$\mathbb{F}$ arb

$0 \neq q \in \mathbb{F} \qquad q^4 \neq 1$

We just proved

Th 110 : the map $\psi : \Delta_q \to \hat{H}_q$ is inj.

From now on we identify each element of Δ_q with its image in $\hat{H}_q$ under ψ.

So we write A, α instead of A^ψ, α^ψ etc

So $A = y + y^{-1}$ etc.

$\alpha = (q^{-1}t_0 + q t_0^{-1}) T_1 + T_2 T_3$

Next general goal: describe the following spaces

$$\hat{H}_q^+ := \{ h \in \hat{H}_q \mid t_0 h = h t_0 \}$$

$$\hat{H}_q^- := \{ h \in \hat{H}_q \mid t_0 h = h t_0^{-1} \}$$

Recall subalgebra

$$\Pi = \langle t_0^{\pm 1}, T_1, T_2, T_3 \rangle$$

$$\cong \mathbb{F}[\lambda_0^{\pm 1}, \lambda_1, \lambda_2, \lambda_3]$$

This is a domain ie $fg = 0 \to f = 0$ or $g = 0 \quad \forall f, g \in \Pi$

LEM 111 For $h \in \hat{H}_q$ and $t \in \mathbb{T}$

(i) Assume $ht = 0$ then $h = 0$ or $t = 0$

(ii) Assume $th = 0$ then $h = 0$ or $t = 0$

pf (i) We assume $t \neq 0$ and show $h = 0$

Recall the YXT-factorization of $\hat{H}_q$ from above LS6:

$$h = \sum_{i,j \in \mathbb{Z}} Y^i X^j t_{ij} \qquad t_{ij} \in \mathbb{T}$$

$$0 = ht = \sum_{i,j \in \mathbb{Z}} Y^i X^j \underbrace{t_{ij} t}_{\in \mathbb{T}}$$

By YXT-factorization

$$t_{ij} t = 0 \qquad \forall\, i,j \in \mathbb{Z}$$

We assume $t \neq 0$ so

$$t_{ij} = 0 \qquad \forall\, i,j \in \mathbb{Z}$$

So

$$h = 0 \qquad \square$$

In Th 52 we obtained a basis for $\hat{H}_q$

We now display 2 related bases that will be useful in our investigation of $\hat{H}_q^{\pm}$

Prop 112 The following is a basis for the $\mathbb{F}$-vector space $\hat{H}_q$:

$$A^i u B^j t_0^k T_1^r T_2^s T_3^t \qquad k \in \mathbb{Z}, \qquad \cancel{*}$$

$$u \in \{1, X, Y, YX\}, \qquad i, j, r, s, t \in \mathbb{N}$$

pf By the YXT - factorization

$$\langle Y^{\pm 1} \rangle \otimes \langle X^{\pm 1} \rangle \otimes \mathbb{T} \longrightarrow \hat{H}_q$$

$$u \otimes v \otimes w \longrightarrow uvw$$

is iso of $\mathbb{F}$-vector spaces

$$\langle Y^{\pm 1} \rangle \cong \mathbb{F}[\lambda^{\pm 1}]$$

has basis $\{Y^i\}_{i \in \mathbb{Z}}$

Another basis

$$(Y+Y^{-1})^i, \quad (Y+Y^{-1})^i Y \qquad i \in \mathbb{N}$$

$$\overset{\shortparallel}{A^i} \qquad A^i Y$$

Sim $\langle X^{\pm 1} \rangle$ has basis

$$B^j, \quad X B^j \qquad j \in \mathbb{N}$$

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so it follows. $\square$

Advantage of basis in Prop 112:

t_0 commutes each of

A, B, t_0, T_1, T_2, T_3

Only difficulty is $u \in \{x, y, yx\}$

— o —

To motivate the next basis let us write

t_0, t_1, t_2, t_3 in the basis from Prop 112

t_0 already in this basis

LEM 113 The elements t_1, t_2, t_3 look as follows in the basis from Prop 112:

$$t_1 = Y t_0 - A t_0 + T_1 \tag{1}$$

$$t_2 = q^{-1} A T_3 - q^{-1} A X t_0^{-1} - q^{-1} Y T_3 + q^{-1} Y X t_0^{-1} \tag{2}$$

$$t_3 = X t_0^{-1} \tag{3}$$

pf In Lem 39 we wrote $t_1 t_2, t_3$ in ess the basis from m52.

The above equations are a minor adjustment

details: By L39

$$t_1 = T_1 - Y^{-1} t_0$$
$$= T_1 - (\underbrace{Y + Y^{-1}}_{A}) t_0 + Y t_0$$

$$t_2 = q^{-1} Y^{-1} T_3 - q^{-1} Y^{-1} X t_0^{-1}$$
$$= q^{-1}(Y+Y^{-1}) T_3 - q^{-1} Y T_3 - q^{-1}(Y+Y^{-1}) X t_0^{-1} + q^{-1} Y X t_0^{-1}$$

$$t_3 = t_3 t_0 t_0^{-1}$$
$$= X t_0^{-1}$$

□

In (1)–(3) let us solve for

X, Y, YX in terms of t_1, t_2, t_3 & viewing

A, t_0, T_1, T_2, T_3 as coeffs:

$$X = t_3 t_0 \qquad (1')$$

$$Y = A + t_1 t_0^{-1} - t_0^{-1} T_1 \qquad (2')$$

$$YX = A t_3 t_0 + q t_2 t_0 + t_1 T_3 - T_1 T_3 \qquad (3')$$

Prop 114 The following is a basis for the $\mathbb{F}$-vector space $\hat{H}_q$:

$$A^i v B^j t_0^k T_1^r T_2^s T_3^t \qquad k \in \mathbb{Z}, \qquad (✱)$$

$$i, j, r, s, t \in \mathbb{N}$$

$$v \in \{1, t_1, t_2, t_3\}$$

pf ✱ spans $\hat{H}_1$ by (1')–(3') and since

$*$ spans $\hat{H}_q$.

Show ✱ is lin indep:

Using (1)–(3) write ✱ in basis $*$.

For $m, n \in \{0,1\}$ put

$$H_{mn} = \operatorname{Span}\left\{ A^i y^m x^n B^j t_0^k T_1^r T_2^s T_3^t \;\middle/\; \begin{array}{l} k \in \mathbb{Z}, \\ i,j,r,s,t \in \mathbb{N} \end{array} \right\}$$

so
$$\hat{H}_q = H_{00} + H_{01} + H_{10} + H_{11} \qquad (\text{ds of vs})$$

Taking $v = 1$ in ✵ we get a basis for H_{00}

For each term
$$A^i t_1 B^j t_0^k T_1^r T_2^s T_3^t$$

in ✵

$$A^i t_1 B^j t_0^k T_1^r T_2^s T_3^t - A^i Y B^j t_0^{k+1} T_1^r T_2^s T_3^t \in H_{00}$$

therefore vectors in ✵ with $v = t_1$ give a basis for a complement of H_{00} in $H_{00} + H_{10}$

By (3)
vectors in ✵ with $v = t_3$ form a basis for H_{01}

For each vector
$$A^i t_2 B^j t_0^k T_1^r T_2^s T_3^t$$

in ✵

$$A^i t_2 B^j t_0^k T_1^r T_2^s T_3^t - q^{-1} A^i Y X B^j t_0^{k-1} T_1^r T_2^s T_3^t$$
$$\in H_{00} + H_{01} + H_{10}$$

therefore the vectors in ✡ with $v = t_2$

form a basis for a complement of

$$H_{00} + H_{01} + H_{10}$$

in $\hat{H}_q$.

therefore ✡ is lin indep. □

— ∘ —

write $\hat{\theta}$ in the bases above

LEM 115 $\hat{\theta}$ looks as follows in the basis for $\hat{H}_q$ from Prop 112:

$$\hat{\theta} = \begin{array}{c|ccc} & 1 & B & X \\ \hline 1 & q^{-1}t_0^{2}T_2 & 0 & T_1 \\ A & T_3 & 0 & -t_0^{-1} \\ Y & -T_3 & t_0 & t_0^{-1} - t_0 \end{array}$$

pf Recall

$$\hat{\theta} = (\gamma - q C)t_0$$

$$= YX^{-1}t_0 - Y^{-1}Xt_0^{-1} + Y^{-1}T_3 + XT_1 + q^{-1}t_0^2 T_2$$

In above equation elim Y^{-1}, X^{-1} using

$$Y^{-1} = A - Y \qquad X^{-1} = B - X$$

LEM 116 $\hat{\theta}$ looks as follows in the basis for $\hat{H}_q$ from Prop 114:

$$\hat{\theta} = q^{-1}t_0^2 T_2 + ABt_0 - BT_1 + t_0 T_1 T_3$$

$$+ \quad t_1 B - t_1 t_0 T_3$$

$$+ \quad q t_2 - q t_2 t_0^2$$

$$+ \quad t_3 t_0 T_1 - A t_3 t_0^2$$

pf In the equation of Lem 115 eval RHS using (1′) – (3′)

To sum up so far

$$\hat{H}_q = \langle A \rangle \underbrace{\langle B \rangle}_{H_{00}} \pi$$

$$+ \langle A \rangle \underbrace{X \langle B \rangle}_{H_{01}} \pi$$

$$+ \langle A \rangle \underbrace{Y \langle B \rangle}_{H_{10}} \pi$$

$$+ \langle A \rangle \underbrace{Y X_i}_{H_{11}} \langle B \rangle \pi$$

direct sum

$$\hat{H}_q = \langle A \rangle \langle B \rangle \pi$$

$$+ \langle A \rangle t_3 \langle B \rangle \pi$$

$$+ \langle A \rangle t_1 \langle B \rangle \pi$$

$$+ \langle A \rangle t_2 \langle B \rangle \pi$$

direct sum

Also

$$\langle A\rangle\langle B\rangle\pi + \langle A\rangle Y\langle B\rangle\pi$$

$$= \langle A\rangle\langle B\rangle\pi + \langle A\rangle t_1\langle B\rangle\pi$$

$$\langle A\rangle\langle B\rangle\pi + \langle A\rangle X\langle B\rangle\pi$$

$$= \langle A\rangle\langle B\rangle\pi + \langle A\rangle t_3\langle B\rangle\pi$$

$$\langle A\rangle\langle B\rangle\pi + \langle A\rangle Y\langle B\rangle\pi + \langle A\rangle X\langle B\rangle\pi$$

$$= \langle A\rangle\langle B\rangle\pi + \langle A\rangle t_1\langle B\rangle\pi + \langle A\rangle t_3\langle B\rangle\pi$$

$\mathbb{F}$ arb

$0 \neq q \in \mathbb{F}$ $\quad q^4 \neq 1$

Recall general goal: describe

$$\hat{H}_q^+ = \{ h \in \hat{H}_q \mid t_0 h = h t_0 \}$$
$$\hat{H}_q^- = \{ h \in \hat{H}_q \mid t_0 h = h t_0^{-1} \}$$

Recall

$$\hat{H}_q = \langle A \rangle \langle B \rangle \mathbb{T} + \langle A \rangle X \langle B \rangle \mathbb{T} + \langle A \rangle Y \langle B \rangle \mathbb{T} + \langle A \rangle YX \langle B \rangle \mathbb{T} \quad \text{ds}$$

$$\mathbb{T} = \langle t_0^{\pm 1}, T_1, T_2, T_3 \rangle$$

$$\hat{H}_q = \langle A \rangle \langle B \rangle \mathbb{T} + \langle A \rangle t_1 \langle B \rangle \mathbb{T} + \langle A \rangle t_2 \langle B \rangle \mathbb{T} + \langle A \rangle t_3 \langle B \rangle \mathbb{T} \quad \text{ds}$$

LEM 117 The following coincide:

(i) $\quad * + \langle A \rangle C \langle B \rangle \mathbb{T},$

(ii) $\quad * + \langle A \rangle YX \langle B \rangle \mathbb{T} (t_0 - t_0^{-1})$

(iii) $\quad * + \langle A \rangle t_2 \langle B \rangle \mathbb{T} (t_0 - t_0^{-1})$

where

$$* = \langle A \rangle \langle B \rangle \mathbb{T} + \langle A \rangle X \langle B \rangle \mathbb{T} + \langle A \rangle Y \langle B \rangle \mathbb{T}$$
$$= \langle A \rangle \langle B \rangle \mathbb{T} + \langle A \rangle t_1 \langle B \rangle \mathbb{T} + \langle A \rangle t_3 \langle B \rangle \mathbb{T}$$

Moreover each sum (i)–(iii) is direct

pf obs

$$\langle A \rangle * \subseteq *,$$

$$* \langle B \rangle \subseteq *$$

$$* \Pi \subseteq *$$

Recall $q C = \gamma - \theta t_0^{-1}$ $\quad \gamma \in \Pi \subseteq *$

By L115

$$\theta + yx(t_0 - t_0^{-1}) \in *$$

By L116

$$\theta + t_2 \, q t_0 (t_0 - t_0^{-1}) \in *$$

So in the quotient vector space $\hat{H}_q / *$

$$C + * = \theta(-q^{-1} t_0^{-1}) + *$$

$$= yx(t_0 - t_0^{-1})(q^{-1} t_0^{-1}) + *$$

$$= t_2(t_0 - t_0^{-1}) + *$$

By this and since $\Pi = t_0^{-1} \Pi$

$$C\Pi + * = \theta \Pi + *$$

$$= yx(t_0 - t_0^{-1}) \Pi + *$$

$$= t_2(t_0 - t_0^{-1}) \Pi + *$$

Follows that (i) – (iii) coincide

Check sums are direct

(ii) is direct since

$$\hat{H}_q = * + \langle A \rangle YX \langle B \rangle \Pi \quad \text{ds}$$

and

$$\langle A \rangle YX \langle B \rangle \Pi \, (t_0 - t_0^{-1}) \subseteq \langle A \rangle YX \langle B \rangle \Pi$$

(iii) is direct since

$$\hat{H}_q = * + \langle A \rangle t_2 \langle B \rangle \Pi \quad \text{ds}$$

and

$$\langle A \rangle t_2 \langle B \rangle \Pi \, (t_0 - t_0^{-1}) \subseteq \langle A \rangle t_2 \langle B \rangle \Pi$$

show (i) is direct

Suppose not. Then $\exists$

$$\sum_{i,j \in \mathbb{Z}} A^i C B^j t_{ij} \in * \qquad t_{ij} \in \Pi \text{ not zero}$$

Then

$$\sum_{i,j \in \mathbb{Z}} A^i t_2 B^j t_{ij} (t_0 - t_0^{-1}) \in *$$

Now

$$\sum_{ij} A^i t_2 B^j t_{ij} (t_0 - t_0^{-1}) = 0 \qquad \text{since (iii) is direct}$$

Now

$$\sum_{ij} A^i t_2 B^j t_{ij} = 0$$

This contradicts Prop 114. □

Comments on $\hat{H}_q^{\pm}$

obs the sum

$$\hat{H}_q^{+} + \hat{H}_q^{-} \quad \text{is direct}$$

- $$\hat{H}_q^{+}\hat{H}_q^{-} \leq \hat{H}_q^{-}$$
$$\hat{H}_q^{-}\hat{H}_q^{+} \leq \hat{H}_q^{-}$$
$$\hat{H}_q^{-}\hat{H}_q^{-} \leq \hat{H}_q^{+}$$

So $\hat{H}_q^{+} + \hat{H}_q^{-}$ is subalgebra of $\hat{H}_q$

$\hat{H}_q^{\pm}$ are components of a $\mathbb{Z}_2$-grading of this subalgebra

LEM 118 $\forall\, h \in \hat{H}_q$

(i) $t_0 h - h t_0 \in \hat{H}_q^{-}$

(ii) $t_0 h - h t_0^{-1} \in \hat{H}_q^{+}$

pf (i)

$$t_0(t_0 h - h t_0) - (t_0 h - h t_0)t_0^{-1}$$

$$= t_0^2 h - t_0 h t_0 - t_0 h t_0^{-1} + h$$

$$= t_0^2 h - t_0 h T_0 + h$$

$$= \underbrace{(t_0^2 - t_0 T_0 + 1)}_{0} h_0$$

$$= 0$$

$\square$

(ii) Sim

Given $h \in H_q$

write

$$u = t_0 h - h t_0^{-1};$$
$$v = t_0 h - h t_0$$

obs

$$(t_0 - t_0^{-1}) h = u + v$$

$$h(t_0 - t_0^{-1}) = u - v$$

$$(t_0 - t_0^{-1}) h + h(t_0 - t_0^{-1}) = 2u$$
$$(t_0 - t_0^{-1}) h - h(t_0 - t_0^{-1}) = 2v$$

By these comments and L118, we get:

LEM 119 $\forall h \in \hat{H}_q$

(i) $h(t_0 - t_0^{-1}) \in \hat{H}_q^+ + \hat{H}_q^-$

(ii) $(t_0 - t_0^{-1}) h \in H_q^+ + \hat{H}_q^-$

(iii) $(t_0 - t_0^{-1}) h + h(t_0 - t_0^{-1}) \in \hat{H}_q^+$

(iv) $(t_0 - t_0^{-1}) h - h(t_0 - t_0^{-1}) \in \hat{H}_q^-$

Consider $t_0 - t_0^{-1}$ carefully

Obs

$$(t_0 - t_0^{-1})^2 = (t_0 + t_0^{-1})^2 - 4$$
$$= T_0^2 - 4 \qquad (\text{Central in } \hat{H}_q)$$

Consider the maps

$$\hat{H}_q \longrightarrow \hat{H}_q \qquad\qquad \hat{H}_q \longrightarrow \hat{H}_q$$
$$h \longrightarrow t_0 h - h t_0^{-1} \qquad\qquad h \longrightarrow t_0 h - h t_0$$

By L118 these maps commute and their composition is 0

We now find the action of these maps on the bases for $\hat{H}_q$ from Prop 112 and Prop 114

Since t_0 commutes with

$A, B, \quad \forall t \in \pi$

suf to consider actions on

$1, X, Y, YX$

and $1, t_1, t_2, t_3$

LEM 120′ We have

h	$t_0 h - h t_0^{-1}$
1	$t_0 - t_0^{-1}$
X	$B t_0 - T_3$
Y	$A t_0 - T_1$
YX	$AB t_0 - \theta + q^{-1} t_0^{-2} T_2$

Note that

$AB t_0 - \theta + q^{-1} t_0^{-2} T_2 =$

	1	B	X
1	0	0	$-T_1$
A	$-T_3$	t_0	t_0^{-1}
Y	T_3	$-t_0$	$t_0 - t_0^{-1}$

pf use LEM 25

LEM 121 We have

h	$t_0 h - h t_0^{-1}$
1	$t_0 - t_0^{-1}$
t_1	$A - t_0^{-1} T_1$
t_2	$C - t_0^{-1} T_2$
t_3	$B - t_0^{-1} T_3$

pf Fn $i = 1, 2, 3$

$$t_0 t_i - t_i t_0^{-1} = t_0 t_i - (T_i - t_i^{-1}) t_0^{-1}$$
$$= t_0 t_i + t_i^{-1} t_0^{-1} - t_0^{-1} T_i$$

Recall
$$A = t_0 t_1 + t_1^{-1} t_0^{-1}$$
$$B = t_0 t_3 + t_3^{-1} t_0^{-1}$$
$$C = t_0 t_2 + t_2^{-1} t_0^{-1}.$$

$\square$

LEM 122 We have

h	$t_0 h - h t_0$
1	0
X	$B t_0 - T_3 - X(t_0 - t_0^{-1})$
Y	$A t_0 - T_1 - Y(t_0 - t_0^{-1})$
YX	$AB t_0 - \theta + q^{-1} t_0^2 T_2 - YX(t_0 - t_0^{-1})$

Note that

$AB t_0 - \theta + q^{-1} t_0^2 T_2 - YX(t_0 - t_0^{-1}) =$

	1	B	X
1	0	0	$-T_1$
A	$-T_3$	t_0	t_0^{-1}
Y	T_3	$-t_0$	0

pf use L120 and comments above L119. $\square$

LEM 123 We have

h	$t_0h - ht_0$
1	0
t_1	$A - t_0^{-1}T_1 - t_1(t_0 - t_0^{-1})$
t_2	$C - t_0^{-1}T_2 - t_2(t_0 - t_0^{-1})$
t_3	$B - t_0^{-1}T_3 - t_3(t_0 - t_0^{-1})$

Note that

$$
\begin{aligned}
C - t_0^{-1}T_2 - t_2(t_0 - t_0^{-1}) &= q^{-1}Bt_0^{-1}T_1 - q^{-1}AB \\
&\quad - q^{-1}t_1Bt_0^{-1} + q^{-1}t_1T_3 \\
&\quad + q^{-1}At_3t_0 - q^{-1}t_3T_1 \\
&\in \langle A\rangle\langle B\rangle\mathbb{T} + \langle A\rangle t_1\langle B\rangle\mathbb{T} \\
&\quad + \langle A\rangle t_3\langle B\rangle\mathbb{T}
\end{aligned}
$$

pf Use L121 and comments above L119 □

LEM 124 Each of the following subspaces is invariant under both $h \to t_0 h - h t_0$ and $h \to t_0 h - h t_0^{T}$

$$\langle A\rangle\langle B\rangle \Pi \tag{1}$$

$$\langle A\rangle\langle B\rangle \Pi + \langle A\rangle X \langle B\rangle \Pi \tag{2}$$

$$\langle A\rangle\langle B\rangle \Pi + \langle A\rangle Y \langle B\rangle \Pi \tag{3}$$

$$\langle A\rangle\langle B\rangle \Pi + \langle A\rangle X \langle B\rangle \Pi + \langle A\rangle Y \langle B\rangle \Pi \tag{4}$$

$\square$

pf By L120, L122

LEM 125 The map $h \to t_0 h - h t_0^{-1}$ acts on quotients as follows

on the space	the map acts as
(1)	$h \to h(t_0 - t_0^{-1})$
(2)/(1)	0
(3)/(1)	0
$\hat{H}_q/(3)$	$h \to h(t_0 - t_0^{-1})$

pf. By L120 $\square$

LEM 126 The map $h \to t_0 h - h t_0$

acts on quotients as follows

on the space	the map acts as
(1)	0
(2)/(1)	$h \to h(t_0 - t_0^{-1})$
(3)/(1)	$h \to h(t_0 - t_0^{-1})$
$\hat{H}_q/(3)$	0

pf Use L 122 $\square$

Next: show

- $\hat{H}_q^+ = \langle A \rangle \langle B \rangle \Pi + \langle A \rangle C \langle B \rangle \Pi$
- $\hat{H}_q^+$ is gen by
  $A, B, C, t_0^{\pm 1}, \{T_i\}_{i=1}^3$

$\mathbb{F}$ arb

$0 \neq q \in \mathbb{F}$ $\quad q^4 \neq 1$

Continue to study Univ DAHA $\hat{H}_q$ of type $(C_1^\vee, C_1)$

Current goal: Describe

$$\hat{H}_q^+ = \{ h \in \hat{H}_q \mid h t_0 = t_0 h \}$$

$$\hat{H}_q^- = \{ h \in \hat{H}_q \mid t_0 h = h t_0^{-1} \}$$

We view

$\hat{H}_q^+$ = kernel of $\mathbb{F}$-lin trans $\quad h \to t_0 h - h t_0$

$\hat{H}_q^-$ = ... $\quad h \to t_0 h - h t_0^{-1}$

Our next specific goal is to prove:

thm 127 The $\mathbb{F}$-algebra $\hat{H}_q$ is generated by

$$A, B, C, t_0^{\pm 1}, \{T_i\}_{i=1}^{3}$$

$$\left[A = Y + Y^{-1}, \quad B = X + X^{-1}, \quad Y = t_0 t_1 \quad X = t_3 t_0 \quad C = t_0 t_2 + (t_0 t_2)^{-1} \right]$$

th 128 the $\mathbb{F}$-vector space $\hat{H}_q$ has basis

$$A^i B^j t_0^k T_1^r T_2^s T_3^t, \quad A^i C B^j t_0^k T_1^r T_2^s T_3^t$$

$$k \in \mathbb{Z}, \quad i, j, r, s, t \in \mathbb{N}$$

th 129 $\quad \hat{H}_q^+ = \langle A \rangle \langle B \rangle \mathbb{T} + \langle A \rangle C \langle B \rangle \mathbb{T}$

$$\left[\begin{array}{l} \langle S \rangle = \text{subalg of } \hat{H}_q \text{ gen by } S \\ \mathbb{T} = \langle t_0^{\pm 1}, T_1, T_2, T_3 \rangle \end{array} \right]$$

Recall that the following sum is direct:

$$\hat{H}_q = \langle A\rangle\langle B\rangle T + \langle A\rangle x \langle B\rangle T + \langle A\rangle y \langle B\rangle T + \langle A\rangle yx \langle B\rangle T \qquad \ast$$

For each of the 4 components in this decomp

we consider the corresp projection map

For $u \in \{1, x, y, yx\}$ define an $\mathbb{F}$-linear

map $\quad \pi_u : \hat{H}_q \to \hat{H}_q$

such that

π_u acts as ident on $\langle A\rangle u \langle B\rangle \pi$

" " 0 on other 3 components of $\ast$

So π_u is projection from $\hat{H}_q$ onto $\langle A\rangle u \langle B\rangle \pi$

For $h \in \hat{H}_q$

$$h = \underset{\langle A\rangle\langle B\rangle\pi}{\underset{\cap}{\pi_1(h)}} + \underset{\langle A\rangle x\langle B\rangle\pi}{\underset{\cap}{\pi_x(h)}} + \underset{\langle A\rangle y\langle B\rangle\pi}{\underset{\cap}{\pi_y(h)}} + \underset{\langle A\rangle yx\langle B\rangle\pi}{\underset{\cap}{\pi_{yx}(h)}}$$

Ex 130

u	$\pi_u(\theta)$
1	$A T_3 + q^{-1} t_0^2 T_2$
X	$X T_1 - A X t_0^{-1}$
Y	$Y B t_0 - Y T_3$
YX	$t_0^{-1} - t_0$

pf this reformulation of L115 $\square$

Ex 130A

u	$\pi_u(C)$
1	$-q^{-1}At_0^{-1}T_3 + q^{-1}\gamma - q^{-2}t_0T_2$
x	$q^{-1}AXt_0^{-2} - q^{-1}Xt_0^{-1}T_1$
y	$-q^{-1}YB + q^{-1}Yt_0^{-1}T_3$
yx	$q^{-1}yX(t_0-t_0^{-1})t_0^{-1}$

DEF 131 let $\tilde{H}_q$ denote the subspace of $\hat{H}_q$ from L117

So

$$\tilde{H}_q = \langle A\rangle\langle B\rangle\pi + \langle A\rangle x\langle B\rangle\pi + \langle A\rangle y\langle B\rangle\pi + \langle A\rangle yx\langle B\rangle\pi \qquad (*)$$

$$= \quad " \quad + \quad " \quad + \quad " \quad + \langle A\rangle c\langle B\rangle\pi \qquad (**)$$

For $h \in \tilde{H}_q$ we have the projections $\pi_u(h)$ from $(*)$

we also have projections for $(**)$

DEF 132 For $v \in \{1, x, y, c\}$ let

$P_v : \tilde{H}_q \to \tilde{H}_q$ denote the $\mathbb{F}$-linear trans such that

P_v acts as ident on $\langle A\rangle v\langle B\rangle\pi$

" " 0 on other 3 components of $**$

[Caution $P_v \neq \pi_v$ for $v = 1, x, y$]

For $h \in \tilde{H}_q$

$$h = \underset{\langle A\rangle\langle B\rangle\pi}{\underset{\cap}{P_1(h)}} + \underset{\langle A\rangle x\langle B\rangle\pi}{\underset{\cap}{P_x(h)}} + \underset{\langle A\rangle y\langle B\rangle\pi}{\underset{\cap}{P_y(h)}} + \underset{\langle A\rangle c\langle B\rangle\pi}{\underset{\cap}{P_c(h)}}$$

For $h \in \widetilde{H}_q$ we now clarify how the $\pi_u(h)$, $P_v(h)$ are related

LEM 133 Given $h \in \widetilde{H}_q$ write

$$P_C(h) = \sum_{i,j \in \mathbb{N}} A^i C B^j t_{ij} \qquad t_{ij} \in \mathbb{T}$$

Then

$$(i) \qquad \pi_1(h) - P_1(h) = \sum_{i,j \in \mathbb{N}} A^i \pi_1(C) B^j t_{ij}$$

$$(ii) \qquad \pi_x(h) - P_x(h) = \sum_{i,j \in \mathbb{N}} A^i \pi_x(C) B^j t_{ij}$$

$$(iii) \qquad \pi_y(h) - P_y(h) = \sum_{i,j \in \mathbb{N}} A^i \pi_y(C) B^j t_{ij}$$

$$(iv) \qquad \pi_{yx}(h) = \sum_{i,j \in \mathbb{N}} A^i y x B^j t_{ij} t_0^{-1}(t_0 - t_0^{-1}) q^{-1}$$

pf

$$h = \pi_1(h) + \pi_x(h) + \pi_y(h) + \pi_{yx}(h)$$

$$h = P_1(h) + P_x(h) + P_y(h) + P_C(h)$$

$$P_C(h) = \sum_{i,j \in \mathbb{N}} A^i \underbrace{C}_{\pi_1(C) + \pi_x(C) + \pi_y(C) + \underbrace{\pi_{yx}(C)}_{\underbrace{-\pi_{yx}(\theta)\, t_0^{\to} q^{\to}}_{y x\, t_0^{\to}(t_0 - t_0^{\to}) q^{\to}}}} B^j\, t_{ij}$$

$$\left[\, qC = \underset{\pi}{\underset{\wedge}{r}} - \theta t_0^{\to} \,\right]$$

Location

$$0 = h - h$$

$$= P_1(h) - \pi_1(h) + \sum_{i,j \in \mathbb{N}} A^i \pi_1(C) B^j t_{ij} \qquad \langle A \rangle \langle B \rangle \pi$$

$$+ \; P_x(h) - \pi_x(h) + \sum_{i,j \in \mathbb{N}} A^i \pi_x(C) B^j t_{ij} \qquad \langle A \rangle x \langle B \rangle \pi$$

$$+ \; P_y(h) - \pi_y(h) + \sum_{i,j \in \mathbb{N}} A^i \pi_y(C) B^j t_{ij} \qquad \langle A \rangle y \langle B \rangle \pi$$

$$-\pi_{yx}(h) + \sum_{i,j \in \mathbb{N}} A^i y x B^j t_0^{\to}(t_0 - t_0^{\to}) q^{\to} t_{ij} \qquad \langle A \rangle yx \langle B \rangle \pi$$

In each row terms must sum to 0

Result follows $\square$

We need a technical lemma

LEM 134 $\forall\, h \in \hat{H}_q$ the following are equiv.

(i) $h \in \langle A\rangle\langle B\rangle\Pi + \langle A\rangle C\langle B\rangle\Pi$

(ii) $h(t_0 - t_0^{-1}) \in \langle A\rangle\langle B\rangle\Pi + \langle A\rangle C\langle B\rangle\Pi$

pf (i) → (ii) Since $t_0 - t_0^{-1} \in \Pi$

(ii) → (i)

Obs $h(t_0 - t_0^{-1}) \in \tilde{H}_q$

Write
$$P_C\left(h(t_0 - t_0^{-1})\right) = \sum_{i,j \in \mathbb{N}} A^i C B^j t_{ij} \qquad t_{ij} \in \Pi$$

strategy: show
$$h \in \tilde{H}_q$$

Obs
$$P_x\left(h(t_0 - t_0^{-1})\right) = 0$$
$$P_y\left(h(t_0 - t_0^{-1})\right) = 0$$

$$\pi_1\left(h(t_0 - t_0^{-1})\right) = \pi_1(h)(t_0 - t_0^{-1})$$
$$\pi_x\left(h(t_0 - t_0^{-1})\right) = \pi_x(h)(t_0 - t_0^{-1})$$
$$\pi_y\left(h(t_0 - t_0^{-1})\right) = \pi_y(h)(t_0 - t_0^{-1})$$
$$\pi_{yx}\left(h(t_0 - t_0^{-1})\right) = \pi_{yx}(h)(t_0 - t_0^{-1})$$

By LB3

$$\pi_{yx}\left(h(t_0 - t_0^{-1})\right) = \sum_{i,j\in\mathbb{N}} A^i yx B^j t_{ij} t_0^{-1}(t_0 - t_0^{-1}) q^{-1}$$

So

$$\pi_{yx}(h) = \sum_{i,j\in\mathbb{N}} A^i yx B^j t_{ij} t_0^{-1} q^{-1}$$

To show $h \in \tilde{H}_q$

show $t_0 - t_0^{-1}$ divides t_{ij} for all $i,j \in \mathbb{N}$

Compute $\pi_y\left(h(t_0 - t_0^{-1})\right)$ in two ways

$$\begin{aligned}\pi_y\left(h(t_0-t_0^{-1})\right) &\overset{\text{Lem } 13.3(iii)}{=} \sum_{i,j\in\mathbb{N}} A^i \underbrace{\pi_y(c)}_{\substack{\| \\ \underbrace{-\pi_y(a) t_0^{-1} q^{-1}}_{q^{-1} Y t_0^{-1} T_3 \;-\; q^{-1} Y B}}} B^j t_{ij} \\ \| \qquad & \qquad \| \\ \pi_y(h)(t_0 - t_0^{-1}) & \qquad \sum_{r,s\in\mathbb{N}} A^r Y B^s \left(q^{-1} t_{rs} t_0^{-1} T_3 - q^{-1} t_{r,s-1}\right)\end{aligned}$$

view as 0 if s=0

Comparing the two sides

$t_0 - t_0^{-1}$ divides $q^{-1} t_{rs} t_0^{-1} T_3 - q^{-1} t_{r,s-1}$ $\quad \forall r,s \in \mathbb{N}$

By this and ind on s

$t_0 - t_0^{-1}$ divides t_{rs} $\quad \forall r,s \in \mathbb{N}$

So $\forall r,s \in \mathbb{N} \quad \exists \; t_{rs} \in \mathbb{I}$

s.t.
$$t_{rs} = t'_{rs}\,(t_0 - t_0^{-1})$$

Now

$$\pi_{yx}(h) = \sum_{i,j \in \mathbb{N}} A^i y x B^j t_{ij}\, t_0^{-1}(t_0 t_0^{-1}) q^{-1}$$
$$\in \langle A \rangle y x \langle B \rangle (t_0 - t_0^{-1})$$

so $\quad h \in \widetilde{H}_q$

Now
$$\underbrace{P_X(h(t_0 - t_0^{-1}))}_{0} = P_X(h)\,(t_0 - t_0^{-1})$$

so $\quad P_X(h) = 0$

$$\underbrace{P_Y(h(t_0 - t_0^{-1}))}_{0} = P_Y(h)\,(t_0 - t_0^{-1})$$

so $\quad P_Y(h) = 0$

Now $\quad h = P_0(h) + P_C(h)$
$$\in \langle A \rangle \langle B \rangle \pi + \langle A \rangle C \langle B \rangle \pi \qquad \square$$

pf of th 129

$\hat{H}_q \supseteq \langle A\rangle\langle B\rangle\Pi + \langle A\rangle C\langle B\rangle\Pi$:

We saw earlier $A, B, C, t_0^{\pm 1}, T_1, T_2, T_3$

commute with t_0

$\subseteq$:

Let $h \in \hat{H}_q^{\,t}$ be given.

First show

$$h(t_0 - t_0^{-1}) \in \langle A\rangle\langle B\rangle\Pi + \langle A\rangle C\langle B\rangle\Pi \qquad (*)$$

By assumption $t_0 h = h t_0$ so

$$h(t_0 - t_0^{-1}) = t_0 h - h t_0^{-1}$$

By L121 image of $\hat{H}_q$ under map $g \to t_0 g - g t_0^{-1}$

is contained in

$$\langle A\rangle\langle B\rangle\Pi + \langle A\rangle C\langle B\rangle\Pi$$

This gives $*$

Now by $*$ and L134

$$h \in \langle A\rangle\langle B\rangle\Pi + \langle A\rangle C\langle B\rangle\Pi \quad \checkmark$$

$\square$

Proof of Th 127 : By Th 129 and since

Π is gen by $t_0^{\pm 1}$, T_1, T_2, T_3 $\square$

Proof of Th 128 : linear independence is by Prop 109

span by Th 127 $\square$

$\mathbb{F}$ arb

$0 \neq q \in \mathbb{F}$ $\quad q^4 \neq 1$

Continue to study univ DAHA $\hat{H}_q$ type $(C_1^\vee C_1)$

Recall
$$\hat{H}_q^+ = \{ h \in \hat{H}_q \mid t_0 h = h t_0 \}$$
$$\hat{H}_q^- = \{ h \in \hat{H}_q \mid t_0 h = h t_0^{-1} \}$$

We saw
$$\hat{H}^+ = \langle A \rangle \langle B \rangle \Pi + \langle A \rangle C \langle B \rangle \Pi$$

Found basis, gen set

We now describe $\hat{H}_q^+$ by gens + rels.

thm 130 The $\mathbb{F}$-alg $\hat{H}_q^+$ is described by gens and rels as follows. The gens are

$$A, B, C, t_0^{\pm 1}, \{T_i\}_{i=1}^3$$

The rels are:

$$t_0 t_0^{-1} = t_0^{-1} t_0 = 1,$$

$$t_0^{\pm 1}, \{T_i\}_{i=1}^3 \text{ are central}$$

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{\alpha}{q+q^{-1}} \qquad \alpha = (q^{-1}t_0 + qt_0^{-1})T_1 + T_2T_3$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{\beta}{q+q^{-1}} \qquad \beta = (q^{-1}t_0 + qt_0^{-1})T_3 + T_1T_2$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{\gamma}{q+q^{-1}} \qquad \gamma = (q^{-1}t_0 + qt_0^{-1})T_2 + T_1T_3$$

$$qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma = \qquad \text{✲}$$
$$(q+q^{-1})^2 - (q^{-1}t_0 + qt_0^{-1})^2 - T_1^2 - T_2^2 - T_3^2 - (q^{-1}t_0 + qt_0^{-1})T_1T_2T_3$$

pf Above rels are the ones used to show the vectors in th 128 span $\hat{H}_q^+$.
Vectors in th 128 form basis for $\hat{H}_q^+$ so no further rels exist in the presentation. $\square$

Note In above presentation we could replace ✲ by any one of the 6 versions of the Casimir element Ω
Because the fact that they are equiv follows from prev 3 rels $\mathbb{Z}_3$-sym AW rels

By LEM 118 under map

$$\hat{H}_q \longrightarrow \hat{H}_q$$
$$h \longrightarrow t_0 h - h t_0^{-1}$$

the image of $\hat{H}_q$ is contained in $\hat{H}_q^+$

LEM 131 Above image is a 2-sided ideal of $\hat{H}_q^+$

pf Write

$$J = \{ t_0 h - h t_0^{-1} \mid h \in \hat{H}_q \}$$

Given $k \in \hat{H}_q^+$ show

$$kJ \subseteq J \qquad Jk \subseteq J$$

$\forall h \in H$

$$k(t_0 h - h t_0^{-1}) = t_0 kh - kh t_0^{-1} \qquad \text{since } t_0 k = k t_0$$
$$\in J$$

$$(t_0 h - h t_0^{-1}) k = t_0 hk - hk t_0^{-1}$$
$$\in J$$

$\square$

Describe J

LEM 132 The ideal J is generated by

$$t_0 - t_0^{-1}, \quad A - t_0^{-1} T_1, \quad B - t_0^{-1} T_3, \quad C - t_0^{-1} T_2$$

pf Need to show

$$J = \hat{H}_q^+ (t_0 - t_0^{-1}) \hat{H}_q^+ + \hat{H}_q^+ (A - t_0^{-1} T_1) \hat{H}_q^+ \quad \ast$$

$$+ \hat{H}_q^+ (B - t_0^{-1} T_3) \hat{H}_q^+ + \hat{H}_q^+ (C - t_0^{-1} T_2) \hat{H}_q^+$$

Recall

$$\hat{H}_q = \langle A \rangle \langle B \rangle \pi_1 + \langle A \rangle t_1 \langle B \rangle \pi +$$

$$\langle A \rangle t_2 \langle B \rangle \pi + \langle A \rangle t_3 \langle B \rangle \pi$$

Apply map $h \mapsto t_0 h - h t_0^{-1}$ and use L121. Get

$$J = \langle A \rangle (t_0 - t_0^{-1}) \langle B \rangle \pi + \langle A \rangle (A - t_0^{-1} T_1) \langle B \rangle \pi$$

$$+ \langle A \rangle (C - t_0^{-1} T_2) \langle B \rangle \pi + \langle A \rangle (B - t_0^{-1} T_3) \langle B \rangle \pi$$

$\square$

$\ast$ follows

Notation

Abel group $\mathbb{Z}_2 = \{1, g\}$ $g^2 = 1$

Group $\mathbb{F}$-algebra

$$\mathbb{F}\mathbb{Z}_2 = \mathbb{F}1 + \mathbb{F}g$$

has basis $1, g$

λ = indet

$$\mathbb{F}\mathbb{Z}_2 \overset{alg}{\underset{iso}{}} \frac{\mathbb{F}[\lambda]}{(\lambda^2-1)} \longleftarrow \text{ideal gen by } \lambda^2-1$$

Let $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ denote mut com indets

obs

$$\frac{\mathbb{F}[\lambda_0, \lambda_1, \lambda_2, \lambda_3]}{(\lambda_0^2-1)} \overset{alg}{\underset{iso}{}} \mathbb{F}\mathbb{Z}_2 \otimes \mathbb{F}[\lambda_1, \lambda_2, \lambda_3]$$

$\mathbb{F}\mathbb{Z}_2 \otimes \mathbb{F}[\lambda_1, \lambda_2, \lambda_3]$ has basis

$g^i \otimes \lambda_1^r \lambda_2^s \lambda_3^t$ $\quad i \in \{0,1\}, \quad r,s,t \in \mathbb{N}$

LEM 133 $\exists$ unique $\mathbb{F}$-alg hom

$$\hat{H}_q^+ \longrightarrow \mathbb{F}\mathbb{Z}_2 \otimes \mathbb{F}[\lambda_1, \lambda_2, \lambda_3]$$

that sends

$$A \to z \otimes \lambda_1$$

$$B \to z \otimes \lambda_3$$

$$C \to z \otimes \lambda_2$$

$$t_0^{\pm 1} \to z \otimes 1$$

$$T_1 \to 1 \otimes \lambda_1$$

$$T_2 \to 1 \otimes \lambda_2$$

$$T_3 \to 1 \otimes \lambda_3$$

this hom is surjective

pf Check the map respects the def rels for $\hat{H}_q^+$ from
Th 130

$$z \otimes \lambda_1 + \frac{q \; z\otimes\lambda_3 \; z\otimes\lambda_2 - q^{-1} \; z\otimes\lambda_2 \; z\otimes\lambda_3}{q^2 - q^{-2}} \stackrel{?}{=}$$

$$\frac{(q^{-1} z\otimes 1 + q \; z\otimes 1) \, 1\otimes\lambda_1 + 1\otimes\lambda_2\lambda_3}{q+q^{-1}}$$

$$LHS = z\otimes\lambda_1 + \frac{1\otimes\lambda_2\lambda_3}{q+q^{-1}} = RHS \quad \checkmark$$

other AW rels sim.

Check Casimir rel:

$$q\ z\otimes\lambda_1\lambda_2\lambda_3 \quad + q^2\, 1\otimes\lambda_1^2 \quad + q^{-2}\, 1\otimes\lambda_3^2 \quad + q^2\, 1\otimes\lambda_2^2$$

$$- q\ z\otimes\lambda_1 \left((q+q^{-1})\, z\otimes\lambda_1 + 1\otimes\lambda_2\lambda_3 \right)$$

$$- q^{-1}\ z\otimes\lambda_3 \left((q+q^{-1})\, z\otimes\lambda_3 + 1\otimes\lambda_1\lambda_2 \right)$$

$$- q\ z\otimes\lambda_2 \left((q+q^{-1})\, z\otimes\lambda_2 + 1\otimes\lambda_1\lambda_3 \right)$$

$$\overset{?}{=} (q+q^{-1})^2 - \left(q^{-1}\, z\otimes 1 + q\, z\otimes 1 \right)^2 - 1\otimes\lambda_1^2 - 1\otimes\lambda_2^2$$

$$-1\otimes\lambda_3^2$$

$$- \left(q^{-1}\, z\otimes 1 + q\, z\otimes 1 \right) 1\otimes\lambda_1\lambda_2\lambda_3$$

LHS − RHS

term	coef				
$z\otimes\lambda_1\lambda_2\lambda_3$	q	$-q$	$-q^{-1}-q$	$+q+q^{-1}$	$=0$
$1\otimes\lambda_1^2$	q^2	$-q(q+q^{-1})$	$+1$		$=0$
$1\otimes\lambda_2^2$	q^2	$-q(q+q^{-1})$	$+1$		$=0$
$1\otimes\lambda_3^2$	q^{-2}	$-q^{-1}(q+q^{-1})$	$+1$		$=0$
$1\otimes 1$	$-q^2-2-q^{-2}$	$+q^2+2+q^{-2}$			$=0$ ✓

Hom exists ✓

unique ✓

surj ✓

□

LEM 134 For the hom in L133

the kernel is J
$\overset{\shortparallel}{K}$

Moreover

$$\hat{H}_q^+ / J \overset{\text{alg}}{\underset{\text{iso}}{\cong}} \mathbb{F}\mathbb{Z}_2 \otimes \mathbb{F}[\lambda_1, \lambda_2, \lambda_3]$$

pf one checks $J \subseteq \ker$
show $\ker \subseteq J$: Consider canon algebra hom

$$\hat{H}_q^+ \longrightarrow \hat{H}^+/J$$

$$h \longrightarrow h + J \quad (= \bar{h})$$

$t_0 - t_0^{-1} \in J$ so

$$\bar{t}_0^{\,2} = 1$$

$A - t_0^{-1} T_1 \in J$ so

$$\bar{A} = \bar{t}_0 \bar{T}_1$$

$\rightarrow$

$$\bar{t}_0 \bar{A} = \bar{A} \bar{t}_0$$
$$\bar{C} = \bar{t}_0 \bar{T}_2$$

Sim $\bar{B} = \bar{t}_0 \bar{T}_3$

$$\bar{t}_0 \bar{B} = \bar{B} \bar{t}_0$$
$$\bar{t}_0 \bar{C} = \bar{C} \bar{t}_0$$

$\bar{t}_0$ central in $\hat{H}_q^+/J$

Now $\bar{A} = \bar{t}_0 \bar{T}_1$ is central in $\hat{H}_q^+/J$

Sim $\bar{B}, \bar{C}$ " "

$\hat{H}_q^+/J$ commutative, gen by $\bar{t}_0, \bar{T}_1, \bar{T}_2, \bar{T}_3$

and $\bar{t}_0^{\,2} = 1$

So $\exists$ alg hom

$\varepsilon \quad \mathbb{F}\mathbb{Z}_2 \otimes \mathbb{F}[\lambda_1, \lambda_2, \lambda_3] \longrightarrow \hat{H}_q^+/J$

that sends

$$\eta \otimes 1 \longrightarrow \overline{E_0}$$
$$1 \otimes \lambda_1 \longrightarrow \overline{T_1}$$
$$1 \otimes \lambda_2 \longmapsto \overline{T_2}$$
$$1 \otimes \lambda_3 \longmapsto \overline{T_3}$$

the diag commutes:

$$\begin{tikzcd} \hat{H}_q^+ \arrow[r] \arrow[dr, "\text{canon}"'] & \mathbb{F}\mathbb{Z}_2 \otimes \mathbb{F}[\lambda_1, \lambda_2, \lambda_3] \arrow[d, "\varepsilon"] \\ & \hat{H}_q^+ / J \end{tikzcd}$$

$\forall h \in K$

$$\begin{tikzcd} h \arrow[r, mapsto] \arrow[dr, dashed, mapsto] & 0 \arrow[d, mapsto] \\ & 0 \end{tikzcd}$$

so $h \in$ ker of canon map $= J$

so $K \subseteq J$

so $K = J$ $\square$

COR 135 The following is a basis for a complement of J in $\hat{H}_q^+$:

$$t_0^i \, T_1^r \, T_2^s \, T_3^t \qquad i \in \{0,1\}, \quad r,s,t \in \mathbb{N}$$

pf By L134 $\square$

$\mathbb{F}$ arb

$0 \neq q \in \mathbb{F} \quad q^4 \neq 1$

$$\hat{H}_q^+ = \{ h \in \hat{H}_q \mid t_0 h = h t_0 \}$$
$$\hat{H}_q^- = \{ h \in \hat{H}_q \mid t_0 h = h t_0^{-1} \}$$

Recall

$$\hat{H}_q = \langle A \rangle \langle B \rangle \Pi + \langle A \rangle t_1 \langle B \rangle \Pi + \langle A \rangle t_2 \langle B \rangle \Pi + \langle A \rangle t_3 \langle B \rangle \Pi$$

Under map $\quad h \longrightarrow t_0 h - h t_0$

Image of $\hat{H}_q$ is

$$\langle A \rangle (t_0 t_1 - t_1 t_0) \langle B \rangle \Pi + \langle A \rangle (t_0 t_2 - t_2 t_0) \langle B \rangle \Pi + \langle A \rangle (t_0 t_3 - t_3 t_0) \langle B \rangle \Pi$$

this image $\subseteq \hat{H}_q^-$

Investigate

$$t_0 t_i - t_i t_0 \qquad i = 1, 2, 3$$

LEM 136

(i) $\quad (t_0 t_1 - t_1 t_0)^2 = A^2 - A T_0 T_1 + T_0^2 + T_1^2 - 4$

(ii) $\quad (t_0 t_2 - t_2 t_0)^2 = C^2 - C T_0 T_2 + T_0^2 + T_2^2 - 4$

(iii) $\quad (t_0 t_3 - t_3 t_0)^2 = B^2 - B T_0 T_3 + T_0^2 + T_3^2 - 4$

pf $\quad$ By L 71 which is about algebra S $\quad \square$

Note The generator ρ of B_3 sends

$$t_0t_3 - t_3t_0 \longrightarrow t_0t_2 - t_2t_0 \longrightarrow t_0t_1 - t_1t_0 \longrightarrow$$

$$t_0^{-1}(t_0t_3 - t_3t_0)t_0 = (t_0t_3 - t_3t_0)t_0^{2} = t_0^{-2}(t_0t_3 - t_3t_0)$$

Find products

$$(t_0t_i - t_it_0)(t_0t_j - t_jt_0) \qquad 1 \le i,j \le 3 \quad i \ne j$$

Recall $\hat{H}_q^-$ $\hat{H}_q^-$ $\subseteq \hat{H}_q^+ = \langle A, B, C, t_0^{\pm 1}, T_1, T_2, T_3 \rangle$

so above products should be polynomials in $A, B, C \ldots$

Prop 137 Abbrev

$A^+ = A - t_0 T_1$ $\quad A^- = A - t_0^{-1} T_1$

$B^+ = B - t_0 T_3$ $\quad B^- = B - t_0^{-1} T_3$

$C^+ = C - t_0 T_2$ $\quad C^- = C - t_0^{-1} T_2$

Then

	$t_0t_1 - t_1t_0$	$t_0t_2 - t_2t_0$	$t_0t_3 - t_3t_0$
$t_0t_1 - t_1t_0$		$A^-C^- + q^{-1}t_0^{-1}(t_0 - t_0^{-1})B^+$	$A^+B^+t_0^{-2} - qt_0^{-1}(t_0 - t_0^{-1})C^-$
$t_0t_2 - t_2t_0$	$C^+A^+ - qt_0(t_0 - t_0^{-1})B^-$		$C^-B^- + q^{-1}t_0^{-1}(t_0 - t_0^{-1})A^+$
$t_0t_3 - t_3t_0$	$B^-A^-t_0^{2} + q^{-1}t_0(t_0 - t_0^{-1})C^+$	$B^+C^+ - qt_0(t_0 - t_0^{-1})A^-$	

pf Recall $A - t_0^{-1}T_1 = t_0t_1 - t_1t_0^{-1}$ etc.

$A - t_0T_1 = -(t_0t_1^{-1} - t_1^{-1}t_0^{-1})$ etc

$(t_0t_1 - t_1t_0)(t_0t_2 - t_2t_0)$

$\overset{?}{=} (t_0t_1 - t_1t_0^{-1})(t_0t_2 - t_2t_0^{-1}) + q^{-1}t_0^{-1}(t_0 - t_0^{-1})(t_0t_3^{-1} - t_3^{-1}t_0^{-1})$

$\cancel{t_0t_1t_0t_2} - \underbrace{t_1t_0^2t_2}_{T_0t_0 - 1} - \underbrace{t_0t_1t_2}_{q^{-1}t_3^{-1}}t_0 + t_1t_0t_2t_0$

$t_1t_2 - T_0t_1t_0t_2$

$q^{-1}t_0^{-1}t_3^{-1}$

$-t_1t_0t_2t_0^{-1}$

$\overset{?}{=}$

$\cancel{t_0t_1t_0t_2} - \underbrace{t_1t_2}_{q^{-1}t_0^{-1}t_3^{-1}} - \underbrace{t_0t_1t_2}_{q^{-1}t_3^{-1}}t_0^{-1} + t_1t_0^{-1}t_2t_0^{-1}$

$t_1T_0t_2t_0^{-1} - t_1t_0^{-1}t_2t_0^{-1}$

$-q^{-1}(1 - t_0^{-2})(t_0t_3^{-1} - t_3^{-1}t_0^{-1})$

$-q^{-1}t_0t_3^{-1} + q^{-1}t_3^{-1}t_0^{-1} + q^{-1}t_0^{-1}t_3^{-1} - q^{-1}t_0^{-2}t_3^{-1}t_0^{-1}$

$-q^{-1}t_0^{-1}t_3^{-1}$

$t_0^{-1}T_0 - 1$

$-q^{-1}t_3^{-1}T_0$

$-q^{-1}t_0^{-1}t_3^{-1}t_0^{-1}T_0$

$+q^{-1}t_3^{-1}t_0^{-1}$

$-t_1t_2t_0^{-1}T_0$

$q^{-1}t_3^{-1}t_0$

$q^{-1}t_3^{-1}T_0$

other cases sim.

$\square$

Recall $\hat{H}_q^+$ has basis

$$A^i B^j t_0^k T_1^r T_2^s T_3^t \qquad A^i C B^j t_0^k T_1^r T_2^s T_3^t$$

$$k \in \mathbb{Z}, \quad i, j, r, s, t \in \mathbb{N}$$

By Prop 137 3 out of 6 products are expr in this basis

We do now the other 3

LEM 138

$$(t_0 t_3 - t_3 t_0)(t_0 t_1 - t_1 t_0) \qquad *$$

=

term	coef
AB	$q^2 t_0^2$
A	$-t_0 T_3$
B	$-t_0 T_1$
C	$q t_0 (q^2 t_0 - q^{-2} t_0^{-1})$
1	$T_1 T_3 - q(q-q^{-1}) t_0^2 \gamma - q^{-1}(t_0 - t_0^{-1}) t_0^2 T_2$

pf In the formula for $*$ in Prop 137

eval using

$$\frac{q AB - q^{-1} BA}{q^2 - q^{-2}} + C = \frac{\gamma}{q+q^{-1}}$$

LEM 139

$$(t_0 t_3 - t_3 t_0)(t_0 t_2 - t_2 t_0)$$

$=$

term	coef
CB	q^{-2}
B	$-t_0 T_2$
C	$-t_0 T_3$
A	$q^{-1} t_0 (q^{-2} t_0^{-1} - q^{2} t_0)$
1	$t_0^{2} T_2 T_3 + q^{-1}(q - q^{-1})\alpha + q(t_0 - t_0^{-1}) T_1$

pf sim to L138 $\square$

$$(t_0 t_2 - t_2 t_0)(t_0 t_1 - t_1 t_0)$$

=

term	coef
AC	q^{-2}
C	$-t_0 T_1$
A	$-t_0 T_2$
B	$q^{-1} t_0 (q^{-2} t_0^{-1} - q^{2} t_0)$
1	$t_0^2 T_1 T_2 + q^{-1}(q - q^{-1})\beta + q(t_0 - t_0^{-1}) T_3$

pf sim to L138 $\square$

LEM 141 the gen ρ of B_3 sends

$$\begin{tikzcd}[column sep=small]
& (t_0t_2 - t_2t_0)(t_0t_3 - t_3t_0) \arrow[dl] & \\
(t_0t_1 - t_1t_0)(t_0t_2 - t_2t_0) \arrow[rr] & & (t_0t_3 - t_3t_0)(t_0t_1 - t_1t_0)\, t_0^{-2} \arrow[ul]
\end{tikzcd}$$

and also sends

$$\begin{tikzcd}[column sep=small]
& (t_0t_3 - t_3t_0)(t_0t_2 - t_2t_0) \arrow[dl] & \\
(t_0t_2 - t_2t_0)(t_0t_1 - t_1t_0) \arrow[rr] & & (t_0t_1 - t_1t_0)(t_0t_3 - t_3t_0)\, t_0^{2} \arrow[ul]
\end{tikzcd}$$

pf use note above L137 $\square$

We now consider how do

$$t_0 t_i - t_i t_0 \qquad i=1,2,3$$

"commute" with A, B, C

We 1st focus on A vs $\{t_0 t_i - t_i t_0\}_{i=1}^{3}$

then using ρ we bring in B, C.

Prop 14.2.

(i) $$A(t_0 t_1 - t_1 t_0) = (t_0 t_1 - t_1 t_0) A$$

(ii) $$q^{-1} A(t_0 t_2 - t_2 t_0) - q(t_0 t_2 - t_2 t_0) A$$
$$= (t_0 t_1 - t_1 t_0) t_0^{-1} T_2 (q - q^{-1}) + (t_0 t_3 - t_3 t_0)(q^2 - q^{-2})$$

(iii) $$q A(t_0 t_3 - t_3 t_0) - q^{-1}(t_0 t_3 - t_3 t_0) A$$
$$= -(t_0 t_1 - t_1 t_0) t_0^{-1} T_3 (q - q^{-1}) - (t_0 t_2 - t_2 t_0)(q^2 - q^{-2})$$

pf (i) Recall $A = t_0 t_1 + (t_0 t_1)^{-1}$ commutes with t_0, t_1.

(iii) Use red rules to verify

$$t_2 + \frac{q A t_3 - q^{-1} t_3 A}{q^2 - q^{-2}} = \frac{t_1^{-1} t_0^{-1} T_3 + q^{-1} T_2}{q + q^{-1}}$$

Now take commutator of each side with t_0 and simplify.

(ii)

$\mathbb{F}$ arb

$0 \neq q \in \mathbb{F} \quad q^4 \neq 1$

$$\hat{H}_q^+ = \{ h \in \hat{H}_q \mid t_0 h = h t_0 \}$$
$$\hat{H}_q^- = \{ h \in \hat{H}_q \mid t_0 h = h t_0^{-1} \}$$

General goal: describe $\hat{H}_q^-$

Motivation Recall our decomp

$$\hat{H}_q = \langle A \rangle \langle B \rangle \mathbb{T} + \langle A \rangle t_1 \langle B \rangle \mathbb{T} + \langle A \rangle t_2 \langle B \rangle \mathbb{T} + \langle A \rangle t_3 \langle B \rangle \mathbb{T} \qquad \text{(ds)}$$

For $i = 1, 2, 3$ describe

$t_i A, \quad B t_i$

From the pt of view of this decomp.

For instance $B t_1$. Write

$$\begin{aligned} B t_1 = & \sum_{i,j \in \mathbb{N}} A^i B^j t_{ij} \\ & + \sum_{i,j \in \mathbb{N}} A^i t_1 B^j t'_{ij} \\ & + \sum_{i,j \in \mathbb{N}} A^i t_2 B^j t''_{ij} \\ & + \sum_{i,j \in \mathbb{N}} A^i t_3 B^j t'''_{ij} \end{aligned}$$

$t_{ij}, t'_{ij}, t''_{ij}, t'''_{ij} \in \mathbb{T} \qquad \forall i, j \in \mathbb{N}$

Find these

We give these for $t_i A, \; B t_i$ shortly

Recall

$$A t_1 = t_1 A \qquad B t_3 = t_3 B \qquad C t_2 = t_2 C$$

LEM 143

(i) $$t_2 + \frac{q A t_3 - q^{-1} t_3 A}{q^2 - q^{-2}} = \frac{t_1^{-1} t_0^{-1} T_3 + q^{-1} T_2}{q + q^{-1}}$$

(ii) $$t_1 + \frac{q B t_2 - q^{-1} t_2 B}{q^2 - q^{-2}} = \frac{t_0^{-1} t_3^{-1} T_2 + q^{-1} T_1}{q + q^{-1}}$$

(iii) $$t_0^{-1} t_3 t_0 + \frac{q C t_1 - q^{-1} t_1 C}{q^2 - q^{-2}} = \frac{t_0^{-1} t_2^{-1} T_1 + q^{-1} T_3}{q + q^{-1}}$$

pf (i) strategy: eval $t_3 A$ using red rules

$$t_3 = X t_0^{-1}$$

$$t_3 A = X(Y + Y^{-1})\, t_0^{-1} \qquad A t_3 = (Y + Y^{-1}) X t_0^{-1}$$

$$t_2 = q^{-1} A T_3 - q^{-1} A X t_0^{-1} - q^{-1} Y T_3 + q^{-1} Y X t_0^{-1}$$

Recall by Prop 30

$$C_0 = q\,(q Y X - q^{-1} X Y)$$

$$C_3 = -(q^{-1} Y^{-1} X - q X Y^{-1})$$

	$T_2 t_0$	$T_3 t_1$	$T_0 t_2$	$T_1 t_3$	$T_0 T_2$	$T_1 T_3$
C_0	q	1	q^{-1}	1	$-q^{-1}$	-1
C_3	1	q^{-1}	1	q	-1	$-q^{-1}$

Elim XY, XY^{-1} in $t_3 A$ and simplify the result.

(ii), (iii) Apply ρ twice to (i) and recall ρ sends

$A \to B \to C \to A$ $\qquad$ $t_3 \to t_2 \to t_1 \to t_0^{-1} t_3 t_0$

$\square$

LEM 144 We have

(i) $$t_3 + \frac{q t_2 A - q^{-1} A t_2}{q^2 - q^{-2}} = \frac{t_1^{-1} t_0^{-1} T_2 + q^{-1} T_3}{q+q^{-1}}$$

(ii) $$t_2 + \frac{q t_1 B - q^{-1} B t_1}{q^2 - q^{-2}} = \frac{t_0^{-1} t_3^{-1} T_1 + q^{-1} T_2}{q+q^{-1}}$$

(iii) $$t_0 t_1 t_0^{-1} + \frac{q t_3 C - q^{-1} C t_3}{q^2 - q^{-2}} = \frac{t_2^{-1} t_0^{-1} T_3 + q^{-1} T_1}{q+q^{-1}}$$

pf (ii) strategy eval $B t_1$ using red rules

$t_1 = t_0^{-1} Y$ $\qquad B = X + X^{-1}$

$$\begin{aligned} B t_1 &= (X + X^{-1}) t_0^{-1} Y \\ &= t_0^{-1} (X + X^{-1}) Y \\ t_1 B &= t_0^{-1} Y (X + X^{-1}) \end{aligned}$$

Elim XY, $X^{-1}Y$ using red rules in a fashion sim to L143

(i), (iii) Apply ρ twice to (ii) □

We now show L143, L144 look
in the basis for $\hat{H}_q$ from Prop 114

LEM 145

(i) $$t_2 + \frac{qAt_3 - q^{-1}t_3A}{q^2-q^{-2}} = \frac{t_0^{-1}T_1T_3 - t_1t_0^{-1}T_3 + q^{-1}T_2}{q+q^{-1}}$$

(ii) $$t_1 + \frac{qBt_2 - q^{-1}t_2B}{q^2-q^{-2}} = \frac{BT_2 - t_3t_0T_2 + q^{-1}T_1}{q+q^{-1}}$$

(iii) $$T_3 - Bt_0 + t_3t_0^2 + \frac{qCt_1 - q^{-1}t_1C}{q^2-q^{-2}}$$
$$= \frac{CT_1 - t_2t_0T_1 + q^{-1}T_3}{q+q^{-1}}$$

pf ~~Use~~ L143 $\square$

LEM 146

(i) $$t_3 + \frac{q t_2 A - q^{-1} A t_2}{q^2 - q^{-2}} = \frac{t_0^{-1} T_1 T_2 - t_1 t_0^{-1} T_2 + q^{-1} T_3}{q + q^{-1}}$$

(ii) $$t_2 + \frac{q t_1 B - q^{-1} B t_1}{q^2 - q^{-2}} = \frac{B T_1 - t_3 t_0 T_1 + q^{-1} T_2}{q + q^{-1}}$$

(iii) $$t_1 t_0^{-2} + (A - t_0^{-1} T_1) t_0^{-1} + \frac{q t_3 C - q^{-1} C t_3}{q^2 - q^{-2}}$$

$$= \frac{t_0^{-1} T_2 T_3 + q^{-1} T_1 - t_2 t_0^{-1} T_3}{q + q^{-1}}$$

pf use L 144. $\square$

Recall our other decomp of $\hat{H}_q$:

$$\hat{H}_q = \langle A \rangle \langle B \rangle \Pi + \langle A \rangle X \langle B \rangle \Pi + \langle A \rangle Y \langle B \rangle \Pi + \langle A \rangle YX \langle B \rangle \Pi \qquad (45)$$

For $u \in \{X, Y, YX\}$

describe uA, Bu

from pt of view of above decomp.

[For completeness we give more gen rels]

LEM 147 $AY = YA$. Moreover

$$\frac{qXA - q^{-1}AX}{q^2 - q^{-2}} = qYX + \frac{t_1^{-1}T_3 - qt_0T_2}{q+q^{-1}}$$

$$\frac{qX^{-1}A - q^{-1}AX^{-1}}{q^2 - q^{-2}} = q^{-1}X^{-1}Y^{-1} + \frac{t_1T_3 - q^{-1}t_0^{-1}T_2}{q+q^{-1}}$$

$$\frac{qAX^{-1} - q^{-1}X^{-1}A}{q^2 - q^{-2}} = qX^{-1}Y + \frac{qt_0^{-1}T_2 - q^2t_1T_3}{q+q^{-1}}$$

$$\frac{qAX - q^{-1}XA}{q^2 - q^{-2}} = q^{-1}Y^{-1}X + \frac{q^{-1}t_0T_2 - q^{-2}t_1^{-1}T_3}{q+q^{-1}}$$

pf In each case write $A = Y + Y^{-1}$. elim all occurrences of $X^{\pm 1}Y^{\pm 1}$ using Prop 30 $\square$

LEM 148 $\quad BX = XB \quad$ Moreover

$$\frac{qBY - q^{-1}YB}{q^2 - q^{-2}} = qYX + \frac{t_3^{-1}T_1 - q t_0 T_2}{q + q^{-1}}$$

$$\frac{qBY^{-1} - q^{-1}Y^{-1}B}{q^2 - q^{-2}} = q^{-1}X^{-1}Y^{-1} + \frac{t_3 T_1 - q^{-1} t_0^{-1} T_2}{q + q^{-1}}$$

$$\frac{qY^{-1}B - q^{-1}BY^{-1}}{q^2 - q^{-2}} = qXY^{-1} + \frac{q t_0^{-1} T_2 - q^{2} t_3 T_1}{q + q^{-1}}$$

$$\frac{qYB - q^{-1}BY}{q^2 - q^{-2}} = q^{-1}YX^{-1} + \frac{q^{-1} t_0 T_2 - q^{-2} t_3^{-1} T_1}{q + q^{-1}}$$

pf $\quad$ Write $B = X + X^{-1}$ $\quad$ Elim all occurances of $X^{\pm 1} Y^{\pm 1}$ using Prop 30 $\quad \square$

LEM 149

(i) $$Y + \frac{qXC - q^{-1}CX}{q^2 - q^{-2}} = \frac{q^{-1}t_0T_1 + t_2^{-1}T_3}{q+q^{-1}}$$

(ii) $$Y^{-1} + \frac{qX^{-1}C - q^{-1}CX^{-1}}{q^2 - q^{-2}} = \frac{qt_0^{-1}T_1 + t_2T_3}{q+q^{-1}}$$

(iii) $$X + \frac{qCY - q^{-1}YC}{q^2 - q^{-2}} = \frac{q^{-1}t_0T_3 + t_2^{-1}T_1}{q+q^{-1}}$$

(iv) $$X^{-1} + \frac{qCY^{-1} - q^{-1}Y^{-1}C}{q^2 - q^{-2}} = \frac{qt_0^{-1}T_3 + t_2T_1}{q+q^{-1}}$$

pf (i), (ii) Apply ρ to Last 2 eqs of L148

recall ρ sends $B \to C$

$Y \to X$ $\quad X \to q^{-1}Y^{-1}t_0X^{-1}t_0$

Simplify using red rules.

(iii), (iv) apply antiaut $\dagger$. Fixes t_0, t_2

swaps t_1, t_3. Swaps X, Y