

$\mathbb{F}$  arb

$0 \neq 1 \in \mathbb{F}$

univ DAHA  $\hat{H}_q$  type  $(C_1, C_1)$

related algebra  $S$  has gens  $\{A_i^{\pm 1}\}_{i=0}^1$  and rels

$$A_i A_i^{-1} = A_i^{-1} A_i = 1 \quad i=0,1$$

$$\underbrace{A_i + A_i^{-1}}_{S_i} \text{ central}$$

let

$$R = A_0 A_1$$

$$\begin{aligned} Q &= A_0 A_1 A_0^{-1} - A_1 \\ &= R A_0^{-1} + R^{-1} A_0 - S_1 \end{aligned}$$

Some variations on  $G$

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LEM 70

$X$	$X^2$
$\Delta_0 \Delta_1 \Delta_0^{-1} - \Delta_1$	$G$
$\Delta_0^{-1} \Delta_1 \Delta_0 - \Delta_1$	$-\Delta_0^{-1} G \Delta_0$
$\Delta_0 \Delta_1^{-1} \Delta_0^{-1} - \Delta_1^{-1}$	$-G$
$\Delta_0^{-1} \Delta_1^{-1} \Delta_0 - \Delta_1^{-1}$	$\Delta_0^{-1} G \Delta_0$
$\Delta_1 \Delta_0 \Delta_1^{-1} - \Delta_0$	$-\Delta_0^{-1} G \Delta_1^{-1}$
$\Delta_1^{-1} \Delta_0 \Delta_1 - \Delta_0$	$\Delta_1^{-1} G \Delta_0$
$\Delta_1 \Delta_0^{-1} \Delta_1^{-1} - \Delta_0^{-1}$	$\Delta_0^{-1} G \Delta_1^{-1}$
$\Delta_1^{-1} \Delta_0^{-1} \Delta_1 - \Delta_0^{-1}$	$-\Delta_1^{-1} G \Delta_0$

In each case

$$X^2 = G^2$$

LEM 71

$$(i) \quad RG = GR^T, \quad R^T G = GR$$

$$(ii) \quad G^2 = (R+R^T)^2 - (R+R^T)S_0 S_1 + S_0^2 + S_1^2 - 4$$

pf (i) follows from

$$A_0 A_1 + (A_0 A_1)^T = A_1 A_0 + (A_1 A_0)^T$$

(ii) Use row rules

$$A_0 R = R^T A_0 + R S_0 - S_1$$

$$A_0 R^T = R A_0 - R S_0 + S_1$$

to write  $G^2$  in the basis for  $S$  from 1A60

Given mut. commuting ideals  $\lambda_0, \lambda_1, \varphi$

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Consider  $\mathbb{F}$ -algebra

$$M = \text{Mat}_2(\mathbb{F}) \otimes \mathbb{F}[\lambda_0^{\pm 1}, \lambda_1^{\pm 1}, \varphi]$$

"  $2 \times 2$  matrices with entries that are polynomials in  $\lambda_0^{\pm 1}, \lambda_1^{\pm 1}, \varphi$  "

•  $\exists$  injective  $\mathbb{F}$ -alg hom

$$S \rightarrow M$$

that sends

$$\lambda_0 \rightarrow \begin{pmatrix} \lambda_0 & 0 \\ 1 & \lambda_0^{-1} \end{pmatrix}$$

$$\lambda_1 \rightarrow \begin{pmatrix} \lambda_1 & \varphi \\ 0 & \lambda_1^{-1} \end{pmatrix}$$

Recall  $R = \lambda_0 \lambda_1$

$$G = \lambda_0 \lambda_1 \lambda_0^{-1} - \lambda_1$$

element of  $S$

image in  $M$

$R$

$$\begin{pmatrix} \lambda_0 \lambda_1 & \lambda_0 \varphi \\ \lambda_1 & \varphi + \lambda_0^{-1} \lambda_1^{-1} \end{pmatrix}$$

$R + R^{-1}$

$$(\lambda_0 \lambda_1 + \lambda_0^{-1} \lambda_1^{-1} + \varphi) I$$

$G$

$$\begin{pmatrix} -\varphi \lambda_0 & \varphi \lambda_0 (\lambda_0 - \lambda_0^{-1}) \\ \lambda_1 \lambda_0^{-1} - \lambda_1^{-1} \lambda_0^{-1} - \varphi & \varphi \lambda_0 \end{pmatrix}$$

$G^2$

$$(\varphi^2 + \varphi (\lambda_0 - \lambda_0^{-1}) (\lambda_1 - \lambda_1^{-1})) I$$

$S_0$

$$(\lambda_0 + \lambda_0^{-1}) I$$

$S_1$

$$(\lambda_1 + \lambda_1^{-1}) I$$

Problem 73

show that

(i) The center  $Z(S)$  has basis

$$(R+R^{-1})^i S_0^j S_1^k \quad i, j, k \in \mathbb{N}$$

(ii)  $Z(S)$  is gen by

$$R+R^{-1}, S_0, S_1$$

(iii)  $Z(S)$  is iso to the poly algebra over  $F$   
in 3 commuting variables

Problem 74. Assume  $F$  is alg closed.

Find all the f.d. mod  $S$ -modules up to iso. [ they all have dim 1 or 2 ]

We will return to Prob 74 when we consider the mod  $\hat{H}_g$ -modules.

Recall Univ AW algebra  $\Delta_q$  from Def 35

Below Def 35 mentioned hom  $\Delta_q \rightarrow \hat{H}_q$ .

Next goal: show this is an injection.

Until further notice

$$q^4 \neq 1$$

Recall  $\Delta_q$  is def by gens  $A, B, C$  and rels:

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} \quad \left( = \frac{\alpha}{q + q^{-1}} \right) \quad (1)$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} \quad \left( = \frac{\beta}{q + q^{-1}} \right) \quad (2)$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} \quad \left( = \frac{\gamma}{q + q^{-1}} \right) \quad (3)$$

is central.

Obs  $\exists$  aut of  $\Delta_q$  that sends

$$A \rightarrow B \rightarrow C \rightarrow A$$

There are more auts.

Going to show  $B_3$  acts on  $\Delta_q$

To see this we need another pres of  $\Delta_q$



LEM 75 The  $\mathbb{F}$ -algebra  $\Delta_q$  is gen by

$A, B, \gamma$ . Moreover

$$(i) \quad C = \frac{\gamma}{q+q^{-1}} - \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

$$(ii) \quad \alpha = \frac{B^2A - (q^2 + q^{-2})BAB + AB^2 + (q^2 - q^{-2})^2 A + (q - q^{-1})^2 B\gamma}{(q - q^{-1})(q^2 - q^{-2})}$$

$$(iii) \quad \beta = \frac{A^2B - (q^2 + q^{-2})ABA + BA^2 + (q^2 - q^{-2})^2 B + (q - q^{-1})^2 A\gamma}{(q - q^{-1})(q^2 - q^{-2})}$$

pf (i) from def of  $\gamma$  (3)

(ii) in (2) elem C using (i)

(iii) in (3) -- C -- (i)

□

Recall notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$n = 0, 1, 2, \dots$

Prop 76 The  $\mathbb{F}$ -alg  $\Delta_q$  has a presentation

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by gens  $A, B, Y$  and rels

$$A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3 = -(q^2 - q^{-2})^2 (AB - BA)$$

$$B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = -(q^2 - q^{-2})^2 (BA - AB)$$

$$A^2B^2 - B^2A^2 + (q^2 + q^{-2})(BABA - ABAB) = -(q - q^{-1})^2 (AB - BA)Y$$

$$YA = AY,$$

$$YB = BY$$

pf Use L75 to express the orig def rels for  $\Delta_q$

in terms of  $A, B, Y$ .

□

Thm 77 The braided gp  $B_3$  acts on  $\Delta_2$  as  
 a group of automorphisms s.t  $\tau(h) = h \forall h \in \Delta_2$  "

and  $\rho, \sigma$  do the following

$u$	$A$	$B$	$C$	$\alpha$	$\beta$	$\gamma$
$\rho(u)$	$B$	$C$	$A$	$\beta$	$\gamma$	$\alpha$
$\sigma(u)$	$B$	$A$	$C + \frac{AB-BA}{q-q^{-1}}$	$\beta$	$\alpha$	$\gamma$

pf By constr  $\exists$  aut  $P$  of  $\Delta_2$  that sends

$$A \rightarrow B \rightarrow C \rightarrow A$$

So  $P^3 = 1$

obs  $P$  sends

$$\alpha \rightarrow \beta \rightarrow \gamma \rightarrow \alpha$$

By Prop 76  $\exists$  aut  $S$  of  $\Delta_2$  that sends

$$A \rightarrow B, \quad B \rightarrow A, \quad \gamma \rightarrow \gamma$$

obs  $S^2 = 1$

Using L75 one checks  $S$  sends

$$\alpha \rightarrow \beta, \quad \beta \rightarrow \alpha, \quad C \rightarrow C + \frac{AB-BA}{q-q^{-1}}$$

Result follows from Def 4  $B_3$ .

□

Note The group  $PSL_2(\mathbb{Z})$  has a pres by gens

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$\rho, \sigma$  and rels  $\rho^3=1, \sigma^2=1$ .

So  $\exists$  surj gp hom

$$B_3 \longrightarrow PSL_2(\mathbb{Z})$$

$$\rho \longrightarrow \rho$$

$$\sigma \longrightarrow \sigma$$

$$\tau \longrightarrow 1.$$

th 77 gives action of  $PSL_2(\mathbb{Z})$  on  $\Delta_g$

We need a certain central el of  $\Delta_q$  called

Casimir element  $\Omega$ .

LEM 78 The following elements of  $\Delta_q$  coincide

Name	Leading term	$A^2$	$B^2$	$C^2$	$A\alpha$	$B\beta$	$C\gamma$
$\Omega_B^+$	$qABC$	$q^2$	$q^{-2}$	$q^2$	$-q$	$-q^{-1}$	$-q$
$\Omega_C^+$	$qBCA$	$q^2$	$q^2$	$q^{-2}$	$-q$	$-q$	$-q^{-1}$
$\Omega_A^+$	$qCAB$	$q^{-2}$	$q^2$	$q^2$	$-q^{-1}$	$-q$	$-q$
$\Omega_B^-$	$q^{-1}CBA$	$q^{-2}$	$q^2$	$q^{-2}$	$-q^{-1}$	$-q$	$-q^{-1}$
$\Omega_C^-$	$q^{-1}ACB$	$q^2$	$q^{-2}$	$q^2$	$-q^{-1}$	$-q^{-1}$	$-q$
$\Omega_A^-$	$q^{-1}BAC$	$q^2$	$q^{-2}$	$q^{-2}$	$-q$	$-q^{-1}$	$-q^{-1}$

Call this common value  $\Omega$

pf

The ant  $\rho$  sends

$$\Omega_B^+ \rightarrow \Omega_C^+ \rightarrow \Omega_A^+$$

$$\Omega_B^- \rightarrow \Omega_C^- \rightarrow \Omega_A^-$$

$\Omega_B^+ - \Omega_C^-$  equals  $(q-q^{-1})A$  times

$$(q+q^{-1})A + \frac{qBC - q^{-1}CB}{q - q^{-1}} - \alpha$$

(\*) is zero so  $\Omega_B^+ = \Omega_C^-$

Apply  $p$  to get

$$\Omega_C^+ = \Omega_A^- \quad \Omega_A^+ = \Omega_B^-$$

$\Omega_B^+ - \Omega_A^-$  equals

$$(q+q^{-1})C + \frac{qAB - q^{-1}BA}{q - q^{-1}} - \gamma$$

times  $(q-q^{-1})C$ .

(\*\*) is zero by def so  $\Omega_B^+ = \Omega_A^-$

Apply  $p$  to get

$$\Omega_C^+ = \Omega_B^- \quad \Omega_A^+ = \Omega_C^-$$

$$\text{Now } \Omega_A^+ = \Omega_B^+ = \Omega_C^+ = \Omega_A^- = \Omega_B^- = \Omega_C^-$$

□

$\mathbb{F}$  arb

$$0 \neq q \in \mathbb{F}$$

$$q^4 \neq 1$$

Univ AW algebra  $\Delta_q$  has gens  $A, B, C$  and rels

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} \quad \text{central} \quad (+ CP)$$

general goal: get injection  $\Delta_q \rightarrow \hat{H}_q$

spec goal: study casimir element  $\Omega$  of  $\Delta_q$

Recall  $\Omega$  is common value of

$$\Omega_A^+ \quad \Omega_B^+ \quad \Omega_C^+ \quad \Omega_A^- \quad \Omega_B^- \quad \Omega_C^-$$

from L78. For instance

$$\Omega_B^+ = qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma$$

$\alpha, \beta, \gamma$  are central els of  $\Delta_q$  from above L75

LEM 79  $\Omega$  is fixed by everything in the braid group  $B_3$

pf Suf to show the  $B_3$  gens  $\rho, \sigma$  fix  $\Omega$

$\rho$  fixes  $\Omega$  since  $\rho$  sends  $\Omega_B^+ \rightarrow \Omega_C^+$  by const.

Show  $\sigma$  fixes  $\Omega$

Show  $\sigma$  sends  $\Omega_B^+ \rightarrow \Omega_A^+$

By 1h 77  $\sigma$  sends

$$A \leftrightarrow B \quad C \rightarrow \underbrace{C + \frac{AB-BA}{q-q^{-1}}}_{C'}$$

$$\alpha \leftrightarrow \beta \quad \gamma \rightarrow \gamma$$

So

$$\sigma(\Omega_B^+) = qBAC' + q^2B^2 + q^{-2}A^2 + q^2(C')^2 - qB\beta - q^2A\alpha - qC'\gamma$$

Also

$$\Omega_A^+ = qCAB + q^2C^2 + q^{-2}A^2 + q^2B^2 - qC\gamma - q^2A\alpha - qB\beta$$

obs

$$\sigma(\Omega_B^+) - \Omega_A^+ = (BA + q^{-1}C + qC' - \gamma) qC' - qC (AB + qC + q^{-1}C' - \gamma)$$

(173)



claim

$$0 = BA + q^2 C + q C' - Y$$

$$C + \frac{AB - BA}{q - q^2} \quad \frac{\delta}{q + q^2} \quad q^2 Y'$$

OK

$$\left( C + \frac{qAB - q^2 BA}{q^2 - q^{-2}} \right) q^2 Y'$$

claim

$$0 = AB + qC + q^{-1}C' - Y$$

Sim checked.

So

$$\sigma(\Omega_B^+) = \Omega_A^+$$

So

$$\sigma \text{ fixes } \Omega$$

□

thm 80  $\Omega$  is central in  $\Delta_q$

pf first show  $A\Omega = \Omega A$

show  $A\Omega_C^+ = \Omega_B^+ A$

Recall

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{\beta}{q + q^{-1}} \quad (\star)$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{\gamma}{q + q^{-1}} \quad (\star)$$

Consider

$$qC \star + q^{-1} \star C - \gamma \star$$

$$+ \beta \star - q^{-1} B \star - q \star B$$

Reduces to

$$A\Omega_C^+ - \Omega_B^+ A = 0$$

(ex).

So far  $A\Omega = \Omega A$

Now apply  $\rho$  twice yet

$$B\Omega = \Omega B,$$

$$C\Omega = \Omega C$$

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□

More symmetries of  $\Delta_q$

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LEM 81      Put  $\epsilon_A, \epsilon_B, \epsilon_C \in \{1, -1\}$

s.t.

$$\epsilon_A \epsilon_B \epsilon_C = 1$$

$\exists$  automorphism of  $\Delta_q$  that sends

" $\mathbb{Z}_2 \times \mathbb{Z}_2$  sym"

$$A \rightarrow \epsilon_A A$$

$$B \rightarrow \epsilon_B B$$

$$C \rightarrow \epsilon_C C$$

this aut sends

$$\alpha \rightarrow \epsilon_A \alpha,$$

$$\beta \rightarrow \epsilon_B \beta,$$

$$\gamma \rightarrow \epsilon_C \gamma$$

pf WLOG  $\epsilon_A = -1, \epsilon_B = -1, \epsilon_C = 1$

One checks  $A \rightarrow -A, B \rightarrow -B, C \rightarrow C$

respects the defining relations for  $\Delta_q$

□

LEM 82  $\Omega$  is fixed by each aut of  $\Delta_q$

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from LEM 81

pf view

$$\Omega = -\Omega_B^+$$

$$= qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma$$

Each term is fixed by the aut

□

LEM 83

 $\exists$  an anti-automorphism  $t$  of  $\Delta_q$ 

7

that sends

$$A \leftrightarrow B$$

$$C \rightarrow C$$

$$\alpha \leftrightarrow \beta$$

$$\gamma \rightarrow \gamma$$

Moreover  $t^2 = 1$ 

pf Define

$$A^+ = B$$

$$B^+ = A$$

$$C^+ = C$$

$$\alpha^+ = \beta$$

$$\beta^+ = \alpha$$

$$\gamma^+ = \gamma$$

show  $A^+, B^+, C^+, \alpha^+, \beta^+, \gamma^+$  sat the def rels for  $\Delta_q$ .in the algebra  $\Delta_q^{op}$ . Need

$$A^+ + \frac{qB^+C^+ - q^{-1}C^+B^+}{q^2 - q^{-2}} \stackrel{?}{=} \frac{\alpha^+}{q + q^{-1}} \quad \text{in } \Delta_q^{op}$$

$$B^+ + \frac{qC^+A^+ - q^{-1}A^+C^+}{q^2 - q^{-2}} \stackrel{?}{=} \frac{\beta^+}{q + q^{-1}} \quad \dots$$

$$C^+ + \frac{qA^+B^+ - q^{-1}B^+A^+}{q^2 - q^{-2}} \stackrel{?}{=} \frac{\gamma^+}{q + q^{-1}} \quad \dots$$

these reduce to orig def rels in  $\Delta_q$ So  $\exists$   $\mathbb{F}$ -alg hom  $t: \Delta_q \rightarrow \Delta_q$  that sends

$$A \rightarrow A^+$$

$$B \rightarrow B^+$$

$$C \rightarrow C^+$$

$$\alpha \rightarrow \alpha^+$$

$$\beta \rightarrow \beta^+$$

$$\gamma \rightarrow \gamma^+$$

By constr  $t^2 = 1$  so  $t$  is bijective.

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□

LEM 84

 $\Omega$  is fixed by the anti-invariant  $\dagger$  of  $\Delta_q$ 

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from L 83.

pf View

$$\Omega = \Omega_B^\dagger = qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma$$

Apply  $\dagger$ 

$$\Omega^\dagger = qCAB + q^2B^2 + q^{-2}A^2 + q^2C^2 - qB\beta - q^{-1}A\alpha - qC\gamma$$

$$= \Omega_A^\dagger$$

$$= \Omega$$

□

$\gamma: \Delta_q \rightarrow \Delta_{q^{-1}}$  that sends

$$A \rightarrow B$$

$$B \rightarrow A$$

$$C \rightarrow C$$

$$\alpha \rightarrow \beta$$

$$\beta \rightarrow \alpha$$

$$\gamma \rightarrow \gamma$$

pf

Notation: Write

$$\varphi = q^{\gamma}$$

Write

$$\hat{A}, \hat{B}, \hat{C}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}$$

generators of  $\Delta_{q^{\gamma}}$

Put

$$A^{\gamma} = \hat{B}$$

$$B^{\gamma} = \hat{A}$$

$$C^{\gamma} = \hat{C}$$

$$\alpha^{\gamma} = \hat{\beta}$$

$$\beta^{\gamma} = \hat{\alpha}$$

$$\gamma^{\gamma} = \hat{\gamma}$$

show

$$A^{\gamma} + \frac{q^{B^{\gamma}C^{\gamma}} - q^{-1}C^{\gamma}B^{\gamma}}{q^2 - q^{-2}} = \alpha^{\gamma} \quad \text{in } \Delta_{q^{\gamma}}$$

$$\hat{B} + \frac{q^{\gamma}\hat{A}\hat{C} - q\hat{C}\hat{A}}{q^{-2} - q^2} = \hat{\beta}$$

$$B^{\gamma} + \frac{qC^{\gamma}A^{\gamma} - q^{-1}A^{\gamma}C^{\gamma}}{q^2 - q^{-2}} = \beta^{\gamma} \quad \text{in } \Delta_{q^{\gamma}}$$

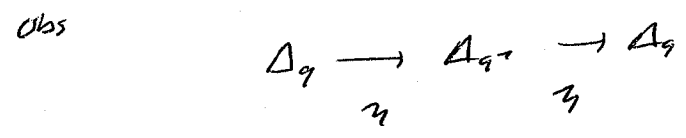
$$\hat{A} + \frac{q^{\gamma}\hat{C}\hat{B} - q\hat{B}\hat{C}}{q^{-2} - q^2} = \hat{\alpha}$$

$$C^{\eta} + \frac{qA^{\eta}B^{\eta} - q^{-1}B^{\eta}A^{\eta}}{q^{\eta} - q^{-\eta}} = \gamma^{\eta} \quad \text{in } \Delta_{q^{\eta}}$$

$$C^{\eta} + \frac{q^{\eta}BA - qAB}{q^{-\eta} - q^{\eta}} = \gamma^{\eta}$$

So  $\exists$   $\mathbb{F}$ -alg hom  $\gamma: \Delta_q \rightarrow \Delta_{q^{\eta}}$  that

leads  $A \rightarrow A^{\eta}$  etc



is ident so  $\gamma$  is bi.

□



LEM 86

The map  $\gamma: \Delta_q \rightarrow \Delta_{q^{-1}}$ 

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from L85 sends the Cas el of  $\Delta_q$  tothe Cas el of  $\Delta_{q^{-1}}$ 

pf

write  $\varphi = q^{-1}$ view  $\Omega = -\Omega_B^+$ 

$$qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma$$

 $\Rightarrow \downarrow$ 

$$\varphi^{-1}BAC + \varphi^{-2}B^2 + \varphi^2A^2 + \varphi^{-2}C^2 - \varphi^{-1}B\beta - \varphi A\alpha - \varphi^{-1}C\gamma$$

this is  $\Omega_A^-$  computed in  $\Delta_\varphi$ 

□

field  $\mathbb{F}$  arb

$0 \neq q \in \mathbb{F}$

$\hat{H}_q$  is  $\mathbb{F}$ -algebra with gens  $\{t_i^{\pm}\}_{i \in \mathbb{I}}$   $\mathbb{I} = \{0, 1, 2, 3\}$   
and rels

$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad i \in \mathbb{I}$

$t_i + t_i^{-1}$  central  $i \in \mathbb{I}$

$t_0 t_1 t_2 t_3 = q^{-1}$

$\hat{H}_q$  is universal DAHA of type  $(C_1^v, C_1)$

Next assume  $q^4 \neq 1$

$\Delta_q$  is  $\mathbb{F}$ -algebra with gens  $A, B, C$  and rels

$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}$  central

$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}$  central

$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$  central

$\Delta_q$  is univ Askey-Wilson algebra

our general goal: show  $\exists$  injection of algebras  $\Delta_q \rightarrow \hat{H}_q$

that sends

$A \rightarrow t_0 t_1 + (t_0 t_1)^{-1}$

$B \rightarrow t_0 t_3 + (t_0 t_3)^{-1}$

$C \rightarrow t_0 t_2 + (t_0 t_2)^{-1}$

By Th 33 we have

Prop 87  $\exists$   $\mathbb{F}$ -algebra homomorphism

$\psi: \Delta_q \rightarrow \hat{H}_q$  that sends

$$A \longrightarrow t_0 t_1 + (t_0 t_1)^{-1}$$

$$B \longrightarrow t_0 t_3 + (t_0 t_3)^{-1}$$

$$C \longrightarrow t_0 t_2 + (t_0 t_2)^{-1}$$

$\psi$  sends

$$\alpha \longrightarrow (q^{\pm} t_0 + q^{\pm} t_0^{-1}) T_1 + T_2 T_3$$

$$(T_i = t_i + t_i^{-1})$$

$$\beta \longrightarrow (q^{\pm} t_0 + q^{\pm} t_0^{-1}) T_3 + T_1 T_2$$

$$\gamma \longrightarrow (q^{\pm} t_0 + q^{\pm} t_0^{-1}) T_2 + T_3 T_1$$

— 0 —

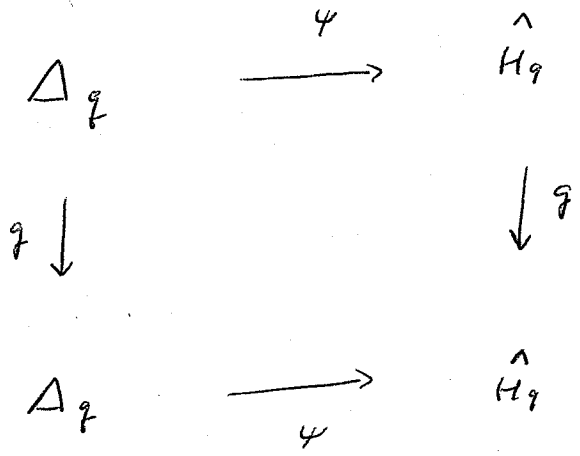
Goal is to show  $\psi$  is injective

Recall Braid group  $B_3$  has gens  $\rho, \sigma$

and rels  $\rho^3 = \sigma^2$

$B_3$  acts on  $\Delta_2$  and  $\hat{H}_2$  as gp of auto

Prop 88  $\forall g \in B_3$  the following diag commutes



pf WLOG  $g = \rho$  or  $g = \sigma$

$g = \rho$ : By 1h77 action of  $\rho$  on  $\Delta_2$  sends

$$A \rightarrow B \rightarrow C \rightarrow A$$

By Prop 32 action of  $\rho$  on  $\hat{H}_2$  sends

$$\begin{aligned}
 t_0 t_1 + (t_0 t_1)^{-1} &\rightarrow t_0 t_2 + (t_0 t_2)^{-1} \rightarrow t_0 t_2 + (t_0 t_2) \\
 &\rightarrow t_0 t_1 + (t_0 t_1)^{-1}
 \end{aligned}$$

$g = \sigma$ :

Recall  $\Delta_2$  is gen by

$A, B, \gamma$

By 77 action of  $\sigma$  on  $\Delta_2$  sends

$A \leftrightarrow B, \quad \gamma \rightarrow \gamma$

Action of  $\sigma$  on  $\hat{H}_g$

swaps

$$t_0 t_1 + (t_0 t_1)^{-1} \leftrightarrow t_0 t_3 + (t_0 t_3)^{-1}$$

(by Prop 32)

fixes  $t_0$  by LEM 9

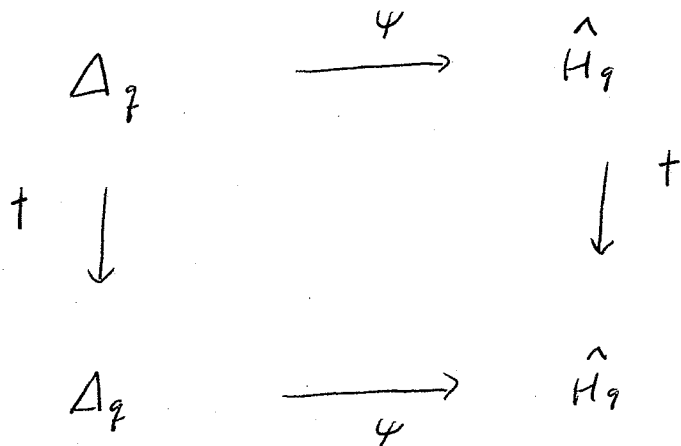
..  $T_1, T_2, T_3$  by Lem 10

so  $\sigma$  fixes

$$(g^{-1} t_0 + g t_0^{-1}) T_2 + T_3 T_1 \quad (= \psi(\gamma))$$

□

Prop 89 The following diag commutes



pf By LB Action of  $\tau$  on  $\Delta_2$  sends

$$A \leftrightarrow B \quad C \rightarrow C$$

By LI Action of  $\tau$  on  $\hat{H}_g$  sends

$$t_0 \rightarrow t_0, \quad t_1 \rightarrow t_3, \quad t_2 \rightarrow t_2, \quad t_3 \rightarrow t_1$$

Chase A, B, C around diagram:

$$\begin{array}{ccc}
 A & \rightarrow & t_0 t_1 + (t_0 t_1)^{-1} \\
 \downarrow & & \downarrow \\
 B & \rightarrow & t_3 t_0 + (t_3 t_0)^{-1} \\
 B & \rightarrow & t_0 t_3 + (t_0 t_3)^{-1} \\
 \downarrow & & \downarrow \\
 A & \rightarrow & t_0 t_1 + (t_0 t_1)^{-1}
 \end{array}$$

)) By Cor 20

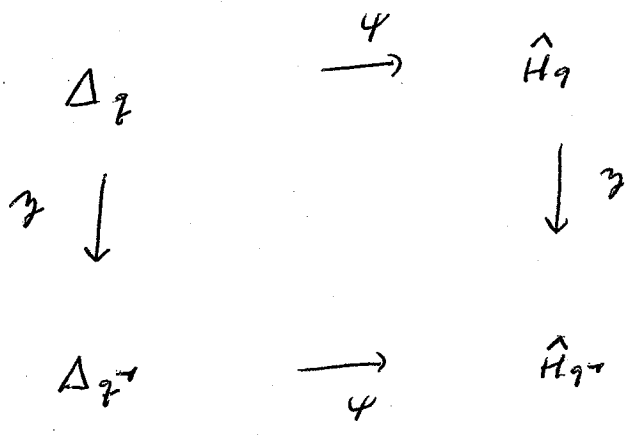
)) By Cor 20

$$\begin{array}{ccc}
 C & \longrightarrow & t_0 t_1 + (t_0 t_2)^{-1} \\
 & & \downarrow \\
 & & t_0 t_1 + (t_2 t_0)^{-1} \\
 & & \parallel \text{ by Cor 20} \\
 C & \longrightarrow & t_0 t_2 + (t_0 t_1)^{-1}
 \end{array}$$

□

Recall algebra iso  $\gamma$  from L12 and L85

Prop 90 The following diagram commutes



pf Action of  $\gamma$  on  $\Delta_2$  sends

$$A \rightarrow B \quad B \rightarrow A \quad C \rightarrow C$$

Action of  $\gamma$  on  $\hat{H}_2$  sends

$$t_0 \rightarrow t_0^{-1}, \quad t_1 \rightarrow t_3^{-1}, \quad t_2 \rightarrow t_2^{-1}, \quad t_3 \rightarrow t_1^{-1}$$

Choose A, B, C around diag:

$$\begin{array}{l}
 A \quad \rightarrow \quad t_0 t_1 + (t_0 t_1)^{-1} \\
 \quad \quad \quad \downarrow \\
 \downarrow \quad \quad \quad t_0^{-1} t_1^{-1} + t_3 t_0 \\
 B \quad \rightarrow \quad t_0 t_3 + (t_0 t_3)^{-1} \quad \quad \quad \parallel \text{ By Cor 20}
 \end{array}$$

$$\begin{array}{l}
 B \quad \rightarrow \quad t_0 t_3 + (t_0 t_3)^{-1} \\
 \quad \quad \quad \downarrow \\
 \downarrow \quad \quad \quad t_0^{-1} t_3^{-1} + t_1 t_0 \\
 A \quad \rightarrow \quad t_0 t_1 + (t_0 t_1)^{-1} \quad \quad \quad \parallel \text{ By Cor 20}
 \end{array}$$

$$\begin{array}{l}
 C \quad \rightarrow \quad t_0 t_2 + (t_0 t_2)^{-1} \\
 \quad \quad \quad \downarrow \\
 \downarrow \quad \quad \quad t_0^{-1} t_2^{-1} + t_2 t_0 \\
 C \quad \rightarrow \quad t_0 t_2 + (t_0 t_2)^{-1} \quad \quad \quad \parallel \text{ By Cor 20}
 \end{array}$$

□

Question: what is max under  $\Psi$  of Casimir el

$$\Omega = gABC + g^2 A^2 + g^{-2} B^2 + g^2 C^2 - gAd - g^2 B\beta - gC\gamma$$

We will ans this question after showing  $\Psi$  is inf



I will display 3 bases for  
the  $\mathbb{F}$ -vector space  $\Delta_q$

We will only use the last one - the first two  
are for completeness

LEM 91 The following is a basis for the  
 $\mathbb{F}$ -vector space  $\Delta_q$ :

$$A^i B^j C^k \Omega^l \beta^a \gamma^b \quad i, j, k, r, s, t \in \mathbb{N}$$

This can be proven using Bergman Diamond Lemma  
or see the paper

Terwilliger: the Univ AW algebra [ArXiv](#)

LEM 92 The following is a basis for the  $\mathbb{F}$ -vector  
space  $\Delta_q$ :

$$A^i B^j C^k \Omega^l \alpha^r \beta^a \gamma^b \quad i, j, k, l, r, s, t \in \mathbb{N}, i+k=0$$

For the proof see above paper. The general idea is to  
start with the basis from L91, and remove all products  
ABC using  $\Omega$

To get third basis start with the basis

in L91. Suppose we eliminated all occurrences

of  $C^2$  using  $\Omega$ . We get the following

Thm 93 The following is a basis for the  $F$ -vector space  $\Delta_q$ :

$$A^i C^j B^k \Omega^l \alpha^r \beta^s \gamma^t$$

$$i, j, k, l, r, s, t \in \mathbb{N}$$

$$j \in \{0, 1\}$$

\*

We will prove Th 93 shortly using the Bergman Diamond Lemma.

Prop 94 The  $F$ -algebra  $\Delta_q$  is presented by  
gens and rels in the following way.

The gens are  $A, B, C, \Omega, \alpha, \beta, \gamma$ .

The rels assert that each of  $\Omega, \alpha, \beta, \gamma$  is central and

$$BA = q^2 AB + q(q^2 - q^{-2})C - q(q - q^{-1})\gamma \quad (1)$$

$$BC = q^{-1} CB - q^{-1}(q^2 - q^{-2})A + q^{-1}(q - q^{-1})\alpha \quad (2)$$

$$CA = q^{-2} AC - q^{-1}(q^2 - q^{-2})B + q^{-1}(q - q^{-1})\beta \quad (3)$$

$$C^2 = q^{-2}\Omega - q^{-3}ACB - q^{-4}A^2 - q^{-4}B^2 + q^{-3}A\alpha + q^{-3}B\beta + q^{-2}C\gamma \quad (4)$$

pf (1)-(3) are reformulations of def rels for  $\Delta_q$

(4) comes from  $\Omega C$  version of  $\Omega$

□

DEF 95 The gens

$A, B, C, \Omega, \alpha, \beta, \gamma$

for  $\Delta_7$  are called balanced

Note 96 Ref to the pres for  $\Delta_7$  from Prop 94

Consider the rels that assert that  $\Omega, \alpha, \beta, \gamma$  are central. these rels can be expressed as

$\Omega A = A \Omega,$	$\Omega B = B \Omega,$	$\Omega C = C \Omega$
$\alpha A = A \alpha$	$\alpha B = B \alpha$	$\alpha C = C \alpha$
$\beta A = A \beta$	$\beta B = B \beta$	$\beta C = C \beta$
$\gamma A = A \gamma$	$\gamma B = B \gamma$	$\gamma C = C \gamma$
$\alpha \Omega = \Omega \alpha$	$\beta \Omega = \Omega \beta$	$\gamma \Omega = \Omega \gamma$
$\beta \alpha = \alpha \beta$	$\gamma \alpha = \alpha \gamma$	$\gamma \beta = \beta \gamma$

DEF 97 By a reduction rule for  $\Delta_g$

we mean an eq from Prop 94 or Note 96  
"1st kind" "2nd kind"

DEF 98 For  $n \in \mathbb{N}$  by a word of length n in  $\Delta_g$

we mean a product  $g_1 g_2 \dots g_n$  such that  $g_i$  is a  
balanced gen of  $\Delta_g$  for  $1 \leq i \leq n$ .

View word of length 0 as identity in  $\Delta_g$

A word is called forbidden whenever it is to the  
left hand side of a red rule.

Each forbidden word has length 2

Forb word is 1st kind if corresp red rule is 1st kind  
2nd -- 2nd --

Def 99 Given a forbidden word  $w$  in  $\Delta_g$  and  
consider the corresp reduction rule. By a descendant  
of  $w$  we mean a word that appears on the RHS  
of that red rule.

Ex The descendants of  $BA$  are

$AB, C, \gamma$

The descendants of  $C^2$  are

$\Omega, ACB, A^2, B^2, A\alpha, B\beta, C\gamma$

$\mathbb{F}$  arb

$0 \neq q \in \mathbb{F}$

$q^4 \neq 1$

Univ. AU alg  $\Delta_q$  has <sup>(balanced)</sup> gens  $A, B, C, \alpha, \beta, \gamma, \Omega$

where Cas  $\Omega = qABC + q^2A^2 + q^2B^2 + q^2C^2 - qA\alpha - q^2B\beta - qC\gamma$

Thm 93 The following is a basis for the  $\mathbb{F}$ -vector space  $\Delta_q$ :

$$A^i C^r B^k \Omega^l \alpha^r \beta^s \gamma^t$$

$$i, k, r, s, t \in \mathbb{N} \quad * \\ j \in \{0, 1\}$$

pf of Th 93

We invoke Bergman's

(11-11) 12

Diamond Lemma.

Let  $g_1 g_2 \dots g_n$  denote a word in  $\Delta q$ .

this word is reducible whenever  $\exists i (2 \leq i \leq n)$

s.t.  $g_i g_{i-1}$  is forbidden. Word is irred if not red.

the list  $X$  consists of the irred words in  $\Delta q$

Let  $w = g_1 g_2 \dots g_n$  denote a word in  $\Delta q$

By an inversion in  $w$  we mean an ordered pair

of integers  $(i, j)$  s.t.  $1 \leq i < j \leq n$  and the word

$g_i g_j$  is forb.

the inv is of 1st kind if  $g_i g_j$  is of first kind

... 2nd kind ... 2nd ...

Let  $W$  denote the set of all words in  $\Delta q$

We def a partial order  $<$  on  $W$  as follows:

Put words  $w, w'$  in  $W$  and write  $w = g_1 g_2 \dots g_n$

We say  $w$  dominates  $w'$  whenever  $\exists i (2 \leq i \leq n)$

s.t.  $g_i g_{i-1}$  is forb and  $w'$  is obtained from  $w$

by replacing  $g_i g_{i-1}$  by one of its descendants.

In this case either

(i)  $w$  has more inversions of the 1st kind than  $w'$

or (ii)  $w, w'$  have same number of inv of 1st kind, but  $w$  has more inv of 2nd kind than  $w'$

Therefore the transitive closure of the dominance relation is a partial order on  $W$  which we denote by  $<$

By constr

- $<$  is semi group partial order
- $<$  satisfies the descending chain condition
- reduction rules are compatible with  $<$

show ambiguities are resolvable;

there are no incl. ambiguities

there are 3 nontriv. overlap ambiguities

$$BCA, \quad BC^2, \quad C^2A$$

Take  $BCA$  for instance.  $BC$  and  $CA$  are forb.

We can reduce  $BCA$  two ways

We could elim  $BC$  first or  $CA$  first. Either way

after 4 steps get same resolution which is

term	$\Omega$	$ACB$	$A^2$	$B^2$	$A\alpha$	$B\beta$	$C\gamma$
coeff	$q^{-3}(q^2 - q^{-2})$	$q^{-6}$	$-q^{-3}(q^2 - q^{-4})$	$-q^{-3}(q^2 - q^{-4})$	$q^{-3}(q^3 - q^{-3})$	$q^{-3}(q^3 - q^{-3})$	$q^{-3}(q - q^{-1})$

The ambiguities  $BC^2, C^2A$  are similarly shown to be resolvable.

Therefore each ambiguity is resolvable so  $X$  is a basis for  $\Delta_g$

11-1  
4

We will discuss coets when an element of  $\Delta_g$  is written in the basis  $*$

To do this we introduce bil form  $\langle \cdot, \cdot \rangle : \Delta_g \times \Delta_g \rightarrow \mathbb{F}$

So  $\langle u, v \rangle = \sum_{i,j} g_{ij} u^i v^j$   $u, v \in *$

So  $\langle \cdot, \cdot \rangle$  is symmetric and  $*$  is orthonormal w.r.t  $\langle \cdot, \cdot \rangle$

For  $w \in \Delta_g$

$$w = \sum \langle w, A^i C^j B^k \Omega^l \alpha^r \beta^s \gamma^t \rangle A^i C^j B^k \Omega^l \alpha^r \beta^s \gamma^t$$

where the sum is over all elements  $A^i C^j B^k \Omega^l \alpha^r \beta^s \gamma^t$  in the basis  $*$ . The sum has finitely many summands



7120111  
5  
Recall our algebra hom  $\psi: \Delta_q \rightarrow \hat{H}_q$

from Prop 87

General goal: show  $\psi$  is isom.

Upcoming steps

(i) Find image of  $\Omega$  under  $\psi$

(ii) Using (i) show images  $\Omega^\psi, \alpha^\psi, \beta^\psi, \gamma^\psi$   
are algebraically indep over  $\mathbb{F}$

(iii) Apply  $\psi$  to the basis for  $\Delta_q$  from Th 93  
and write images in the basis for  $\hat{H}_q$  from  
Thm 52. Using (ii) show these images are  
lin indep.

Find  $\Omega^\psi$

Work with  $\Omega = \Omega \bar{C}$

$$\Omega = q^{-1}ACB + q^{-2}A^2 + q^{-2}B^2 + q^2C^2 - q^2A\alpha - q^2B\beta - qC\gamma$$

Apply  $\psi$  and write image in the basis for  $\hat{H}_q$  from A52.  
Recall

$$A^\psi = Y + Y^{-1}$$

$$Y = t_0 t_1$$

$$B^\psi = X + X^{-1}$$

$$X = t_1 t_0$$

$$\alpha^\psi = (q^{-1}t_0 + q t_0^{-1}) T_1 + T_2 T_3$$

$$T_i = t_i + t_i^{-1}$$

$$\beta^\psi = (q^{-1}t_0 + q t_0^{-1}) T_3 + T_1 T_2$$

$$\gamma^\psi = (q^{-1}t_0 + q t_0^{-1}) T_2 + T_3 T_1$$

$$C^\psi = q^2 \gamma^\psi - q^2 \theta t_0^{-1} \quad \text{where}$$

$$\theta = YX^{-1}t_0 - Y^{-1}Xt_0^{-1} + Y^{-1}T_3 + XT_1 + q^{-1}t_0^2 T_2$$

Keep in mind each of the following commutes with  $t_0^2$ :

$$A^\psi, B^\psi, C^\psi, \theta$$

To find  $\Omega^\psi$  only difficult term is  $(C^\psi)^2$

Write everything in terms of  $\theta$  instead of  $C^\psi$

$\mathbb{F}$  arb

$$0 \neq q \in \mathbb{F} \quad q^4 \neq 1$$

Recall our algebra hom  $\psi: \Delta_q \rightarrow \hat{\mathcal{H}}_q$ General goal: show  $\psi$  is injectivecurrent step: find  $\Omega^\psi$   $\Omega = \text{Casimir el of } \Delta_q$ 

Recall:

$$\Omega = q^{-1}ACB + q^{-2}A^2 + q^{-2}B^2 + q^2C^2 - q^{-1}A\alpha - q^{-1}B\beta - qC\gamma$$

$$A^\psi = Y + Y^{-1}$$

$$B^\psi = X + X^{-1}$$

$$\alpha^\psi = (q^{-1}t_0 + q t_0^{-1}) T_1 + T_2 T_3$$

$$\beta^\psi = (q^{-1}t_0 + q t_0^{-1}) T_3 + T_1 T_2$$

$$\gamma^\psi = (q^{-1}t_0 + q t_0^{-1}) T_2 + T_3 T_1$$

$$C^\psi = q^{-1}\gamma^\psi - q^{-1}\theta t_0^{-1}$$

$$\theta = YX^{-1}t_0 - Y^{-1}Xt_0^{-1} + Y^{-1}T_3 + XT_1 + q^{-1}t_0^{-2}T_2$$

keep in mind  $t_0$  commutes with

$$A^\psi, B^\psi, C^\psi, \theta$$

terms in  $\Omega$

image under  $\psi$

$$q^2 ACB$$

$$q^{-2} (Y+Y^{-1})(X+X^{-1}) Y^4$$
  
$$- q^{-2} (Y+Y^{-1}) \theta (X+X^{-1}) t_0^{-1}$$

$$q^{-2} A^2$$

$$q^{-2} (Y+Y^{-1})^2$$

$$q^{-2} B^2$$

$$q^{-2} (X+X^{-1})^2$$

$$-q^{-1} A \alpha$$

$$-q^{-1} (Y+Y^{-1}) \alpha^4$$

$$-q^{-1} B \beta$$

$$-q^{-1} (X+X^{-1}) \beta^4$$

$$q^2 C^2 - q CY$$

$$\theta^2 t_0^{-2} - \theta t_0^{-1} Y^4$$

$$\theta^2 = \left( YX^{-1}t_0 - Y^{-1}Xt_0^{-1} + Y^{-1}T_3 + Xt_1 + q^{-1}t_0^2 T_2 \right) \theta$$

$$= YX^{-1}\theta t_0 - Y^{-1}X\theta t_0^{-1} + Y^{-1}\theta T_3 + X\theta T_1 + q^{-1}\theta t_0^2 T_2$$

Find  $X^{-1}\theta$  and  $X\theta$

First find  $X^{-1}\theta$  in terms of  $X\theta$

LEM 100

$$X^{-1}\theta = q^{-2}\theta(X+X^{-1}) - X\theta + (q^2 - q^{-2})(Y+Y^{-1})t_0 + q^{-1}(q^{-1})(X+X^{-1})\gamma^4 t_0 - (q^{-1})\alpha^4 t_0$$

pf Recall

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{\alpha}{q + q^{-1}}$$

Apply  $\Psi$

$$Y+Y^{-1} + \frac{q(X+X^{-1})C^\Psi - q^{-1}C^\Psi(X+X^{-1})}{q^2 - q^{-2}} = \frac{\alpha^\Psi}{q + q^{-1}}$$

Now elim  $C^\Psi$  using  $qC^\Psi = \gamma^\Psi - \theta t_0^{-1}$  and solve for  $X^{-1}\theta$

LEM 101

We have

( ... )

$$\Omega^4 =$$

term	coef
$\theta$	$q^{-2}T_2 - t_0^{-2}\delta^4$
$\gamma^{-1}\theta$	$t_0^{-2}T_3$
$\gamma^{-1}\theta(x+x^{-1})$	$-q^{-2}t_0^{-1}$
$x\theta$	$t_0^{-2}T_1$
$\gamma x\theta$	$-t_0^{-1}$
$\gamma^{-1}x\theta$	$-t_0^{-3}$

$$1 \quad \left( q^{-2}(\gamma+\gamma^{-1})(x+x^{-1})\delta^4 + q^{-2}(\gamma+\gamma^{-1})^2 + q^{-2}(x+x^{-1})^2 \right. \\ \left. - q^{-1}(\gamma+\gamma^{-1})\alpha^4 - q^{-1}(x+x^{-1})\beta^4 + (q^2-q^{-2})\gamma(\gamma+\gamma^{-1}) \right. \\ \left. + q^{-1}(q-q^{-1})(x+x^{-1})\delta^4 - (q-q^{-1})\gamma\alpha^4 \right)$$

pf collect terms so far.

□

R

"Remainder"

Matrix rep  $\Theta$  is

	$x^{-2}$	$x^{-1}$	1	$x$	$x^2$
$y^{-2}$	0	0	0	0	0
$y^{-1}$	0	0	$T_3$	$-t_0^{-1}$	0
1	0	0	$q^{-1} t_0^{-2} T_2$	$T_1$	0
$y$	0	$t_0$	0	0	0
$y^2$	0	0	0	0	0

" "

(-)

Notation Given a matrix  $M$

with rows indexed by  $\{y^i\}_{i \in \mathbb{Z}}$

and cols ...  $\{x^i\}_{i \in \mathbb{Z}}$

Matrix  $M^{\uparrow}$  is obtained from  $M$  by moving each entry 1 step North

$M^{\rightarrow}$  ... East

$M^{\downarrow}$  ... South

$M^{\leftarrow}$  ... West

For instance

$$M = \begin{array}{c|ccc} & x^{\uparrow} & 1 & x \\ \hline y^{\uparrow} & 0 & T_1 & 0 \\ 1 & t_0 & 0 & 0 \\ y & 0 & 0 & 0 \end{array}$$

$$M^{\rightarrow} = \begin{array}{c|ccc} & x^{\uparrow} & 1 & x \\ \hline y^{\uparrow} & 0 & 0 & T_1 \\ 1 & 0 & t_0 & 0 \\ y & 0 & 0 & 0 \end{array}$$

$$M^{\downarrow} = \begin{array}{c|ccc} & x^{\uparrow} & 1 & x \\ \hline y^{\uparrow} & 0 & 0 & 0 \\ 1 & 0 & T_1 & 0 \\ y & t_0 & 0 & 0 \end{array}$$



LEM 102

(i) Matrix rep  $\gamma^{\mu} \theta$  is  $\ominus^{\uparrow}$

(ii) ..  $\gamma^{\mu} \theta (x+x^{\dagger})$  is  $\ominus^{\uparrow} + \ominus^{\leftarrow}$

pf (i) clear

(ii) Recall  $x+x^{\dagger}$  commutes with  $t_0$  □

LEM 103

We have

$X\theta =$

	$X^{-2}$	$X^{-1}$	$1$	$X$	$X^2$
$q^{-2}$	0	0	0	0	0
$q^{-1}$	0	$-q^{-2}t_0^2 T_3$	$t_0(q^{-2} + q^{-2}t_0^2 + T_3^2)$	$-q^{-1}(q^{-1}t_0 + q^{-1}t_0^2)T_3$	$q^{-2}t_0$
$1$	0	$q^{-1}t_0^2 T_2$	$-t_0(t_0 T_1 + t_2 T_2 T_3)$	$q^4 t_0$	0
$q$	0	0	$q^2 t_0$	0	0
$q^2$	0	0	0	0	0

" $\ominus_2$ "

pf  $X\theta = XYX^{-1}t_0 - XY^{-1}Xt_0^{-1} + XY^{-1}T_3 + X^2T_1$

$+ q^{-1}Xt_0^2T_2$

Use the reduction rules to express the Right-hand side in the basis from Th. 52 Routine