

F arb

$$0 \neq q \in F \quad q^4 \neq 1$$

We just proved

th 110: the map $\psi: \Delta_q \rightarrow \hat{H}_q$ is inj.

From now on we identify each element of Δ_q with its image in \hat{H}_q under ψ .

So we write A, α instead of A^ψ, α^ψ etc

So $A = y + y^{-1}$ etc

$$\alpha = (q^{-1}t_0 + qt_0^{-1})T_1 + T_2T_3$$

Next general goal: describe the following spaces

$$H_q^+ := \{ h \in \hat{H}_q \mid t_0 h = h t_0 \}$$

$$H_q^- := \{ h \in \hat{H}_q \mid t_0 h = h t_0^{-1} \}$$

Real subalgebra

$$\mathbb{T} = \langle t_0^{\pm 1}, T_1, T_2, T_3 \rangle$$

$$\cong F[\lambda_0^{\pm 1}, \lambda_1, \lambda_2, \lambda_3]$$

This is a domain $\Leftarrow f q = 0 \rightarrow f = 0 \text{ or } q = 0 \quad \forall f, q \in \mathbb{T}$

LEM III For $h \in \hat{H}_g$ and $t \in \mathbb{T}$

(i) Assume $ht=0$ then $h=0$ or $t=0$

(ii) Assume $th=0$ then $h=0$ or $t=0$

pf (i) We assume $t \neq 0$ and show $h=0$

Recall the YXT-factorization of \hat{H}_g from above L56:

$$h = \sum_{i \in \mathbb{Z}} \gamma^i x^i t_{i\tau} \quad t_{i\tau} \in \mathbb{T}$$

$$0 = ht = \sum_{i \in \mathbb{Z}} \gamma^i x^i \underbrace{t_{i\tau} t}_{\in \mathbb{T}}$$

By YXT-factorization

$$t_{i\tau} t = 0 \quad \forall i \in \mathbb{Z}$$

We assume $t \neq 0$ so

$$t_{i\tau} = 0 \quad \forall i \in \mathbb{Z}$$

So

$$h=0$$

□

In Th 52 we obtained a basis for \hat{H}_g

We now display 2 related bases that will be useful in our investigation of \hat{H}_g^{\pm}

Prop 112 The following is a basis for the \mathbb{F} -vector space \hat{H}_g :

$$A^i u B^j t_0^k T_1^r T_2^s T_3^t \quad k \in \mathbb{Z}, \quad *$$

$$u \in \{1, x, y, yx\},$$

$$i, j, r, s, t \in \mathbb{N}$$

pf By the YXT-factorization

$$\begin{array}{ccccccc} \langle Y^{\pm 1} \rangle & \otimes & \langle X^{\pm 1} \rangle & \otimes & \Pi & \rightarrow & \hat{H}_g \\ u & \otimes & v & \otimes & w & \rightarrow & uvw \end{array}$$

is iso of \mathbb{F} -vector spaces

$$\langle Y^{\neq 1} \rangle \cong \mathbb{F}[\lambda^{\neq 1}]$$

has basis $\{Y^i\}_{i \in \mathbb{Z}}$

Another basis

$$(Y + Y^{-1})^i, \quad (Y + Y^{-1})^{i_1} Y^{i_2} \quad i \in \mathbb{N}$$

$$A^i \quad A^{i_1} Y^{i_2}$$

Sim $\langle X^{\neq 1} \rangle$ has basis

$$B^j, \quad X B^j \quad j \in \mathbb{N}$$

it follows

□

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Advantage of basis in Prop 112:

t_0 commutes each of

A, B, t_0, T_1, T_2, T_3

Only difficulty is $u \in \{x, y, yx\}$

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To motivate the next basis let us write
 t_0, t_1, t_2, t_3 in the basis from Prop 112

t_0 already in the basis

LEM 113

The elements t_1, t_2, t_3

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Look as follows in the basis from Prop 112:

$$t_1 = Y t_0 - A t_0 + T_1 \quad (1)$$

$$t_2 = q^{-1} A T_3 - q^{-1} A X t_0 - q^{-1} Y T_3 + q^{-1} Y X t_0 \quad (2)$$

$$t_3 = X t_0 \quad (3)$$

pf In Lem 39 we wrote t_1, t_2, t_3 in ess the basis from M52.

The above equations are a minor adjustment

details: By L39

$$t_1 = T_1 - Y^{-1} t_0$$

$$= T_1 - \underbrace{(Y + Y^{-1})}_{A} t_0 + Y t_0$$

$$t_2 = q^{-1} Y^{-1} T_3 - q^{-1} Y^{-1} X t_0$$

$$= q^{-1} (Y + Y^{-1}) T_3 - q^{-1} Y T_3$$

$$- q^{-1} (Y + Y^{-1}) X t_0 + q^{-1} Y X t_0$$

$$t_3 = t_3 t_0 t_0$$

$$= X t_0$$

□

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In (1)-(3) let us solve for

X, Y, YX in terms of t_1, t_2, t_3 & viewing
 A, t_0, T_1, T_2, T_3 as coeffs?

$$X = t_3 t_0 \quad (1')$$

$$Y = A + t_1 t_0^{-1} - t_0^{-1} T_1 \quad (2')$$

$$YX = A t_3 t_0 + t_1 t_3 - T_1 T_3 \quad (3')$$

Prop 114 The following is a basis for the F -vector
space \hat{H}_g :

$$A^i V B^j t_0^k T_1^r T_2^s T_3^t$$

$$k \in \mathbb{Z},$$

$$i, j, r, s, t \in \mathbb{N}$$

$$v \in \{1, t_1, t_2, t_3\}$$

pf \star spans \hat{H}_1 by (1')-(3') and since

\star spans \hat{H}_g .

Show \star is lin indep:

Using (1)-(3) write \star in basis \star .

For $m, n \in \{0, 1\}$ put

$$H_{mn} = \text{Span} \left\{ A^i Y^m X^n B^j t_0^k T_1^r T_2^s T_3^t \mid k \in \mathbb{Z}, i, j, r, s, t \in \mathbb{N} \right\}$$

So $\hat{H}_g = H_{00} + H_{01} + H_{10} + H_{11}$ (ds + vs)

Taking $v=1$ in $*$ we get a basis for H_{00}

For each term

$$A^i t_1 B^j t_0^k T_1^r T_2^s T_3^t$$

in $*$

$$A^i t_1 B^j t_0^k T_1^r T_2^s T_3^t - A^i Y B^j t_0^{k+1} T_1^r T_2^s T_3^t \in H_{00}$$

therefore vectors in $*$ with $v=t_1$ give a basis for a complement of H_{00} in $H_{00} + H_{10}$

By (3)

vectors in $*$ with $v=t_3$ form a basis for H_{01}

For each vector

$$A^i t_2 B^j t_0^k T_1^r T_2^s T_3^t$$

in $*$

$$A^i t_2 B^j t_0^k T_1^r T_2^s T_3^t - 2 A^i Y X B^j t_0^{k+1} T_1^r T_2^s T_3^t \in H_{00} + H_{01} + H_{10}$$

(74)

therefore the vectors in \mathcal{B} with $v = t_2$

form a basis for a complement of

$$H_{00} + H_{01} + H_{10}$$

in \hat{H}_g .

therefore \mathcal{B} is lin indep. □

— • —

Write $\hat{\theta}$ in the bases above

LEM 115 $\hat{\theta}$ looks as follows in the basis for \hat{H}_g from Prop 112:

$$\hat{\theta} =$$

	I	B	X
I	$q^{-1}t_0^2 t_2$	0	T_1
A	T_3	0	$-t_0^1$
Y	$-T_3$	t_0	$t_0^1 - t_0$

pf Recall

$$\hat{\theta} = (\gamma - \gamma C) / t_0$$

$$= \gamma X^{-1} t_0 - \gamma^{-1} X t_0^{-1} + \gamma^{-1} T_3 + X T_1 + \gamma^{-1} t_0^{-2} T_2$$

In above equation elim γ^{-1}, X^{-1} using

$$\gamma^{-1} = A - \gamma \quad X^{-1} = B - X$$

LEM 116 $\hat{\theta}$ looks as follows in the basis
 fr \hat{H}_q from Prop 114 :

$$\hat{\theta} = \gamma^{-1} t_0^{-2} T_2 + A B t_0 - B T_1 + t_0 T_1 T_3$$

$$+ t_1 B - t_1 t_0 T_3$$

$$+ \gamma t_2 - \gamma t_2 t_0^{-2}$$

$$+ t_3 t_0 T_1 - A t_3 t_0^{-2}$$

pf In the equation of Lem 115
 eval RHS using (1') - (3')

To sum up so far

$$\begin{aligned} \hat{H}_g &= \langle A \rangle \underbrace{\langle B \rangle}_{H_{00}} \pi \\ &+ \langle A \rangle \underbrace{X \langle B \rangle}_{H_{01}} \pi \\ &+ \langle A \rangle \underbrace{Y \langle B \rangle}_{H_{10}} \pi \\ &+ \langle A \rangle \underbrace{YX \langle B \rangle}_{H_{11}} \pi \end{aligned}$$

direct
sum

$$\begin{aligned} \hat{H}_g &= \langle A \rangle \langle B \rangle \pi \\ &+ \langle A \rangle t_3 \langle B \rangle \pi \\ &+ \langle A \rangle t_1 \langle B \rangle \pi \\ &+ \langle A \rangle t_2 \langle B \rangle \pi \end{aligned}$$

direct
sum

Also

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$$\langle A \rangle \langle B \rangle \pi + \langle A \rangle \gamma \langle B \rangle \pi$$
$$= \langle A \rangle \langle B \rangle \pi + \langle A \rangle \epsilon_1 \langle B \rangle \pi$$

$$\langle A \rangle \langle B \rangle \pi + \langle A \rangle \times \langle B \rangle \pi$$
$$= \langle A \rangle \langle B \rangle \pi + \langle A \rangle \epsilon_3 \langle B \rangle \pi$$

$$\langle A \rangle \langle B \rangle \pi + \langle A \rangle \gamma \langle B \rangle \pi$$
$$+ \langle A \rangle \times \langle B \rangle \pi$$

$$= \langle A \rangle \langle B \rangle \pi + \langle A \rangle \epsilon_1 \langle B \rangle \pi$$
$$+ \langle A \rangle \epsilon_3 \langle B \rangle \pi$$

\mathbb{F} arb

$$0 \neq q \in \mathbb{F} \quad q^4 \neq 1$$

Recall general goal: describe

$$\hat{H}_q^+ = \{ h \in \hat{H}_q \mid t_0 h = h t_0 \}$$

$$\hat{H}_q^- = \{ h \in \hat{H}_q \mid t_0 h = h t_0^{-1} \}$$

Recall

$$\begin{aligned} \hat{H}_q &= \langle A \rangle \langle B \rangle \Pi + \langle A \rangle x \langle B \rangle \Pi \\ &\quad + \langle A \rangle y \langle B \rangle \Pi + \langle A \rangle yx \langle B \rangle \Pi \end{aligned} \quad ds$$

$$\Pi = \langle t_0^{\pm 1}, T_1, T_2, T_3 \rangle$$

$$\begin{aligned} \hat{H}_q &= \langle A \rangle \langle B \rangle \Pi + \langle A \rangle t_1 \langle B \rangle \Pi \\ &\quad + \langle A \rangle t_2 \langle B \rangle \Pi + \langle A \rangle t_3 \langle B \rangle \Pi \end{aligned} \quad ds$$

LEM 117 The following coincide:

- (i) $*$ + $\langle A \rangle \subset \langle B \rangle \Pi$,
- (ii) $*$ + $\langle A \rangle yx \langle B \rangle \Pi (t_0 - t_0^{-1})$
- (iii) $*$ + $\langle A \rangle t_2 \langle B \rangle \Pi (t_0 - t_0^{-1})$

where

$$\begin{aligned} * &= \langle A \rangle \langle B \rangle \Pi + \langle A \rangle x \langle B \rangle \Pi \\ &\quad + \langle A \rangle y \langle B \rangle \Pi \\ &= \langle A \rangle \langle B \rangle \Pi + \langle A \rangle t_1 \langle B \rangle \Pi \\ &\quad + \langle A \rangle t_3 \langle B \rangle \Pi \end{aligned}$$

Moreover each sum (i)-(iii) is direct

pf obs

$$\langle A \rangle * \subseteq *$$

$$* \langle B \rangle \subseteq *$$

$$* \Pi \subseteq *$$

Recall

$$q C = \gamma - \theta t_0^{-1} \quad \forall \gamma \in \Pi \subseteq *$$

By L115

$$\theta + \gamma x (t_0 - t_0^{-1}) \in *$$

By L116

$$\theta + t_2 q t_0 (t_0 - t_0^{-1}) \in *$$

So in the quotient vector space $\hat{H}_q / *$

$$\begin{aligned}
 C + * &= \theta (-q^{-1} t_0^{-1}) + * \\
 &= \gamma x (t_0 - t_0^{-1}) (q^{-1} t_0^{-1}) + * \\
 &= t_2 (t_0 - t_0^{-1}) + *
 \end{aligned}$$

By this and since $\Pi = t_0^{-1} \Pi$

$$\begin{aligned}
 C \Pi + * &= \theta \Pi + * \\
 &= \gamma x (t_0 - t_0^{-1}) \Pi + * \\
 &= t_2 (t_0 - t_0^{-1}) \Pi + *
 \end{aligned}$$

Follows that (i) - (iii) coincide

Check sums are direct

(ii) is direct since

$$\hat{H}_g = * + \langle A \rangle \times \langle B \rangle \pi \quad ds$$

and

$$\langle A \rangle \times \langle B \rangle \pi (t_0 - t_0') \in \langle A \rangle \times \langle B \rangle \pi$$

(iii) is direct since

$$\hat{H}_g = * + \langle A \rangle t_2 \langle B \rangle \pi \quad ds$$

and

$$\langle A \rangle t_2 \langle B \rangle \pi (t_0 - t_0') \in \langle A \rangle t_2 \langle B \rangle \pi$$

show (c) is direct

Suppose not. Then \exists

$$\sum_{i \in \mathbb{Z}} A^i C B^j t_{ij} \in *$$

$t_{ij} \in \pi$
not all 0

then

$$\sum_{i \in \mathbb{Z}} A^i t_2 B^j t_{ij} (t_0 - t_0') \in *$$

Now

$$\sum_{i \in \mathbb{Z}} A^i t_2 B^j t_{ij} (t_0 - t_0') = 0$$

since (iii)
is direct

Now

$$\sum_{i \in \mathbb{Z}} A^i t_2 B^j t_{ij} = 0$$

this contradicts Prop 114.

□

Comments on \hat{H}_g^{\pm}

obs the sum

$$\hat{H}_g^+ + \hat{H}_g^-$$

is direct

$$\hat{H}_g^+ \hat{H}_g^- \subseteq \hat{H}_g^-$$

$$\hat{H}_g^- \hat{H}_g^+ \subseteq \hat{H}_g^-$$

$$\hat{H}_g^- \hat{H}_g^- \subseteq \hat{H}_g^+$$

So

$$\hat{H}_g^+ + \hat{H}_g^-$$

is subalgebra of \hat{H}_g

\hat{H}_g^{\pm}

are components of a \mathbb{Z}_2 -grading of the subalgebra

LEM 18 $\forall h \in \hat{H}_g$

$$(i) \quad t_0 h - h t_0 \in \hat{H}_g^-$$

$$(ii) \quad t_0 h - h t_0 \in \hat{H}_g^+$$

pf (i)

$$\begin{aligned} & t_0 (t_0 h - h t_0) - (t_0 h - h t_0) t_0 \\ &= t_0^2 h - t_0 h t_0 - t_0 h t_0 + h \\ &= t_0^2 h - t_0 h t_0 + h \\ &= (t_0^2 - t_0 t_0 + 1) h \\ &= 0 \end{aligned}$$

□

(ii) Sim

Given $h \in H_1$

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write

$$u = t_0 h - h t_0^{-1}$$

$$v = t_0 h - h t_0$$

obs

$$(t_0 - t_0^{-1})h = u + v$$

$$h(t_0 - t_0^{-1}) = u - v$$

$$(t_0 - t_0^{-1})h + h(t_0 - t_0^{-1}) = 2u$$

$$(t_0 - t_0^{-1})h - h(t_0 - t_0^{-1}) = 2v$$

By these comments and L118, we get:

LEM 119 $\forall h \in \hat{H}_1$

$$(i) \quad h(t_0 - t_0^{-1}) \in \hat{H}_1^+ + \hat{H}_1^-$$

$$(ii) \quad (t_0 - t_0^{-1})h \in \hat{H}_1^+ + \hat{H}_1^-$$

$$(iii) \quad (t_0 - t_0^{-1})h + h(t_0 - t_0^{-1}) \in \hat{H}_1^+$$

$$(iv) \quad (t_0 - t_0^{-1})h - h(t_0 - t_0^{-1}) \in \hat{H}_1^-$$

Consider $t_0 - t_0^{-1}$ carefully

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obs

$$\begin{aligned}(t_0 - t_0^{-1})^2 &= (t_0 + t_0^{-1})^2 - 4 \\ &= T_0^2 - 4 \quad (\text{Central in } \hat{H}_g)\end{aligned}$$

Consider the maps

$$\hat{H}_g \longrightarrow \hat{H}_g$$

$$h \longrightarrow t_0 h - h t_0^{-1}$$

$$\hat{H}_g \longrightarrow \hat{H}_g$$

$$h \longrightarrow t_0 h - h t_0$$

By 1118 these maps commute and their composition is 0

We now find the action of these maps on the bases for \hat{H}_g from Prop 112 and Prop 114

Since t_0 commutes with

$$A, B, \quad \forall t \in T$$

sof t_0 consider actions on

$$1, X, Y, YX$$

and

$$1, t_1, t_2, t_3$$

LEM 120

We have

h	$to_h - hto^{\rightarrow}$
1	$to - to^{\rightarrow}$
X	$Bto - T_3$
Y	$Ato - T_1$
YX	$ABto - \theta + q^{\rightarrow}to^2 T_2$

Note that

$$ABto - \theta + q^{\rightarrow}to^2 T_2 =$$

	1	B	X
1	0	0	$-T_1$
A	$-T_3$	to	to^{\rightarrow}
Y	T_3	$-to$	$to - to^{\rightarrow}$

pf use LEM 25

h	$t_0 h - h t_0^{\rightarrow}$
1	$t_0 - t_0^{\rightarrow}$
t_1	$A - t_0^{\rightarrow} T_1$
t_2	$C - t_0^{\rightarrow} T_2$
t_3	$B - t_0^{\rightarrow} T_3$

pf $F_n \quad i=1,2,3$

$$\begin{aligned}
 t_0 t_i - t_i t_0^{\rightarrow} &= t_0 t_i - (T_i - t_i^{\rightarrow}) t_0^{\rightarrow} \\
 &= t_0 t_i + t_i^{\rightarrow} t_0^{\rightarrow} - t_0^{\rightarrow} T_i
 \end{aligned}$$

Recall

$$\begin{aligned}
 A &= t_0 t_1 + t_1^{\rightarrow} t_0^{\rightarrow} \\
 B &= t_0 t_3 + t_3^{\rightarrow} t_0^{\rightarrow} \\
 C &= t_0 t_2 + t_2^{\rightarrow} t_0^{\rightarrow}
 \end{aligned}$$

□

LEM 122

We have

h	$t_0 h - h t_0$
1	0
X	$B t_0 - T_3 - X (t_0 - t_0^{-1})$
Y	$A t_0 - T_1 - Y (t_0 - t_0^{-1})$
YX	$AB t_0 - \theta + q^{-1} t_0^2 T_2 - YX (t_0 - t_0^{-1})$

Note that

$$AB t_0 - \theta + q^{-1} t_0^2 T_2 - YX (t_0 - t_0^{-1}) =$$

	1	B	X
1	0	0	$-T_1$
A	$-T_3$	t_0	t_0^{-1}
Y	T_3	$-t_0$	0

pf use L120 and comments above L119.

□
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LEM 123 We have

h	$t_0 h - h t_0$
1	0
t_1	$A - t_0^{-1} T_1 - t_1 (t_0 - t_0^{-1})$
t_2	$C - t_0^{-1} T_2 - t_2 (t_0 - t_0^{-1})$
t_3	$B - t_0^{-1} T_3 - t_3 (t_0 - t_0^{-1})$

Note that

$$\begin{aligned}
C - t_0^{-1} T_2 - t_2 (t_0 - t_0^{-1}) &= q^{-1} B t_0^{-1} T_1 - q^{-1} A B \\
&\quad - q^{-1} t_1 B t_0^{-1} + q^{-1} t_1 T_3 \\
&\quad + q^{-1} A t_3 t_0 - q^{-1} t_3 T_1 \\
&\in \langle A \rangle \langle B \rangle \Pi + \langle A \rangle t_1 \langle B \rangle \Pi \\
&\quad + \langle A \rangle t_3 \langle B \rangle \Pi
\end{aligned}$$

pf Use L121 and comments above L119

□

LEM 124

Each of the following
subspaces is invariant under both
 $h \rightarrow tch - hto$ and $h \rightarrow tch - hto^*$

$$\langle A \rangle \langle B \rangle \pi \quad (1)$$

$$\langle A \rangle \langle B \rangle \pi + \langle A \rangle \times \langle B \rangle \pi \quad (2)$$

$$\langle A \rangle \langle B \rangle \pi + \langle A \rangle \gamma \langle B \rangle \pi \quad (3)$$

$$\langle A \rangle \langle B \rangle \pi + \langle A \rangle \times \langle B \rangle \pi + \langle A \rangle \gamma \langle B \rangle \pi \quad (4)$$

pf By L120, L122

□

LEM 125 The map $h \rightarrow tch - ht\bar{a}$
 acts on quotients as follows

v2

on the space	the map acts as
(1)	$h \rightarrow h(t\bar{a} - t\bar{a})$
(2)/(1)	0
(3)/(1)	0
$\hat{H}_4 / (3)$	$h \rightarrow h(t\bar{a} - t\bar{a})$

pf

By L120

□

LEM 126 the map $h \rightarrow t_0 h - h t_0$
acts on quotients as follows

on the space	the map acts as
(1)	0
(2)/(1)	$h \rightarrow h(t_0 - t_0^{-1})$
(3)/(1)	$h \rightarrow h(t_0 - t_0^{-1})$
$\hat{H}_g / (3)$	0

pf Use L 122

□

Next: show

- $\hat{H}_g^+ = \langle A \rangle \langle B \rangle \pi + \langle A \rangle C \langle B \rangle \pi$

- \hat{H}_g^+ is gen by

$$A, B, C, t_0^{\pm 1}, \{T_i\}_{i=1}^3$$

\mathbb{F} arb

$$0 \neq q \in \mathbb{F} \quad q^4 \neq 1$$

Continue to study Univ DANA \hat{H}_q of type (C_1^1, C_1)

Current goal: describe

$$\hat{H}_q^+ = \{ h \in \hat{H}_q \mid h t_0 = t_0 h \}$$

$$\hat{H}_q^- = \{ h \in \hat{H}_q \mid t_0 h = h t_0^{-1} \}$$

We view

$$\hat{H}_q^+ = \text{kernel of } \mathbb{F}\text{-lin trans } h \mapsto t_0 h - h t_0$$

$$\hat{H}_q^- = \text{kernel of } h \mapsto t_0 h - h t_0^{-1}$$

Our next specific goal is to prove:

thm 127 The \mathbb{F} -algebra \hat{H}_q is generated by

$$A, B, C, t_0^{\pm 1}, \{T_i\}_{i=1}^3$$

$$[A = x + x^{-1}, B = x + x^{-1}, y = t_0 t_1 \quad x = t_0 t_0 \quad C = t_0 t_2 + (t_0 t_2)^{-1}]$$

th 128 the \mathbb{F} -vector space \hat{H}_q has basis

$$A^i B^j t_0^k T_1^r T_2^s T_3^t, \quad A^i C^j t_0^k T_1^r T_2^s T_3^t$$

$$k \in \mathbb{Z}, \quad i, j, r, s, t \in \mathbb{N}$$

th 129 $\hat{H}_q = \langle A \rangle \langle B \rangle \Pi + \langle A \rangle C \langle B \rangle \Pi$

$$\left[\begin{array}{l} \langle \mathcal{B} \rangle = \text{subalg of } \hat{H}_q \text{ gen by } \mathcal{B} \\ \Pi = \langle t_0^{\pm 1}, T_1, T_2, T_3 \rangle \end{array} \right]$$

Recall that the following sum is direct:

$$\hat{H}_q = \langle A \rangle \langle B \rangle^\perp + \langle A \rangle \times \langle B \rangle^\perp + \langle A \rangle \perp \langle B \rangle^\perp + \langle A \rangle \perp \times \langle B \rangle^\perp \quad *$$

For each of the 4 components in this decomp
we consider the corresp projection map

For $u \in \{I, \times, \perp, \perp \times\}$ define an \mathbb{F} -linear
map $\pi_u : \hat{H}_q \rightarrow \hat{H}_q$

such that

π_u acts as ident on $\langle A \rangle u \langle B \rangle^\perp$
... \circ on other 3 components of *

So π_u is projection from \hat{H}_q onto $\langle A \rangle u \langle B \rangle^\perp$

For $h \in \hat{H}_q$

$$h = \underbrace{\pi_I(h)}_{\langle A \rangle \langle B \rangle^\perp} + \underbrace{\pi_\times(h)}_{\langle A \rangle \times \langle B \rangle^\perp} + \underbrace{\pi_\perp(h)}_{\langle A \rangle \perp \langle B \rangle^\perp} + \underbrace{\pi_{\perp \times}(h)}_{\langle A \rangle \perp \times \langle B \rangle^\perp}$$

$$\langle A \rangle \langle B \rangle^\perp \quad \langle A \rangle \times \langle B \rangle^\perp \quad \langle A \rangle \perp \langle B \rangle^\perp \quad \langle A \rangle \perp \times \langle B \rangle^\perp$$

u	$\pi_u(\theta)$
1	$A T_3 + q^{-1} t_0^2 T_2$
x	$X T_1 - A X t_0^{-1}$
y	$Y B t_0 - Y T_3$
$q x$	$t_0^{-1} - t_0$

pf this reformulation of L115

□

EX 130A

u	$\pi_u(c)$
1	$-q^T A t_0^T T_3 + q^T \gamma - q^{-2} t_0^T T_2$
x	$q^T A X t_0^{-2} - q^T X t_0^T T_1$
y	$-q^T Y B + q^T Y t_0^T T_3$
yx	$q^T Y X (t_0 - t_0^T) t_0^T$

DEF 131 Let \tilde{H}_q denote the subspace of \hat{H}_q from L117 4

So

$$\begin{aligned} \tilde{H}_q &= \langle A \rangle \langle B \rangle \pi + \langle A \rangle x \langle B \rangle \pi + \langle A \rangle y \langle B \rangle \pi + \langle A \rangle yx \langle B \rangle \pi \quad (\text{Go to } *) \\ &= \dots + \dots + \dots + \langle A \rangle c \langle B \rangle \pi \quad (***) \end{aligned}$$

For $h \in \tilde{H}_q$ we have the projections $\pi_u(h)$ from (*)

we also have projections for (**)

DEF 132 For $v \in \{1, x, y, c\}$ let

$P_v : \tilde{H}_q \rightarrow \tilde{H}_q$ denote the \mathbb{F} -linear trans such that

P_v acts as identity on $\langle A \rangle v \langle B \rangle \pi$

... 0 on other 3 components of \tilde{H}_q

[Caution $P_v \neq \pi_v$ for $v = 1, x, y$]

For $h \in \tilde{H}_q$

$$h = \underbrace{P_1(h)}_{\pi} + \underbrace{P_x(h)}_{\pi} + \underbrace{P_y(h)}_{\pi} + \underbrace{P_c(h)}_{\pi}$$

$$\langle A \rangle \langle B \rangle \pi \quad \langle A \rangle x \langle B \rangle \pi \quad \langle A \rangle y \langle B \rangle \pi \quad \langle A \rangle c \langle B \rangle \pi$$

For $h \in \tilde{H}_g$ we now clarify
how the $\pi_u(h)$, $P_v(h)$ are related

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LEM 133 Given $h \in \tilde{H}_g$ write

$$P_C(h) = \sum_{i,j \in \mathbb{N}} A^i C B^j t_{ij} \quad t_{ij} \in \mathbb{T}$$

then

$$(i) \quad \pi_1(h) - P_1(h) = \sum_{i,j \in \mathbb{N}} A^i \pi_1(C) B^j t_{ij}$$

$$(ii) \quad \pi_x(h) - P_x(h) = \sum_{i,j \in \mathbb{N}} A^i \pi_x(C) B^j t_{ij}$$

$$(iii) \quad \pi_y(h) - P_y(h) = \sum_{i,j \in \mathbb{N}} A^i \pi_y(C) B^j t_{ij}$$

$$(iv) \quad \pi_{yx}(h) = \sum_{i,j \in \mathbb{N}} A^i y_x B^j t_{ij} t_0^{-1} (t_0 - t_0^{-1}) q^j$$

pf

$$h = \pi_i(h) + \pi_x(h) + \pi_y(h) + \pi_{yx}(h)$$

$$h = p_i(h) + p_x(h) + p_y(h) + p_c(h)$$

$$p_c(h) = \sum_{i,j \in \mathbb{N}} A^i C B^j t_{ij}$$

$$\parallel \pi_i(c) + \pi_x(c) + \pi_y(c) + \underbrace{\pi_{yx}(c)}_{\parallel}$$

$$\underbrace{-\pi_{yx}(c) t_0^{-1} q^{-1}}_{\parallel}$$

$$\parallel q \times t_0^{-1} (t_0 - t_0^{-1}) q^{-1}$$

$$\left[qC = \underbrace{y}_{\parallel} - \theta t_0^{-1} \right]$$

$$a = h - h$$

$$= p_i(h) - \pi_i(h) + \sum_{i,j \in \mathbb{N}} A^i \pi_i(c) B^j t_{ij}$$

Location

$\langle A \rangle \langle B \rangle \Pi$

$$+ p_x(h) - \pi_x(h) + \sum_{i,j \in \mathbb{N}} A^i \pi_x(c) B^j t_{ij}$$

$\langle A \rangle \times \langle B \rangle \Pi$

$$+ p_y(h) - \pi_y(h) + \sum_{i,j \in \mathbb{N}} A^i \pi_y(c) B^j t_{ij}$$

$\langle A \rangle \gamma \langle B \rangle \Pi$

$$- \pi_{yx}(h) + \sum_{i,j \in \mathbb{N}} A^i \gamma_x B^j t_0^{-1} (t_0 - t_0^{-1}) q^{-1} t_{ij}$$

$\langle A \rangle \gamma_x \langle B \rangle \Pi$

In each row terms must sum to 0

Result follows

□

We need a technical Lemma

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LEM 134 $\forall h \in \hat{H}_g$ the following are equiv.

$$(i) \quad h \in \langle A \rangle \langle B \rangle \Pi + \langle A \rangle C \langle B \rangle \Pi$$

$$(ii) \quad h(t_0 - t_0^{-1}) \in \langle A \rangle \langle B \rangle \Pi + \langle A \rangle C \langle B \rangle \Pi$$

pf (i) \rightarrow (ii) Since $t_0 - t_0^{-1} \in \Pi$

(ii) \rightarrow (i)

Obs. $h(t_0 - t_0^{-1}) \in \tilde{H}_g$

Write

$$P_C \left(h(t_0 - t_0^{-1}) \right) = \sum_{i_1 \in \mathbb{N}} A^{i_1} C B^{i_1} t_{i_1} \quad t_{i_1} \in \Pi$$

strategy: show

$$h \in \tilde{H}_g$$

Obs

$$P_x \left(h(t_0 - t_0^{-1}) \right) = 0$$

$$P_y \left(h(t_0 - t_0^{-1}) \right) = 0$$

$$\pi_x \left(h(t_0 - t_0^{-1}) \right) = \pi_x(h) (t_0 - t_0^{-1})$$

$$\pi_x \left(h(t_0 - t_0^{-1}) \right) = \pi_x(h) (t_0 - t_0^{-1})$$

$$\pi_y \left(h(t_0 - t_0^{-1}) \right) = \pi_y(h) (t_0 - t_0^{-1})$$

$$\pi_{yx} \left(h(t_0 - t_0^{-1}) \right) = \pi_{yx}(h) (t_0 - t_0^{-1})$$

By L33

$$\pi_{yx}(h(t_0 - t_0^{-1})) = \sum_{i \in \mathbb{N}} A^i y x B^i t_0^{-1} t_0^{-1} (t_0 - t_0^{-1}) q^{-i}$$

So

$$\pi_{yx}(h) = \sum_{i \in \mathbb{N}} A^i y x B^i t_0^{-1} t_0^{-1} q^{-i}$$

To show $h \in \tilde{H}_q$

show $t_0 - t_0^{-1}$ divides t_0^{-1} for all $i \in \mathbb{N}$

Compute $\pi_y(h(t_0 - t_0^{-1}))$ in two ways

$$\begin{aligned} \pi_y(h(t_0 - t_0^{-1})) &= \sum_{i \in \mathbb{N}} A^i \underbrace{\pi_y(c)}_{= -\pi_y(t_0^{-1} t_0^{-1} q^{-i})} B^i t_0^{-1} \\ &= \sum_{i \in \mathbb{N}} A^i y x B^i t_0^{-1} t_0^{-1} q^{-i} - \sum_{i \in \mathbb{N}} A^i y x B^i t_0^{-1} t_0^{-1} q^{-i} \\ &= \sum_{i \in \mathbb{N}} A^i y x B^i (q^{-i} t_0^{-1} t_0^{-1} T_3 - q^{-i} y x B) \end{aligned}$$

$$\sum_{r,s \in \mathbb{N}} A^r y B^s (q^r t_{rs} t_0^{-1} T_3 - q^r t_{rs})$$

↑
view as
LHS

Comparing the two sides

$$t_0 - t_0^{-1} \text{ divides } q^r t_{rs} t_0^{-1} T_3 - q^r t_{rs} \quad \forall r,s \in \mathbb{N}$$

By the ind on \mathbb{Z}

$$t_0 - t_0^{-1} \text{ divides } t_{rs} \quad \forall r,s \in \mathbb{N}$$

So $\forall r, s \in \mathbb{N} \quad \exists t_{rs} \in \mathbb{H}$

Set. $t_{rs} = t_{rs} (t_0 - t_0^{-1})$

Now

$$\pi_{Y \times X}(h) = \sum_{i, j \in \mathbb{N}} A^i Y X B^j t_{ij} t_0^{-1} (t_0 - t_0^{-1}) q^{-i}$$
$$\in \langle A \rangle Y X \langle B \rangle (t_0 - t_0^{-1})$$

So $h \in \tilde{H}_Z$

Now

$$P_X \left(\underbrace{h(t_0 - t_0^{-1})}_{=0} \right) = P_X(h) (t_0 - t_0^{-1})$$

so $P_X(h) = 0$

$$P_Y \left(\underbrace{h(t_0 - t_0^{-1})}_{=0} \right) = P_Y(h) (t_0 - t_0^{-1})$$

so $P_Y(h) = 0$

Now

$$h = P_0(h) + P_c(h)$$

$$\in \langle A \rangle \langle B \rangle \pi + \langle A \rangle C \langle B \rangle \pi$$

□

$$\hat{H}_g \supseteq \langle A \rangle \langle B \rangle \pi + \langle A \rangle C \langle B \rangle \pi$$

We saw earlier $A, B, C, t_0^{\pm 1}, T_1, T_2, T_3$
commute with t_0

$$\subseteq \dots$$

Let $h \in \hat{H}_g$ be given

First show

$$h(t_0 t_0^{-1}) \in \langle A \rangle \langle B \rangle \pi + \langle A \rangle C \langle B \rangle \pi \quad (*)$$

By assumption $t_0 h t_0^{-1} = h$ so

$$h(t_0 t_0^{-1}) = t_0 h t_0^{-1}$$

By L121 image of \hat{H}_g under map $g \rightarrow t_0 g t_0^{-1}$
is contained in

$$\langle A \rangle \langle B \rangle \pi + \langle A \rangle C \langle B \rangle \pi$$

This gives *

Now by * and L134

$$h \in \langle A \rangle \langle B \rangle \pi + \langle A \rangle C \langle B \rangle \pi \quad \cup$$

□

Proof of Th 127 : By Th 129 and since

Π is gen by $t_0^{\neq 1}, T_1, T_2, T_3$

□

Proof of Th 128 :

Linear independence is by Prop 109

Span by Th 127

□

\mathbb{F} arb

$$0 \neq q \in \mathbb{F} \quad q^4 \neq 1$$

Continue to study univ DAVA \hat{H}_q type $(C_4^2 C_2)$

Recall
$$\hat{H}_q^+ = \{ h \in \hat{H}_q \mid t_0 h = h t_0 \}$$

$$\hat{H}_q^- = \{ h \in \hat{H}_q \mid t_0 h = h t_0^{-1} \}$$

We saw
$$\hat{H}^+ = \langle A \rangle \langle B \rangle \Pi + \langle A \rangle C \langle B \rangle \Pi$$

Found basis, gen set

We now describe \hat{H}_q^+ by gens + rels.

Thm 130 The \mathbb{F} -alg \hat{H}_q^+ is described by
 gens and rels as follows. The gens are

$$A, B, C, t_0^{\pm 1}, \{T_i\}_{i=1}^3$$

The rels are:

$$t_0 t_0^{-1} = t_0^{-1} t_0 = 1,$$

$t_0^{\pm 1}, \{T_i\}_{i=1}^3$ are central

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{\alpha}{q + q^{-1}} \quad \alpha = (q^{-1}t_0 + qt_0^{-1})T_1 + T_2T_3$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{\beta}{q + q^{-1}} \quad \beta = (q^{-1}t_0 + qt_0^{-1})T_3 + T_0T_2$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{\gamma}{q + q^{-1}} \quad \gamma = (q^{-1}t_0 + qt_0^{-1})T_2 + T_1T_3$$

$$qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma = (q + q^{-1})^2 - (q^{-1}t_0 + qt_0^{-1})^2 - T_1^2 - T_2^2 - T_3^2 - (q^{-1}t_0 + qt_0^{-1})T_1T_2T_3$$

pf Above rels are the ones used to show the
 vectors in Th 128 span \hat{H}_q^+ .
 Vectors in Th 128 form basis for \hat{H}_q^+ so no further rels
 exist in the presentation. \square

Note In above presentation we could replace α, β, γ by any
 one of the 6 versions of the Casimir element Ω
 Because the fact that they are equiv follows from
 prev 3 rels E_3 -sym AW rels

By LEM 118 under map

$$\begin{aligned} \hat{H}_g &\longrightarrow \hat{H}_g \\ h &\longrightarrow toh - hto^{-1} \end{aligned}$$

The image of \hat{H}_g is contained in \hat{H}_g^+

LEM 131 Above image is a 2-sided ideal of \hat{H}_g^+

pf Write

$$J = \{ toh - hto^{-1} \mid h \in \hat{H}_g \}$$

Given $k \in \hat{H}_g^+$ show

$$kJ \subseteq J \qquad Jk \subseteq J$$

$\forall h \in H$

$$k(toh - hto^{-1}) = tokh - khto^{-1} \in J$$

since $to k = k to$

$$(toh - hto^{-1})k = tohk - hkt o^{-1} \in J$$

□

Describe J

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LEM 132 The ideal J is generated by

$$t_0 - t_0^{\rightarrow}, \quad A - t_0^{\rightarrow} T_1, \quad B - t_0^{\rightarrow} T_3, \quad C - t_0^{\rightarrow} T_2$$

pf Need to show

$$J = \hat{H}_1^+ (t_0 - t_0^{\rightarrow}) \hat{H}_2^+ + \hat{H}_2^+ (A - t_0^{\rightarrow} T_1) \hat{H}_9^+ + \hat{H}_9^+ (B - t_0^{\rightarrow} T_3) \hat{H}_9^+ + \hat{H}_9^+ (C - t_0^{\rightarrow} T_2) \hat{H}_9^+ \quad *$$

Recall

$$\hat{H}_9^+ = \langle A \rangle \langle B \rangle \pi_1 + \langle A \rangle t_1 \langle B \rangle \pi +$$

$$\langle A \rangle t_2 \langle B \rangle \pi + \langle A \rangle t_3 \langle B \rangle \pi$$

Apply map $h \mapsto t_0 h - h t_0^{\rightarrow}$ and use L121. Get

$$J = \langle A \rangle (t_0 - t_0^{\rightarrow}) \langle B \rangle \pi + \langle A \rangle (A - t_0^{\rightarrow} T_1) \langle B \rangle \pi + \langle A \rangle (C - t_0^{\rightarrow} T_2) \langle B \rangle \pi + \langle A \rangle (B - t_0^{\rightarrow} T_3) \langle B \rangle \pi$$

* follows

□

Notation

Abel group $Z_2 = \{1, y\}$ $y^2 = 1$

Group \mathbb{F} -algebra

$$\mathbb{F} Z_2 = \mathbb{F} 1 + \mathbb{F} y$$

has basis $1, y$

$\mathbb{F} Z_2$ alg iso $\frac{\mathbb{F}[\lambda]}{(\lambda^2 - 1)}$ \leftarrow ideal gen by $\lambda^2 - 1$

$\lambda = \text{indets}$

Let $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ denote mult com indets

obs $\frac{\mathbb{F}[\lambda_0, \lambda_1, \lambda_2, \lambda_3]}{(\lambda_0^2 - 1)}$ alg iso $\mathbb{F} Z_2 \otimes \mathbb{F}[\lambda_1, \lambda_2, \lambda_3]$

$\mathbb{F} Z_2 \otimes \mathbb{F}[\lambda_1, \lambda_2, \lambda_3]$ has basis

$$y^c \otimes \lambda_1^r \lambda_2^s \lambda_3^t$$

$$c \in \{0, 1\}, r, s, t \in \mathbb{N}$$

LEM 133 \exists unique \mathbb{F} -alg hom

$$\hat{H}_9^+ \rightarrow \mathbb{F} \mathbb{Z}_2 \otimes \mathbb{F}[\lambda_1, \lambda_2, \lambda_3]$$

that sends

$$A \rightarrow z \otimes \lambda_1$$

$$B \rightarrow z \otimes \lambda_3$$

$$C \rightarrow z \otimes \lambda_2$$

$$t_0^{\pm 1} \rightarrow z \otimes 1$$

$$T_1 \rightarrow 1 \otimes \lambda_1$$

$$T_2 \rightarrow 1 \otimes \lambda_2$$

$$T_3 \rightarrow 1 \otimes \lambda_3$$

this hom is surjective

pf Check the map respects the def rels for \hat{H}_9^+ from 130

$$z \otimes \lambda_1 + \frac{z \otimes \lambda_3 \quad z \otimes \lambda_2 \quad -q^{-1} z \otimes \lambda_2 \quad z \otimes \lambda_3}{q^2 - q^{-2}} = ?$$

$$\frac{(q^{-1} z \otimes 1 + q z \otimes 1) 1 \otimes \lambda_1 + 1 \otimes \lambda_2 \lambda_3}{q + q^{-1}}$$

$$\text{LHS} = z \otimes \lambda_1 + \frac{1 \otimes \lambda_2 \lambda_3}{q + q^{-1}} = \text{RHS} \quad \checkmark$$

other def rels sim.

Check Casimir rels:

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$$q \ 3 \otimes \lambda_1 \lambda_2 \lambda_3 + q^2 \ 1 \otimes \lambda_1^2 + q^{-2} \ 1 \otimes \lambda_3^2 + q^2 \ 1 \otimes \lambda_2^2$$

$$- q \ 3 \otimes \lambda_1 \left((q+q^{-1}) \ 3 \otimes \lambda_1 + 1 \otimes \lambda_2 \lambda_3 \right)$$

$$- q^{-1} \ 3 \otimes \lambda_3 \left((q+q^{-1}) \ 3 \otimes \lambda_3 + 1 \otimes \lambda_1 \lambda_2 \right)$$

$$- q \ 3 \otimes \lambda_2 \left((q+q^{-1}) \ 3 \otimes \lambda_2 + 1 \otimes \lambda_1 \lambda_3 \right)$$

$$\stackrel{?}{=} (q+q^{-1})^2 - \left(q^{-1} \ 3 \otimes 1 + q \ 3 \otimes 1 \right)^2 - 1 \otimes \lambda_1^2 - 1 \otimes \lambda_2^2 - 1 \otimes \lambda_3^2$$

$$- \left(q^{-1} \ 3 \otimes 1 + q \ 3 \otimes 1 \right) \otimes \lambda_1 \lambda_2 \lambda_3$$

LHS-RHS

term	coef			
$3 \otimes \lambda_1 \lambda_2 \lambda_3$	q	$-q$	$-q^{-1} - q$	$+q+q^{-1}$
$1 \otimes \lambda_1^2$	q^2	$-q(q+q^{-1})$	$+1$	$=0$
$1 \otimes \lambda_2^2$	q^2	$-q(q+q^{-1})$	$+1$	$=0$
$1 \otimes \lambda_3^2$	q^{-2}	$-q^{-1}(q+q^{-1})$	$+1$	$=0$
$1 \otimes 1$	$-q^2 - 2 - q^{-2}$	$+q^2 + 2 + q^{-2}$		$=0$

Hom exists ✓
 unique ✓
 surj ✓

□
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LEM 134 For the hom in L133

the kernel is J

Moreover

$$\hat{H}_9^+ / J \cong_{\text{alg}} \mathbb{F}Z_2 \otimes \mathbb{F}[\lambda_1, \lambda_2, \lambda_3]$$

one checks JSker
pf \Rightarrow Consider canon algebra hom
show

$$\begin{aligned} \hat{H}_9^+ &\rightarrow \hat{H}^+ / J \\ h &\rightarrow h + J \quad (= \bar{h}) \end{aligned}$$

$$\begin{aligned} t_0 - t_0^* &\in J \text{ so} \\ \bar{t}_0^2 &= 1 \end{aligned}$$

$$A - t_0^* T_1 \in J \text{ so}$$

$$\bar{A} = \bar{t}_0 \bar{T}_1$$

\rightarrow

$$\begin{aligned} \bar{t}_0 \bar{A} &= \bar{A} \bar{t}_0 \\ \bar{C} &= \bar{t}_0 \bar{T}_2 \\ \bar{t}_0 \bar{B} &= \bar{B} \bar{t}_0 \\ \bar{t}_0 \bar{C} &= \bar{C} \bar{t}_0 \end{aligned}$$

Sum

\bar{t}_0 central in \hat{H}_9^+ / J

Now $\bar{A} = \bar{t}_0 \bar{T}_1$ is central in \hat{H}_9^+ / J

Sum

$$\bar{B}, \bar{C} \quad \dots$$

\hat{H}_9^+ / J commutative, gen by $\bar{t}_0, \bar{T}_1, \bar{T}_2, \bar{T}_3$ and $\bar{t}_0^2 = 1$

So \exists algebra

$$\mathbb{F}Z_2 \otimes \mathbb{F}[\lambda_1, \lambda_2, \lambda_3] \rightarrow \hat{H}_9^+ / J$$

that sends

$$\begin{aligned} \lambda_0 &\rightarrow \bar{E}_0 \\ 1 \otimes \lambda_1 &\rightarrow \bar{T}_1 \\ 1 \otimes \lambda_2 &\rightarrow \bar{T}_2 \\ 1 \otimes \lambda_3 &\rightarrow \bar{T}_3 \end{aligned}$$

the diagram commutes:

$$\begin{array}{ccc} \hat{\Lambda}^+ H_9 & \xrightarrow{\quad} & \mathbb{F} Z_2 \otimes \mathbb{F}[\lambda_1, \lambda_2, \lambda_3] \\ & \searrow \text{canon.} & \downarrow \varepsilon \\ & & \hat{\Lambda}^+ H_9 / J \end{array}$$

check

$$\begin{array}{ccc} h & \xrightarrow{\quad} & 0 \\ & \searrow & \downarrow \\ & & 0 \end{array}$$

so $h \in \ker \text{ of canon map}$
 $= J$

so $K \subseteq J$

so $K = J$

□

COR 135 the following is a basis
for a complement of J in H_9^+ :

$$t_0^i, T_1^r, T_2^s, T_3^t$$

$$i \in \{0,1\}, r,s,t \in \mathbb{N}$$

pf By LV34

□

\mathbb{F} arb

$$a \neq q \in \mathbb{F} \quad q^4 \neq 1$$

$$\hat{H}_q^+ = \{h \in \hat{H}_q \mid t_0 h = h t_0\}$$

$$\hat{H}_q^- = \{h \in \hat{H}_q \mid t_0 h = h t_0^{-1}\}$$

Recall

$$\hat{H}_q = \langle A \rangle \langle B \rangle \Pi + \langle A \rangle t_1 \langle B \rangle \Pi + \langle A \rangle t_2 \langle B \rangle \Pi + \langle A \rangle t_3 \langle B \rangle \Pi$$

Under map $h \rightarrow t_0 h - h t_0$

Image of \hat{H}_q is

$$\langle A \rangle (t_0 t_1 - t_1 t_0) \langle B \rangle \Pi + \langle A \rangle (t_0 t_2 - t_2 t_0) \langle B \rangle \Pi + \langle A \rangle (t_0 t_3 - t_3 t_0) \langle B \rangle \Pi$$

This image $\subseteq \hat{H}_q^-$

Investigate

$$t_0 t_i - t_i t_0 \quad i=1,2,3$$

LEM 136

$$\begin{aligned} \text{(i)} \quad (t_0 t_1 - t_1 t_0)^2 &= A^2 - A T_0 T_1 + T_0^2 + T_1^2 - 4 \\ \text{(ii)} \quad (t_0 t_2 - t_2 t_0)^2 &= C^2 - C T_0 T_2 + T_0^2 + T_2^2 - 4 \\ \text{(iii)} \quad (t_0 t_3 - t_3 t_0)^2 &= B^2 - B T_0 T_3 + T_0^2 + T_3^2 - 4 \end{aligned}$$

pf By L 71 which is about algebra S

□

Note The generator ρ of B_3 sends

$$t_0 t_3 - t_3 t_0 \rightarrow t_0 t_2 - t_2 t_0 \rightarrow t_0 t_1 - t_1 t_0$$

$$t_0^{-1} (t_0 t_3 - t_3 t_0) t_0 = (t_0 t_3 - t_3 t_0) t_0^2 = t_0^{-2} (t_0 t_3 - t_3 t_0)$$

Find products

$$(t_0 t_i - t_i t_0) (t_0 t_j - t_j t_0) \quad 1 \leq i, j \leq 3 \quad i \neq j$$

Recall $\hat{H}_q^- \hat{H}_q^- \subseteq \hat{H}_q^+ = \langle A, B, C, t_0^{\pm 1}, T_1, T_2, T_3 \rangle$

so above products should be polynomials in A, B, C, \dots

Prop 137 Abbrev

$$\begin{aligned} A^+ &= A - t_0 T_1 & A^- &= A - t_0^{-1} T_1 & \text{Then} \\ B^+ &= B - t_0 T_3 & B^- &= B - t_0^{-1} T_3 \\ C^+ &= C - t_0 T_2 & C^- &= C - t_0^{-1} T_2 \end{aligned}$$

	$t_0 t_1 - t_1 t_0$	$t_0 t_2 - t_2 t_0$	$t_0 t_3 - t_3 t_0$
$t_0 t_1 - t_1 t_0$		$A^- C^- + t_0^{-1} (t_0 - t_0^{-1}) B^+$	$A^+ B^+ t_0^{-2} - t_0^{-1} (t_0 - t_0^{-1}) C^-$
$t_0 t_2 - t_2 t_0$	$C^+ A^+ - t_0 (t_0 - t_0^{-1}) B^-$		$C^- B^- + t_0^{-1} (t_0 - t_0^{-1}) A^+$
$t_0 t_3 - t_3 t_0$	$B^- A^- t_0^2 + t_0^{-1} (t_0 - t_0^{-1}) C^+$	$B^+ C^+ - t_0 (t_0 - t_0^{-1}) A^-$	

pf

Recall

$$A - t_0 T_1 = t_0 t_1 - t_1 t_0 \text{ etc.}$$

$$A - t_0 T_2 = -(t_0 t_1 - t_1 t_0) \text{ etc.}$$

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$$(t_0 t_1 - t_1 t_0) \parallel (t_0 t_2 - t_2 t_0)$$

$$= (t_0 t_1 - t_1 t_0) (t_0 t_2 - t_2 t_0) \stackrel{?}{=} q^{-1} t_0^{-1} (t_0 - t_0^{-1}) (t_0 t_2 - t_2 t_0)$$

$$\cancel{t_0 t_1 t_0 t_2} - \underbrace{t_1 t_0^2 t_2}_{\text{Total}} - \underbrace{t_0 t_1 t_2 t_0}_{q^{-1} t_1^2} + \underbrace{t_1 t_0 t_2 t_0}_{-t_1 t_0 t_2 t_0^{-1}}$$

$$\stackrel{?}{=} \underbrace{t_1 t_2}_{q^{-1} t_0^{-1} t_1^2} - \underbrace{t_0 t_1 t_2 t_0^{-1}}_{q^{-1} t_1^2} + \underbrace{t_1 t_0^{-1} t_2 t_0^{-1}}_{t_1 T_0 t_0^{-1}} - \underbrace{t_0 t_0^{-1} t_2 t_0^{-1}}_{t_0 t_0^{-1} t_2 t_0^{-1}}$$

$$\cancel{t_0 t_1 t_0 t_2} - \underbrace{t_1 t_2}_{q^{-1} t_0^{-1} t_1^2} - \underbrace{t_0 t_1 t_2 t_0^{-1}}_{q^{-1} t_1^2} + \underbrace{t_1 t_0^{-1} t_2 t_0^{-1}}_{t_1 T_0 t_0^{-1}} - \underbrace{t_0 t_0^{-1} t_2 t_0^{-1}}_{t_0 t_0^{-1} t_2 t_0^{-1}}$$

$$= -q^{-1} (1 - t_0^{-2}) (t_0 t_2 - t_2 t_0)$$

$$\underbrace{-q^{-1} t_0 t_2^{-1}}_{-q^{-1} t_0^{-1} t_2^{-1}} + \underbrace{q^{-1} t_2^{-1} t_0^{-1}}_{q^{-1} t_2^{-1} t_0^{-1}} + \underbrace{q^{-1} t_0^{-1} t_2^{-1}}_{q^{-1} t_0^{-1} t_2^{-1}} - \underbrace{q^{-1} t_0^{-2} t_2^{-1} t_0^{-1}}_{t_0^{-1} t_0^{-1}}$$

$$\underbrace{-q^{-1} t_2^{-1} T_0}_{-q^{-1} t_0^{-1} t_2^{-1} T_0} - \underbrace{t_1 t_2 t_0^{-1} T_0}_{-t_1 t_2 t_0^{-1} T_0} + \underbrace{q^{-1} t_2^{-1} t_0^{-1}}_{q^{-1} t_2^{-1} t_0^{-1}} + \underbrace{q^{-1} t_2^{-1} T_0}_{q^{-1} t_2^{-1} T_0}$$

Other cases sum.

□

Recall H_1^+ has basis

$$A^i B^j t_0^k T_1^r T_2^s T_3^t \quad A^i C B^j t_0^k T_1^r T_2^s T_3^t$$

$$k \in \mathbb{Z}, \quad i, j, r, s, t \in \mathbb{N}$$

By Prop 137 3 out of 6 products are exp in this basis

We do now the other 3

LEM 138

$$(t_0 t_3 - t_3 t_0)(t_0 t_1 - t_1 t_0)$$

*

term	coef
AB	$q^2 t_0^2$
A	$-t_0 T_3$
B	$-t_0 T_1$
C	$q t_0 (q^2 t_0 - q^{-2} t_0^{-1})$
I	$T_1 T_3 - q(q - q^{-1}) t_0^2 \gamma - q^{-1} (t_0 - t_0^{-1}) t_0^2 T_2$

pf In the formula for * in Prop 137 eval using

$$\frac{q AB - q^{-1} BA}{q^2 - q^{-2}} + C = \frac{\gamma}{q + q^{-1}}$$