cor 189 The following is a basis for
a complement of K in \hat{H}_9 :

$$y^i x^j t^k$$

$$i, j \in \mathbb{Z}$$

$$k \in \{0, 1\}$$

pf Compare L187, L188. □

Thm 190 The following is a basis for the Fréchet
space $H(k_0, k_1, k_2, k_3; \varrho)$

$$y^i x^j t^k$$

$$i, j \in \mathbb{Z}$$

$$k \in \{0, 1\}$$

pf By Cor 189 and since

$$H(k_0, k_1, k_2, k_3; \varrho) \cong \hat{H}_9 / K$$
□

\mathbb{F} arb $0 \neq 1 \in \mathbb{F}$ Some two-sided ideals of $M_n(\mathbb{F})$ Recall that $M_n(\mathbb{F})$ contains the 2-sided ideal

$$J = \{ t_0 h - h t_0^T \mid h \in M_n(\mathbb{F}) \}.$$

ideal J gen by

$$A - t_0^T T_1,$$

$$B - t_0^T T_3,$$

$$C - t_0^T T_2, \quad t_0 - t_0^T$$

We will discuss the 2-sided ideal J^2 this has gen set

$$\{ uv \mid u, v \in J \}$$

Consider subspace

$$[\hat{H}_9^+, \hat{H}_9^+] = \text{Span} \left\{ uv - vu \mid u, v \in \hat{H}_9^+ \right\}$$

$\hat{H}_9^+ [\hat{H}_9^+, \hat{H}_9^+] \hat{H}_9^+$ is 2-sided ideal of \hat{H}_9^+

generated by $[\hat{H}_9^+, \hat{H}_9^+]$

Similar to L175 we have:

Lem 191. Let L denote any 2-sided ideal of \hat{H}_9^+

TFAE

$$(i) \quad L \supseteq \hat{H}_9^+ [\hat{H}_9^+, \hat{H}_9^+] \hat{H}_9^+$$

(ii) the quot \mathbb{F} -alg $\hat{H}_9^+ / \hat{H}_9^+ [\hat{H}_9^+, \hat{H}_9^+] \hat{H}_9^+$ is commutative.

LEM 192 The ideal

$$H_9^{\wedge+} [H_9^{\wedge+}, H_9^{\wedge+}] H_9^{\wedge+}$$

is gen by

$$AB-BA, \quad AC-CA, \quad BC-CB$$

pf sim to L76

□

Prop 193

$$H_9^+ \left[\begin{array}{c} H_9^+ \\ H_9^+ \end{array} \right] H_9^+ \subseteq J^2$$

pf

$$[A, B] = \left[\begin{array}{c} \downarrow J \\ A - t_0^{-1} T_1, \end{array} \begin{array}{c} \downarrow J \\ B - t_0^{-1} T_3 \end{array} \right]$$

since t_0, T_1, T_3 com with A, B

$$\in J^2$$

Sim

$$[B, C], [A, C] \in J^2$$

Done by L192. □

Cor 194 The H -alg

$$H_9^+ / J^2$$

is commutative.

pf By L191 and P 193 □

Now consider the subspace

$$\left(\hat{H}_g^-\right)^2 = \text{Span} \left\{ hk \mid h, k \in \hat{H}_g^- \right\}$$

LEM 199. $\left(\hat{H}_g^-\right)^2$ is a 2-sided ideal of \hat{H}_g^+

pf We saw earlier

$$\left(\hat{H}_g^-\right)^2 \subseteq \hat{H}_g^+$$

$\left(\hat{H}_g^-\right)^2$ is a left ideal of \hat{H}_g^+ since

$$\hat{H}_g^+ \left(\hat{H}_g^-\right)^2 = \underbrace{\hat{H}_g^+ \hat{H}_g^-}_{\subseteq \hat{H}_g^-} \hat{H}_g^- \subseteq \left(\hat{H}_g^-\right)^2$$

Similarly

$$\left(\hat{H}_g^-\right)^2 \text{ is a right ideal of } \hat{H}_g^+$$

□

LEM 196 As a 2-sided ideal of H_9^+

$(\hat{H}_9^-)^2$ is gen by

$$(t_{0i} - t_{i0}) \quad (t_{0j} - t_{j0}) \quad 1 \leq i, j \leq 3$$

pf By L74.

Prop 19 ~~7~~

$$\left(\hat{H}_q^- \right)^2 \subseteq J^2$$

pf. By L196 suf to show

$$(t_{0i} - t_i t_0) (t_{0j} - t_j t_0) \in J^2 \quad *$$

for $1 \leq i, j \leq 3$

We found these products in L136, p 137.

Case $i=3$ By L136

$$\begin{aligned} (t_{03} - t_3 t_0)^2 &= A^2 - A T_0 T_1 + T_0^2 + T_1^2 - 4 \\ &= (A - t_0^{-1} T_1)(A - t_0 T_1) + (t_0 - t_0^{-1})^2 \end{aligned}$$

$$\left[\begin{array}{l} A - t_0 T_1 = A - t_0^{-1} T_1 - (t_0 - t_0^{-1}) \\ A - t_0^{-1} T_1 \in J \\ t_0 - t_0^{-1} \in J \end{array} \right]$$

$$\in J^2$$

Similarly

$$(t_{0i} - t_i t_0)^2 \in J^2 \quad i=2, 3.$$

take $\epsilon=1, \eta=2$

$$(t_0 t_1 - t_1 t_0) (t_0 t_2 - t_2 t_0) =$$

$$(A - t_0^{-1} T_1) (C - t_0^{-1} T_2) + q^{-2} t_0^{-1} (t_0 - t_0^{-1}) (B - t_0 T_3)$$

 \uparrow
J \uparrow
J \uparrow
J \uparrow
J

$$\in J^2$$

Other values of ϵ, η are similar.

□

Summary List of 2-sided ideals for \hat{H}_q^+

- J^2
- $\hat{H}_q^+ [\hat{H}_q^+, \hat{H}_q^+] \hat{H}_q^+$
- $(\hat{H}_q^-)^2$
- intersection of \hat{H}_q^+ with $\hat{H}_q [\hat{H}_q, \hat{H}_q] \hat{H}_q$
- \dots $\hat{H}_q \hat{H}_q^- \hat{H}_q$
- \dots K
- sums / intersections of the above

(i) How are the above ideals related?

For each ideal find a basis

For each containment $U \subseteq V$ find a basis

for a complement of U in V

(ii) Find the center of \hat{H}_g^+ . Probably the center is \mathbb{T} .

(iii) Is it true that

$$\hat{H}_g^+ [\hat{H}_g^+, \hat{H}_g^+] \hat{H}_g^+ = \hat{H}_g^+ \cap \hat{H}_g [\hat{H}_g, \hat{H}_g] \hat{H}_g$$

(iv) Is it true that

$$(\hat{H}_g^-)^2 = \hat{H}_g^+ \cap \hat{H}_g \hat{H}_g^- \hat{H}_g$$

(v) How are the following 8 subspaces related?

$$\{ h \in \hat{H}_g \mid t_i h = h t_i \} \quad i \in \mathbb{I}$$

$$\{ h \in \hat{H}_g \mid t_i h = h t_i^{-1} \} \quad i \in \mathbb{I}$$

(vi) Describe the B_3 -orbit containing t_0 . Are the elements of this orbit also under?

