Lec 7 Friday

$\mathbb{F}$ arb

$0 \neq q \in \mathbb{F}$

univ DAHA $\hat{H}_q$ type $(C_1^\vee, C_1)$

Related algebra S has gens $\{s_i^{\pm 1}\}_{i=0}^{1}$ and rels

$$s_i s_i^{-1} = s_i^{-1} s_i = 1 \qquad i = 0, 1$$

$$\underbrace{s_i + s_i^{-1}}_{S_i} \text{ central}$$

let

$$R = s_0 s_1$$

$$G = s_0 s_1 s_0^{-1} - s_1$$

$$= R s_0^{-1} + R^{-1} s_0 - S_1$$

Some variations on G

LEM 70

X	$X =$
$A_0 A_1 A_0^{-1} - A_1$	G
$A_0^{-1} A_1 A_0 - A_1$	$-A_0^{-1} G A_0$
$A_0 A_1^{-1} A_0^{-1} - A_1^{-1}$	$-G$
$A_0^{-1} A_1^{-1} A_0 - A_1^{-1}$	$A_0^{-1} G A_0$
$A_1 A_0 A_1^{-1} - A_0$	$-A_0^{-1} G A_1^{-1}$
$A_1^{-1} A_0 A_1 - A_0$	$A_1^{-1} G A_0$
$A_1 A_0^{-1} A_1^{-1} - A_0^{-1}$	$A_0^{-1} G A_1^{-1}$
$A_1^{-1} A_0^{-1} A_1 - A_0^{-1}$	$-A_1^{-1} G A_0$

In each case

$$X^2 = G^2$$

LEM 71

(i) $\quad RG = GR^{-1}, \qquad R^{-1}G = GR$

(ii) $\quad G^2 = (R+R^{-1})^2 - (R+R^{-1})S_0S_1 + S_0^2 + S_1^2 - 4$

pf (i) Follows from

$$s_0s_1 + (s_0s_1)^{-1} = s_1s_0 + (s_1s_0)^{-1}$$

(ii) Use red rules

$$s_0R = R^{-1}s_0 + RS_0 - S_1$$

$$s_0R^{-1} = Rs_0 - RS_0 + S_1$$

to write G^2 in the basis for S from Th 60

Given mut. commuting indets $\lambda_0, \lambda_1, \varphi$

Consider $\mathbb{F}$-algebra

$$M = Mat_2(\mathbb{F}) \otimes \mathbb{F}[\lambda_0^{\pm 1}, \lambda_1^{\pm 1}, \varphi]$$

" 2×2 matrices with entries that are polys in $\lambda_0^{\pm 1}, \lambda_1^{\pm 1}, \varphi$ "

- $\exists$ injective $\mathbb{F}$-alg hom

$$S \longrightarrow M$$

that sends

$$s_0 \longrightarrow \begin{pmatrix} \lambda_0 & 0 \\ 1 & \lambda_0^{-1} \end{pmatrix}$$

$$s_1 \longrightarrow \begin{pmatrix} \lambda_1 & \varphi \\ 0 & \lambda_1^{-1} \end{pmatrix}$$

Recall $R = s_0 s_1$

$G = s_0 s_1 s_0^{-1} - s_1$

element of S	image in M
R	$\begin{pmatrix} \lambda_0\lambda_1 & \lambda_0\varphi \\ \lambda_1 & \varphi + \lambda_0^{-1}\lambda_1^{-1} \end{pmatrix}$
$R+R^{-1}$	$(\lambda_0\lambda_1 + \lambda_0^{-1}\lambda_1^{-1} + \varphi)\, I$
G	$\begin{pmatrix} -\varphi\lambda_0 & \varphi\lambda_0(\lambda_0 - \lambda_0^{-1}) \\ \lambda_1\lambda_0^{-1} - \lambda_1^{-1}\lambda_0^{-1} - \varphi & \varphi\lambda_0 \end{pmatrix}$
G^2	$\left(\varphi^2 + \varphi(\lambda_0 - \lambda_0^{-1})(\lambda_1 - \lambda_1^{-1})\right) I$
S_0	$(\lambda_0 + \lambda_0^{-1})\, I$
S_1	$(\lambda_1 + \lambda_1^{-1})\, I$

Problem 7 3 Show that

(i) The center $Z(S)$ has basis

$$(R+R^{-1})^i S_0^j S_1^k \qquad i,j,k \in \mathbb{N}$$

(ii) $Z(S)$ is gen by

$$R+R^{-1}, \ S_0, \ S_1$$

(iii) $Z(S)$ is iso to the poly algebra over $\mathbb{F}$ in 3 commuting variables

Problem 74. Assume $\mathbb{F}$ is alg closed.

Find all the f.d. irred S-modules up to iso. [they all have dim 1 or 2]

We will return to Prob 74 when we consider the irred $\hat{H}_q$-modules.

Recall Univ AW algebra Δ_q from Def 35

Below Def 35 mentioned hom $\Delta_q \to \hat{H}_q$.

Next goal: show this is an injection.

Until further notice

$q^4 \neq 1$

Recall Δ_q is def by gens A, B, C and rels:

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} \qquad \left(= \frac{\alpha}{q+q^{-1}}\right) \qquad (1)$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} \qquad \left(= \frac{\beta}{q+q^{-1}}\right) \qquad (2)$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} \qquad \left(= \frac{\gamma}{q+q^{-1}}\right) \qquad (3)$$

is central.

Obs $\exists$ aut of Δ_q that sends

$A \to B \to C \to A$

There are more auts.

Going to show B_3 acts on Δ_q

To see this we need another pres of Δ_q

LEM 75 The $\mathbb{F}$-algebra Δ_q is gen by A, B, γ. Moreover

(i) $$C = \frac{\gamma}{q+q^{-1}} - \frac{qAB - q^{-1}BA}{q^2-q^{-2}}$$

(ii) $$\alpha = \frac{B^2A - (q^2+q^{-2})BAB + AB^2 + (q^2-q^{-2})^2 A + (q-q^{-1})^2 B\gamma}{(q-q^{-1})(q^2-q^{-2})}$$

(iii) $$\beta = \frac{A^2B - (q^2+q^{-2})ABA + BA^2 + (q^2-q^{-2})^2 B + (q-q^{-1})^2 A\gamma}{(q-q^{-1})(q^2-q^{-2})}$$

pf (i) from def of γ (3)

(ii) in (2) elim C using (i)

(iii) in (3) " C " (i)

$\square$

——— o ———

Recall notation

$$[n]_q = \frac{q^n - q^{-n}}{q-q^{-1}} \qquad n = 0, 1, 2 \ldots$$

Prop 76 The $\mathbb{F}$-alg Δ_q has a presentation by gens A, B, Υ and rels

$$A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3 = -(q^2-q^{-2})^2(AB-BA)$$

$$B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = -(q^2-q^{-2})^2(BA-AB)$$

$$A^2B^2 - B^2A^2 + (q^2+q^{-2})(BABA - ABAB) = -(q-q^{-1})^2(AB-BA)\Upsilon$$

$$\Upsilon A = A\Upsilon, \qquad \Upsilon B = B\Upsilon$$

pf Use L75 to express the orig def rels for Δ_q in terms of A, B, Υ. $\square$

Thm 77 The braid gp B_3 acts on Δ_q as a group of automorphisms s.t $\tau(h) = h \;\forall h \in \Delta_q$ and ρ, σ do the following

u	A	B	C	α	β	γ
$\rho(u)$	B	C	A	β	γ	α
$\sigma(u)$	B	A	$C + \frac{AB-BA}{q-q^{-1}}$	β	α	γ

pf By constr $\exists$ aut P of Δ_q that sends

$$A \to B \to C \to A$$

So $P^3 = 1$

Obs P sends

$$\alpha \to \beta \to \gamma \to \alpha$$

By Prop 76 $\exists$ aut S of Δ_q that sends

$$A \to B, \qquad B \to A, \qquad \gamma \to \gamma$$

Obs $S^2 = 1$

Using L75 one checks S sends

$$\alpha \to \beta, \qquad \beta \to \alpha, \qquad C \to C + \frac{AB-BA}{q-q^{-1}}$$

Result follows from Def of B_3.

$\square$

Note The group $PSL_2(\mathbb{Z})$ has a pres by gens ρ, σ and rels $\rho^3=1$, $\sigma^2=1$.

So $\exists$ surj gp hom

$$\begin{aligned} B_3 &\longrightarrow PSL_2(\mathbb{Z}) \\ \rho &\to \rho \\ \sigma &\to \sigma \\ \tau &\to 1. \end{aligned}$$

Th 77 gives action of $PSL_2(\mathbb{Z})$ on Δ_q

We need a certain central el of Δ_q called Casimir element Ω.

LEM 78 The following elements of Δ_q coincide

Name	Leading term	A^2	B^2	C^2	$A\alpha$	$B\beta$	$C\gamma$
Ω_B^+	$qABC$	q^2	q^{-2}	q^2	$-q$	$-q^{-1}$	$-q$
Ω_C^+	$qBCA$	q^2	q^2	q^{-2}	$-q$	$-q$	$-q^{-1}$
Ω_A^+	$qCAB$	q^{-2}	q^2	q^2	$-q^{-1}$	$-q$	$-q$
Ω_B^-	$q^{-1}CBA$	q^{-2}	q^2	q^{-2}	$-q^{-1}$	$-q$	$-q^{-1}$
Ω_C^-	$q^{-1}ACB$	q^{-2}	q^{-2}	q^2	$-q^{-1}$	$-q^{-1}$	$-q$
Ω_A^-	$q^{-1}BAC$	q^2	q^{-2}	q^{-2}	$-q$	$-q^{-1}$	$-q^{-1}$

Call this common value Ω

pf The aut ρ sends

$$\Omega_B^+ \to \Omega_C^+ \to \Omega_A^+$$

$$\Omega_B^- \to \Omega_C^- \to \Omega_A^-$$

$\Omega_B^+ - \Omega_C^-$ equals $(q-q^{-1})A$ times

$$(q+q^{-1})A + \frac{qBC - q^{-1}CB}{q-q^{-1}} - \alpha \qquad *$$

$(*)$ is zero so $\Omega_B^+ = \Omega_C^-$

Apply ρ to get
$\Omega_C^+ = \Omega_A^-$. $\qquad \Omega_A^+ = \Omega_B^-$

$\Omega_B^+ - \Omega_A^-$ equals

$$(q+q^{-1})C + \frac{qAB - q^{-1}BA}{q-q^{-1}} - \gamma \qquad **$$

times $(q-q^{-1})C$.
$(**)$ is zero by def so $\Omega_B^+ = \Omega_A^-$

Apply ρ to get

$$\Omega_C^+ = \Omega_B^- \qquad \Omega_A^+ = \Omega_C^-$$

Now $\Omega_A^+ = \Omega_B^+ = \Omega_C^+ = \Omega_A^- = \Omega_B^- = \Omega_C^-$ $\qquad \square$

$\mathbb{F}$ arb

$0 \neq q \in \mathbb{F}$ $\quad q^4 \neq 1$

Univ AW algebra Δ_q has gens A, B, C and rels

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} \quad \text{central} \qquad (+\ \text{CP})$$

general goal: get injection $\Delta_q \to \hat{H}_q$

spec goal: study casimir element Ω of Δ_q

Recall Ω is common value of

$$\Omega_A^+ \quad \Omega_B^+ \quad \Omega_C^+ \quad \Omega_A^- \quad \Omega_B^- \quad \Omega_C^-$$

from L78. For instance

$$\Omega_B^+ = qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma$$

α, β, γ are central els of Δ_q from above L75

LEM 79 Ω is fixed by everything in the braid group B_3

pf Suf to show the B_3 gens ρ, σ fix Ω

ρ fixes Ω since ρ sends $\Omega_B^+ \to \Omega_C^+$ by constr.

Show σ fixes Ω

Show σ sends $\Omega_B^+ \to \Omega_A^+$

By lm 77 σ sends

$$A \leftrightarrow B \qquad C \to \underbrace{C + \frac{AB - BA}{q - q^{-1}}}_{C'}$$

$$\alpha \leftrightarrow \beta \qquad \gamma \to \gamma$$

So

$$\sigma(\Omega_B^+) = qBAC' + q^2B^2 + q^{-2}A^2 + q^2(C')^2 - qB\beta - q^{-1}A\alpha - qC'\gamma$$

Also

$$\Omega_A^+ = qCAB + q^2C^2 + q^{-2}A^2 + q^2B^2 - qC\gamma - q^{-1}A\alpha - qB\beta$$

Obs

$$\sigma(\Omega_B^+) - \Omega_A^+ =$$

$$(BA + q^{-1}C + qC' - \gamma)\, qC' - qC\,(AB + qC + q^{-1}C' - \gamma)$$

claim

$$0 = BA + q^{-1}C + q\,C' - \gamma$$

$$C' = C + \frac{AB - BA}{q - q^{-1}} \qquad \gamma = \frac{\gamma}{q+q^{-1}}\, q+q^{-1} = \left(C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} \right) q+q^{-1}$$

ok

claim

$$0 = AB + q\,C + q^{-1}C' - \gamma$$

Sim checked

So

$$\sigma(\Omega^+_B) = \Omega^+_A$$

So σ fixes Ω $\square$

thm 80 $\quad \Omega$ is central in Δ_q

pf $\quad$ first show $\quad A\Omega = \Omega A$

show $\quad A\Omega_C^+ = \Omega_B^+ A$

Recall

$$\beta + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{\beta}{q+q^{-1}} \qquad (\star)$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{\gamma}{q+q^{-1}} \qquad (*)$$

Consider

$$q\,C\,\star \quad + q^{-1}\,\star\,C \quad - \gamma\,\star$$

$$+ \quad \beta\,* \quad - q^{-1}\,B\,* \quad - q\,*\,B$$

Reduces to

$$A\Omega_C^+ - \Omega_B^+ A = 0$$

(ex).

So far $\quad A\Omega = \Omega A$

Now apply ρ twice get

$B\Omega = \Omega B, \qquad C\Omega = \Omega C$

$\square$

More symmetries of Δ_q

LEM 81 Pick $\varepsilon_A, \varepsilon_B, \varepsilon_C \in \{1, -1\}$

s.t. $\varepsilon_A \varepsilon_B \varepsilon_C = 1$

$\exists$ automorphism of Δ_q that sends "$\mathbb{Z}_2 \times \mathbb{Z}_2$ sym"

$$A \longrightarrow \varepsilon_A A$$
$$B \longrightarrow \varepsilon_B B$$
$$C \longrightarrow \varepsilon_C C$$

This aut sends

$$\alpha \to \varepsilon_A \alpha, \qquad \beta \to \varepsilon_B \beta, \qquad \gamma \to \varepsilon_C \gamma$$

pf WLOG $\varepsilon_A = -1, \quad \varepsilon_B = -1, \quad \varepsilon_C = 1$

One checks $A \to -A, \quad B \to -B, \quad C \to C$

respects the defining relations for Δ_q $\square$

LEM 82 Ω is fixed by each aut of Δ_q

from LEM 81

pf View

$$\Omega = \Omega_B^+$$

$$= qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma$$

Each term is fixed by the aut $\square$

LEM 83 $\exists$ an antiaut $\dagger$ of Δ_q that sends

$$A \leftrightarrow B \qquad C \to C$$
$$\alpha \leftrightarrow \beta \qquad \gamma \to \gamma$$

Moreover $\dagger^2 = 1$

pf Define

$$A^\dagger = B \qquad B^\dagger = A \qquad C^\dagger = C$$
$$\alpha^\dagger = \beta \qquad \beta^\dagger = \alpha \qquad \gamma^\dagger = \gamma$$

Show $A^\dagger, B^\dagger, C^\dagger, \alpha^\dagger, \beta^\dagger, \gamma^\dagger$ sat the def rels for Δ_q in the algebra Δ_q^{op}. Need

$$A^\dagger + \frac{q B^\dagger C^\dagger - q^{-1} C^\dagger B^\dagger}{q^2 - q^{-2}} \stackrel{?}{=} \frac{\alpha^\dagger}{q+q^{-1}} \quad \text{in } \Delta_q^{op}$$

$$B^\dagger + \frac{q C^\dagger A^\dagger - q^{-1} A^\dagger C^\dagger}{q^2 - q^{-2}} \stackrel{?}{=} \frac{\beta^\dagger}{q+q^{-1}} \quad \text{"}$$

$$C^\dagger + \frac{q A^\dagger B^\dagger - q^{-1} B^\dagger A^\dagger}{q^2 - q^{-2}} \stackrel{?}{=} \frac{\gamma^\dagger}{q+q^{-1}} \quad \text{"}$$

these reduce to orig def rels in Δ_q ✓

So $\exists$ $\mathbb{F}$-alg hom $\dagger : \Delta_q \to \Delta_q$ that sends

$$A \to A^\dagger \qquad B \to B^\dagger \qquad C \to C^\dagger$$
$$\alpha \to \alpha^\dagger \qquad \beta \to \beta^\dagger \qquad \gamma \to \gamma^\dagger$$

By constr $\dagger^2 = 1$ so $\dagger$ is bijection. $\square$

LEM 84 Ω is fixed by the antiaut $\dagger$ of Δ_q

from L 83

pf View

$$\Omega = \Omega_B^{\dagger} = qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma$$

Apply $\dagger$

$$\Omega^{\dagger} = qCAB + q^2B^2 + q^{-2}A^2 + q^2C^2 - qB\beta - q^{-1}A\alpha - qC\gamma$$

$$= \Omega_A^{\dagger}$$

$$= \Omega \qquad \square$$

LEM 85 $\exists$ ISO of $\mathbb{F}$-algebras

$\eta : \Delta_q \to \Delta_{q^{-1}}$ that sends

$$A \to B \qquad B \to A \qquad C \to C$$

$$\alpha \to \beta \qquad \beta \to \alpha \qquad \gamma \to \gamma$$

pf Notation: Write

$$\varphi = q^{-1}$$

write $\hat{A}, \hat{B}, \hat{C}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}$

for gens of $\Delta_{q^{-1}}$

Put

$$A^\eta = \hat{B} \qquad B^\eta = \hat{A} \qquad C^\eta = \hat{C}$$

$$\alpha^\eta = \hat{\beta} \qquad \beta^\eta = \hat{\alpha} \qquad \gamma^\eta = \hat{\gamma}$$

show

$$\underbrace{A^\eta + \frac{qB^\eta C^\eta - q^{-1}C^\eta B^\eta}{q^2 - q^{-2}}}_{\hat{B} + \frac{\varphi^{-1}\hat{A}\hat{C} - \varphi\hat{C}\hat{A}}{\varphi^{-2} - \varphi^2}} \overset{?}{=} \underset{\hat{\beta}}{\alpha^\eta} \quad \text{in } \Delta_{q^{-1}}$$

✓

$$\underbrace{B^\eta + \frac{qC^\eta A^\eta - q^{-1}A^\eta C^\eta}{q^2 - q^{-2}}}_{\hat{A} + \frac{\varphi^{-1}\hat{C}\hat{B} - \varphi\hat{B}\hat{C}}{\varphi^{-2} - \varphi^2}} \overset{?}{=} \underset{\hat{\alpha}}{\beta^\eta} \quad \text{in } \Delta_{q^{-1}}$$

✓

$$\underbrace{C^3 + \frac{qA^3B^3 - q^{-1}B^3A^3}{q^2 - q^{-2}}}_{\hat{C} + \frac{Q^{-1}BA - QAB}{Q^{-2} - Q^2}} \overset{?}{=} \gamma^3 \quad \text{in } \Delta_{q^{-1}}$$

$$\overset{?}{=} \hat{\gamma}$$

✓

So $\exists$ $\mathbb{F}$-alg hom $\eta : \Delta_q \to \Delta_{q^{-1}}$ that sends $A \to A^3$ etc

Obs

$$\Delta_q \xrightarrow[\eta]{} \Delta_{q^{-1}} \xrightarrow[\eta]{} \Delta_q$$

is ident so η is bij. $\square$

LEM 86 the map $\natural : \Delta_q \to \Delta_{q^{-1}}$

from L85 sends the Cas el of Δ_q to

the Cas el of $\Delta_{q^{-1}}$

pf Write $\varphi = q^{-1}$ View $\Omega = \Omega_B^+$

$$q ABC + q^2 A^2 + q^{-2} B^2 + q^2 C^2 - qA\alpha - q^{-1} B\beta - q C\gamma$$

$\natural \downarrow$

$$\varphi^{-1} BAC + \varphi^{-2} B^2 + \varphi^2 A^2 + \varphi^{-2} C^2 - \varphi^{-1} B\beta - \varphi A\alpha - \varphi^{-1} C\gamma$$

this is Ω_A^- computed in Δ_φ

$\square$

Lec 11 Wednesday Sept 28

9/28/11

field $\mathbb{F}$ arb

$0 \neq q \in \mathbb{F}$

$\hat{H}_q$ is $\mathbb{F}$-algebra with gens $\{t_i^{\pm}\}_{i \in \mathbb{I}}$ $\quad \mathbb{I} = \{0,1,2,3\}$

and rels

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \qquad i \in \mathbb{I}$$

$$t_i + t_i^{-1} \quad \text{central} \qquad i \in \mathbb{I}$$

$$t_0 t_1 t_2 t_3 = q^{-1}$$

$\hat{H}_q$ is <u>universal DAHA of type $(C_1^\vee, C_1)$</u>

Next assume $q^4 \neq 1$

Δ_q is $\mathbb{F}$-algebra with gens A, B, C and rels

$\left(\frac{\alpha}{q+q^{-1}} = \right)$ $\qquad A + \dfrac{qBC - q^{-1}CB}{q^2 - q^{-2}}$ central

$\left(\frac{\beta}{q+q^{-1}} = \right)$ $\qquad B + \dfrac{qCA - q^{-1}AC}{q^2 - q^{-2}}$ central

$\left(\frac{\gamma}{q+q^{-1}} = \right)$ $\qquad C + \dfrac{qAB - q^{-1}BA}{q^2 - q^{-2}}$ central

Δ_q is <u>univ Askey-Wilson algebra</u>

our general goal: show $\exists$ injection of algebras $\Delta_q \to \hat{H}_q$

that sends

$$A \longrightarrow t_0 t_1 + (t_0 t_1)^{-1}$$

$$B \longrightarrow t_0 t_3 + (t_0 t_3)^{-1}$$

$$C \longrightarrow t_0 t_2 + (t_0 t_2)^{-1}$$

By Th 33 we have

Prop 87 $\exists$ $\mathbb{F}$-algebra homomorphism

$\psi : \Delta_q \to \hat{H}_q$ that sends

$$A \longrightarrow t_0 t_1 + (t_0 t_1)^{-1}$$

$$B \longrightarrow t_0 t_3 + (t_0 t_3)^{-1}$$

$$C \longrightarrow t_0 t_2 + (t_0 t_2)^{-1}$$

ψ sends

$$\alpha \longrightarrow (q^{-1} t_0 + q t_0^{-1}) T_1 + T_2 T_3$$

$$(T_i = t_i + t_i^{-1})$$

$$\beta \longrightarrow (q^{-1} t_0 + q t_0^{-1}) T_3 + T_1 T_2$$

$$\gamma \longrightarrow (q^{-1} t_0 + q t_0^{-1}) T_2 + T_3 T_1$$

— o —

Goal is to show ψ is injective

Recall Braid group B_3 has gens ρ, σ

and rels $\rho^3 = \sigma^2$

B_3 acts on Δ_q and $\hat{H}_q$ as gp of auts

Prop 88 $\forall g \in B_3$ the following diag commutes

$$\begin{array}{ccc} \Delta_q & \xrightarrow{\psi} & \hat{H}_q \\ g \downarrow & & \downarrow g \\ \Delta_q & \xrightarrow[\psi]{} & \hat{H}_q \end{array}$$

pf WLOG $g = \rho$ or $g = \sigma$

$g = \rho$: By Th 77 action of ρ on Δ_q sends

$$A \to B \to C \to A$$

By Prop 32 action of ρ on $\hat{H}_q$ sends

$$t_0 t_1 + (t_0 t_1)^{-1} \to t_0 t_3 + (t_0 t_3)^{-1} \to t_0 t_2 + (t_0 t_2)$$

$$\to t_0 t_1 + (t_0 t_1)^{-1}$$

$g = \sigma$: Recall Δ_q is gen by

A, B, γ

By 77 action of σ on Δ_q sends

$A \leftrightarrow B, \qquad \gamma \to \gamma$

Action of σ on $\hat{H}_q$

swaps $\quad t_0 t_1 + (t_0 t_1)^{-1} \leftrightarrow t_0 t_3 + (t_0 t_3)^{-1}$

(by Prop 32)

fixes t_0 by LEM 9

" T_1, T_2, T_3 by Lem 10

so σ fixes

$(q^{-1} t_0 + q t_0^{-1}) T_2 + T_3 T_1 \qquad (= \psi(\gamma))$

$\square$

Prop 89 The following diag commutes

$$
\begin{tikzcd}
\Delta_q \arrow[r, "\psi"] \arrow[d, "t"'] & \hat{H}_q \arrow[d, "t"] \\
\Delta_q \arrow[r, "\psi"'] & \hat{H}_q
\end{tikzcd}
$$

pf By L13 Action of t on Δ_q sends

$$A \leftrightarrow B \qquad C \to C$$

By L11 Action of t on $\hat{H}_q$ sends

$$t_0 \to t_0, \quad t_1 \to t_3 \qquad t_2 \to t_2 \qquad t_3 \to t_1$$

Chase A, B, C around diagram:

$$
\begin{tikzcd}
A \arrow[r] \arrow[dd] & t_0t_1 + (t_0t_1)^{-1} \arrow[d] \\
 & t_3t_0 + (t_3t_0)^{-1} \arrow[d, equal, "\text{By Cor 20}"] \\
B \arrow[r] & t_0t_3 + (t_0t_3)^{-1}
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r] \arrow[dd] & t_0t_3 + (t_0t_3)^{-1} \arrow[d] \\
 & t_1t_0 + (t_1t_0)^{-1} \arrow[d, equal, "\text{By Cor 20}"] \\
A \arrow[r] & t_0t_1 + (t_0t_1)^{-1}
\end{tikzcd}
$$

$$\begin{array}{ccc} C & \longrightarrow & t_0t_2 + (t_0t_2)^{-1} \\ & & \downarrow \\ \downarrow & & t_2t_0 + (t_2t_0)^{-1} \\ & & \| \text{ By Cor 20} \\ C & \longrightarrow & t_0t_2 + (t_0t_2)^{-1} \end{array}$$

$\square$

Recall algebra iso η from L12 and L85

Prop 90 The following diagram commutes

$$\begin{array}{ccc} \Delta_q & \xrightarrow{\psi} & \hat{H}_q \\ \eta \downarrow & & \downarrow \eta \\ \Delta_{q^{-1}} & \xrightarrow[\psi]{} & \hat{H}_{q^{-1}} \end{array}$$

pf Action of η on Δ_q sends

$A \to B \qquad B \to A \qquad C \to C$

Action of η on $\hat{H}_q$ sends

$t_0 \to t_0^{-1}, \quad t_1 \to t_3^{-1}, \quad t_2 \to t_2^{-1}, \quad t_3 \to t_1^{-1}$

Chase A, B, C around diag:

$$\begin{tikzcd}
A \arrow[r] \arrow[dd] & t_0t_1 + (t_0t_1)^{-1} \arrow[d] \\
& t_0^{-1}t_3^{-1} + t_3t_0 \arrow[d, equal, "\text{By Cor 20}"] \\
B \arrow[r] & t_0t_3 + (t_0t_3)^{-1}
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r] \arrow[dd] & t_0t_3 + (t_0t_3)^{-1} \arrow[d] \\
& t_0^{-1}t_1^{-1} + t_1t_0 \arrow[d, equal, "\text{By Cor 20}"] \\
A \arrow[r] & t_0t_1 + (t_0t_1)^{-1}
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r] \arrow[dd] & t_0t_2 + (t_0t_2)^{-1} \arrow[d] \\
& t_0^{-1}t_2^{-1} + t_2t_0 \arrow[d, equal, "\text{By Cor 20}"] \\
C \arrow[r] & t_0t_2 + (t_0t_2)^{-1}
\end{tikzcd}$$

$\square$

Question: what is image under φ of Casimir el

$$\Omega = qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-2}B\beta - qC\gamma$$

We will ans this question after showing φ is inj

I will display 3 bases for the $\mathbb{F}$-vector space Δ_q

We will only use the Last one - the first two are for completeness

LEM 91 The following is a basis for the $\mathbb{F}$-vector space Δ_q:

$$A^i B^j C^k \alpha^r \beta^s \gamma^t \qquad i,j,k,r,s,t \in \mathbb{N}$$

this can be proven using Bergman Diamond Lemma or see the paper

Terwilliger: the Univ AW algebra ArXiv

LEM 92 The following is a basis for the $\mathbb{F}$-vector space Δ_q:

$$A^i B^j C^k \Omega^l \alpha^r \beta^s \gamma^t \qquad i,j,k,l,r,s,t \in \mathbb{N}, \quad ijk = 0$$

For the proof see above paper. The general idea is to start with the basis from L91, and remove all products ABC using Ω

To get third basis start with the basis in L91. Suppose we eliminated all occurrences of C^2 using Ω. We get the following

Thm 93 The following is a basis for the $\mathbb{F}$-vector space Δ_q:

$$A^i C^j B^k \Omega^l \alpha^r \beta^s \gamma^t \qquad i,k,l,r,s,t \in \mathbb{N} \quad j \in \{0,1\} \qquad \ast$$

We will prove Th 93 shortly using the Bergman Diamond Lemma.

Prop 94 The $\mathbb{F}$-algebra Δ_q is presented by gens and rels in the following way.

The gens are $A, B, C, \Omega, \alpha, \beta, \gamma$.

The rels assert that each of $\Omega, \alpha, \beta, \gamma$ is central and

$$BA = q^2 AB + q(q^2 - q^{-2})C - q(q - q^{-1})\gamma \tag{1}$$

$$BC = q^{-2} CB - q^{-1}(q^2 - q^{-2})A + q^{-1}(q - q^{-1})\alpha \tag{2}$$

$$CA = q^{-2} AC - q^{-1}(q^2 - q^{-2})B + q^{-1}(q - q^{-1})\beta \tag{3}$$

$$C^2 = q^{-2}\Omega - q^{-3}ACB - q^{-4}A^2 - q^{-4}B^2 + q^{-3}A\alpha + q^{-3}B\beta + q^{-2}C\gamma \tag{4}$$

pf (1)–(3) are reformulations of def rels for Δ_q

(4) comes from $\Omega_{\bar{C}}$ version of Ω $\square$

DEF 95 The gens

$A, B, C, \Omega, \alpha, \beta, \gamma$

for Δ_q are called <u>balanced</u>

Note 96 Ref to the pres for Δ_q from Prop 94

Consider the rels that assert that $\Omega, \alpha, \beta, \gamma$ are central.

These rels can be expressed as

$\Omega A = A\Omega$,	$\Omega B = B\Omega$,	$\Omega C = C\Omega$
$\alpha A = A\alpha$	$\alpha B = B\alpha$	$\alpha C = C\alpha$
$\beta A = A\beta$	$\beta B = B\beta$	$\beta C = C\beta$
$\gamma A = A\gamma$	$\gamma B = B\gamma$	$\gamma C = C\gamma$
$\alpha\Omega = \Omega\alpha$	$\beta\Omega = \Omega\beta$	$\gamma\Omega = \Omega\gamma$
$\beta\alpha = \alpha\beta$	$\gamma\alpha = \alpha\gamma$	$\gamma\beta = \beta\gamma$

DEF 97 By a reduction rule for Δ_q

we mean an eq from Prop 94 or Note 96
"1st kind" "2nd kind"

DEF 98 For $n \in \mathbb{N}$ by a word of length n in Δ_q

we mean a product $g_1 g_2 \cdots g_n$ such that g_i is a

balanced gen of Δ_q for $1 \le i \le n$.

View word of length 0 as identity in Δ_q

A word is called forbidden whenever it is the

left-hand side of a red rule

Each forbidden word has length 2

Forb word is 1st kind if corresp red rule is 1st kind
2nd .. .. 2nd ..

Def 99 Given a forbidden word w in Δ_q and

consider the corresp reduction rule. By a descendant

of w we mean a word that appears on the RHS

of that red rule.

Ex the descendents of BA are

AB, C, γ

the descendents of C^2 are

Ω, ACB, A^2, B^2, $A\alpha$, $B\beta$. $C\gamma$

$\mathbb{F}$ arb

$0 \neq q \in \mathbb{F}$

$q^4 \neq 1$

Univ AW alg Δ_q has $\overset{\text{(balanced)}}{\text{gens}}$ $A, B, C, \alpha, \beta, \gamma, \Omega$

where Cas $\Omega = qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma$

Thm 93 The following is a basis for the $\mathbb{F}$-vector space Δ_q:

$$A^i C^j B^k \Omega^\ell \alpha^r \beta^s \gamma^t \qquad i, k, \ell, r, s, t \in \mathbb{N} \qquad *$$
$$j \in \{0, 1\}$$

<u>pf of Th 93</u> We invoke Bergman's Diamond Lemma.

Let $g_1 g_2 \cdots g_n$ denote a word in Δ_q.

This word is <u>reducible</u> whenever $\exists\, i$ $(2 \le i \le n)$ s.t. $g_{i-1} g_i$ is forbidden. Word is <u>irred</u> if not red.

The list $*$ consists of the irred words in Δ_q

Let $w = g_1 g_2 \cdots g_n$ denote a word in Δ_q

By an <u>inversion</u> in w we mean an ordered pair of integers (i, j) s.t. $1 \le i < j \le n$ and the word $g_i g_j$ is forb.

The inv is of <u>1st kind</u> if $g_i g_j$ is of first kind
-- 2nd kind -- 2nd --

Let W denote the set of all words in Δ_q

We def a partial order $<$ on W as follows:

Pick words w, w' in W and write $w = g_1 g_2 \cdots g_n$

We say w <u>dominates</u> w' whenever $\exists\, i$ $(2 \le i \le n)$ s.t. $g_{i-1} g_i$ is forb and w' is obtained from w by replacing $g_{i-1} g_i$ by one of its descendents.

In this case either

(i) w has more inversions of the 1st kind than w'

or (ii) w, w' have same number of inv of 1st kind, but w has more inv of 2nd kind than w' 135

Therefore the transitive closure of the dominance relation is a partial order on W which we denote by $<$

By constr

- $<$ is semigroup partial order
- $<$ satisfies the descending chain condition
- reduction rules are compatible with $<$

show ambiguities are resolvable:

there are no incl ambiguities

there are 3 nontriv overlapp ambiguities

$$BCA, \qquad BC^2 \qquad C^2A$$

Take BCA for instance. BC and CA are forb.
We can reduce BCA two ways
We could elim BC first or CA first. Either way
after 4 steps get same resolution which is

term	Ω	ACB	A^2	B^2	$A\alpha$	$B\beta$	$C\gamma$
coeff	$q^{-3}(q^2-q^{-2})$	q^{-6}	$-q^{-3}(q^4-q^{-4})$	$-q^{-3}(q^4-q^{-4})$	$q^{-3}(q^3-q^{-3})$	$q^{-3}(q^3-q^{-3})$	$q^{-3}(q-q^{-1})$

the ambiguities BC^2, C^2A are similarly shown to be resolvable.

Therefore each ambiguity is resolvable so X is a basis for Δ_q $\square$

We will discuss coefs when an element of Δ_q

is written in the basis $*$

To do this we introduce bil form $\langle , \rangle : \Delta_q \times \Delta_q \to \mathbb{F}$

s.t $\langle u, v \rangle = \delta_{uv}$ $u, v \in *$

So $\langle , \rangle$ is symmetric and $*$ is orthonormal rel $\langle , \rangle$

For $w \in \Delta_q$

$$w = \sum \langle w, A^i C^j B^k \Omega^l \alpha^r \beta^s \gamma^t \rangle A^i C^j B^k \Omega^l \alpha^r \beta^s \gamma^t$$

where the sum is over all elements $A^i C^j B^k \Omega^l \alpha^r \beta^s \gamma^t$

in the basis $*$. The sum has fi many nonzero summands

Recall our algebra hom $\psi : \Delta_q \to \hat{H}_q$

from Prop 87

General goal: show ψ is inj.

upcoming steps

(i) Find image of Ω under ψ

(ii) Using (i) show images $\Omega^\psi, \alpha^\psi, \beta^\psi, \gamma^\psi$ are algebraically indep over $\mathbb{F}$

(iii) Apply ψ to the basis for Δ_q from Th 93 and write images in the basis for $\hat{H}_q$ from Thm 52. Using (ii) show these images are lin indep.

Find Ω^{ψ}

Work with $\Omega = \Omega_{\bar{C}}$

$$\Omega = q^{-1}ACB + q^{-2}A^2 + q^{-2}B^2 + q^{2}C^2 - q^{-1}A\alpha - q^{-1}B\beta - qC\gamma$$

Apply ψ and write image in the basis for $\hat{H}_q$ from p 52.

Recall

$$A^{\psi} = Y + Y^{-1} \qquad Y = t_0 t_1$$

$$B^{\psi} = X + X^{-1} \qquad X = t_3 t_0$$

$$\alpha^{\psi} = (q^{-1}t_0 + q t_0^{-1})T_1 + T_2 T_3 \qquad T_i = t_i + t_i^{-1}$$

$$\beta^{\psi} = (q^{-1}t_0 + q t_0^{-1})T_3 + T_1 T_2$$

$$\gamma^{\psi} = (q^{-1}t_0 + q t_0^{-1})T_2 + T_3 T_1$$

$$C^{\psi} = q^{-1}\gamma^{\psi} - q^{-1}\Theta t_0^{-1} \qquad \text{where}$$

$$\Theta = YX^{-1}t_0 - Y^{-1}Xt_0^{-1} + Y^{-1}T_3 + XT_1 + q^{-1}t_0^{2}T_2$$

Keep in mind each of the following commutes with t_0:

$$A^{\psi},\ B^{\psi},\ C^{\psi},\ \Theta$$

To find Ω^{ψ} only difficult term is $(C^{\psi})^2$

write everything in terms of Θ instead of C^{ψ}

$\mathbb{F}$ arb
$0 \neq q \in \mathbb{F}$ $\quad q^4 \neq 1$

Recall our algebra hom $\psi : \Delta_q \to \hat{H}_q$

General goal: show ψ is injective

current step: find Ω^ψ $\quad$ Ω = Casimir el of Δ_q

Recall:

$$\Omega = q^{-1}ACB + q^{-2}A^2 + q^{-2}B^2 + q^2C^2 - q^{-1}A\alpha - q^{-1}B\beta - qC\gamma$$

$$A^\psi = Y + Y^{-1}$$
$$B^\psi = X + X^{-1}$$
$$\alpha^\psi = (q^{-1}t_0 + q t_0^{-1})T_1 + T_2T_3$$
$$\beta^\psi = (q^{-1}t_0 + q t_0^{-1})T_3 + T_1T_2$$
$$\gamma^\psi = (q^{-1}t_0 + q t_0^{-1})T_2 + T_3T_1$$

$$C^\psi = q^{-1}\gamma^\psi - q^{-1}\theta t_0^{-1}$$

$$\theta = YX^{-1}t_0 - Y^{-1}Xt_0^{-1} + Y^{-1}T_3 + XT_1 + q^{-1}t_0^2 T_2$$

Keep in mind t_0 commutes with

A^ψ, B^ψ, C^ψ, θ

terms in Ω	image under ψ
$q^{-1}ACB$	$q^{-2}(Y+Y^{-1})(X+X^{-1})\gamma^{\psi}$ $- q^{-2}(Y+Y^{-1})\theta(X+X^{-1})t_0^{-1}$
$q^{-2}A^2$	$q^{-2}(Y+Y^{-1})^2$
$q^{-2}B^2$	$q^{-2}(X+X^{-1})^2$
$-q^{-1}A\alpha$	$-q^{-1}(Y+Y^{-1})\alpha^{\psi}$
$-q^{-1}B\beta$	$-q^{-1}(X+X^{-1})\beta^{\psi}$
$q^2C^2-qC\gamma$	$\theta^2 t_0^{-2} - \theta t_0^{-1}\gamma^{\psi}$

$$\theta^2 = \left(YX^{-1}t_0 - Y^{-1}Xt_0^{-1} + Y^{-1}T_3 + XT_1 + q^{-1}t_0^2T_2 \right)\theta$$

$$= YX^{-1}\theta t_0 - Y^{-1}X\theta t_0^{-1} + Y^{-1}\theta T_3$$

$$+ X\theta T_1 + q^{-1}\theta t_0^2 T_2$$

Find $X^{-1}\theta$ and $X\theta$

First find $X^{-1}\theta$ in terms of $X\theta$

LEM 100

$$X^{-1}\theta = q^{-2}\theta(X+X^{-1}) - X\theta + (q^2-q^{-2})(Y+Y^{-1})t_0$$

$$+ q^{-1}(q-q^{-1})(X+X^{-1})\gamma^{\psi}t_0 - (q-q^{-1})\alpha^{\psi}t_0$$

pf Recall

$$A + \frac{qBC - q^{-1}CB}{q^2-q^{-2}} = \frac{\alpha}{q+q^{-1}}$$

Apply ψ

$$Y+Y^{-1} + \frac{q(X+X^{-1})C^{\psi} - q^{-1}C^{\psi}(X+X^{-1})}{q^2-q^{-2}} = \frac{\alpha^{\psi}}{q+q^{-1}}$$

Now elim C^{ψ} using $qC^{\psi} = \gamma^{\psi} - \theta t_0^{-1}$

and solve for $X^{-1}\theta$

 $\square$

LEM 101 We have

$$\Omega^{\Psi} =$$

term	coef
θ	$q^{-1}T_2 - t_0^{-1}\gamma^{\Psi}$
$Y^{-1}\theta$	$t_0^{-2}T_3$
$Y^{-1}\theta(X+X^{-1})$	$-q^{-2}t_0^{-1}$
$X\theta$	$t_0^{-2}T_1$
$YX\theta$	$-t_0^{-1}$
$Y^{-1}X\theta$	$-t_0^{-3}$
1	R

$$\underbrace{\begin{aligned} &q^{-2}(Y+Y^{-1})(X+X^{-1})\gamma^{\Psi} + q^{-2}(Y+Y^{-1})^2 + q^{-2}(X+X^{-1})^2 \\ &-q^{-1}(Y+Y^{-1})\alpha^{\Psi} - q^{-1}(X+X^{-1})\beta^{\Psi} + (q^2-q^{-2})Y(Y+Y^{-1}) \\ &+q^{-1}(q-q^{-1})Y(X+X^{-1})\gamma^{\Psi} - (q-q^{-1})Y\alpha^{\Psi} \end{aligned}}_{R}$$

"remainder"

pf collect terms so far. $\square$

Matrix rep Θ is

	x^{-2}	x^{-1}	1	x	x^{2}
y^{-2}	0	0	0	0	0
y^{-1}	0	0	T_3	$-t_0^{-1}$	0
1	0	0	$q^{-1}t_0^{2}T_2$	T_1	0
y	0	t_0	0	0	0
y^{2}	0	0	0	0	0

" " $-$ "

Notation Given a matrix M

with rows indexed by $\{y^i\}_{i \in \mathbb{Z}}$

and cols .. $\{x^i\}_{i \in \mathbb{Z}}$

Matrix $M^{\uparrow}$ is obtained from M by moving each entry 1 step North

$\overrightarrow{M}$ .. East

$M^{\downarrow}$ .. South

$\overleftarrow{M}$ .. West

For instance

$$M = \begin{array}{c|ccc} & x^{-1} & 1 & x \\ \hline y^{-1} & 0 & T_1 & 0 \\ 1 & t_0 & 0 & 0 \\ y & 0 & 0 & 0 \end{array}$$

$$\overrightarrow{M} = \begin{array}{c|ccc} & x^{-1} & 1 & x \\ \hline y^{-1} & 0 & 0 & T_1 \\ 1 & 0 & t_0 & 0 \\ y & 0 & 0 & 0 \end{array}$$

$$M^{\downarrow} = \begin{array}{c|ccc} & x^{-1} & 1 & x \\ \hline y^{-1} & 0 & 0 & 0 \\ 1 & 0 & T_1 & 0 \\ y & t_0 & 0 & 0 \end{array}$$

LEM 102

(i) Matrix rep $Y^{-1}\theta$ is $\ominus^{\uparrow}$

(ii) " " $Y^{-1}\theta(X+X^{-1})$ is $\ominus^{\rightarrow} + \ominus^{\leftarrow}$

pf (i) clear

(ii) Recall $X+X^{-1}$ commutes with t_0 $\square$

LEM 10.3 We have

$X\Theta =$

	X^{-2}	X^{-1}	1	X	X^{2}
Y^{-2}	0	0	0	0	0
Y^{-1}	0	$-q^{-2}t_0^2T_3$	$t_0(q^{-2}+q^{-2}t_0^2+T_3^2)$	$-q^{-1}(q^{-1}t_0+qt_0^{-1})T_3t_0$	$q^{-2}t_0$
1	0	$q^{-1}t_0^2T_2$	$-t_0(t_0T_1+qT_2T_3)$	$\gamma^{4}t_0$	0
Y	0	0	q^2t_0	0	0
Y^{2}	0	0	0	0	0

"Θ_2"

pf $\quad X\Theta = XYX^{-1}t_0 - XY^{-1}Xt_0^{-1} + XY^{-1}T_3 + X^2T_1$

$\qquad + q^{-1}Xt_0^2T_2$

Use the reduction rules to express the Right-hand side in the basis from Th. 5.2

Routine $\square$

LEM 104

(i) Matrix rep $\Psi \times \Theta$ is $\Theta_z^{\downarrow}$

(ii) - - $\Psi^{\to} \times \Theta$ is $\Theta_z^{\uparrow}$

pf clear. □

The matrix rep K is

$$
\begin{array}{c|ccccc}
 & x^{-2} & x^{-1} & 1 & x & x^{2} \\
\hline
y^{-2} & 0 & 0 & q^{-2} & 0 & 0 \\
y^{-1} & 0 & q^{-2}\gamma^{4} & -q^{-1}\alpha^{4} & q^{-2}\gamma^{4} & 0 \\
1 & q^{-2} & -q^{-1}\beta^{4} & q^{2}+3q^{-2} & -q^{-1}\beta^{4} & q^{-2} \\
y & 0 & \gamma^{4} & -q\alpha^{4} & \gamma^{4} & 0 \\
y^{2} & 0 & 0 & q^{2} & 0 & 0
\end{array}
$$

"$\mathcal{R}$"

th 106 For the hom $\psi : \Delta_q \to H_q$

the image of the Casimir element Ω is

$$(q+q^{-1})^2 - (q^{-1}t_0 + q t_0^{-1})^2 - T_1^2 - T_2^2 - T_3^2 \qquad \ast$$

$$- (q^{-1}t_0 + q t_0^{-1}) T_1 T_2 T_3$$

pf Find the coef matrix for Ω^ψ

Ω^ψ given in L101

For each term in L101 we found the corresp coef matrix

Coef matrix for Ω^ψ is

$$\ominus \, (q^{-1} T_2 - t_0^{-1} \gamma^\psi) + \ominus^{\uparrow} (t_0^{-2} T_3)$$

$$- (\ominus^{\ulcorner\rightarrow} + \ominus^{\leftarrow}) (q^{-2} t_0^{-1})$$

$$+ \ominus_2 (t_0^{-2} T_1) - \ominus_2^{\downarrow} (t_0^{-1}) - \ominus_2^{\uparrow} (t_0^{-3})$$

$$+ \mathcal{R}$$

By elem lin algebra the above matrix has

(1,1)-entry $\ast$ and all other entries 0

	x^{-2}	x^{-1}	1	x	x^2
y^{-2}	0	0	0	0	0
y^{-1}	0	0	0	0	0
1	0	0	$\ast$	0	0
y	0	0	0	0	0
y^2	0	0	0	0	0

Result follows

10/3/11
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$\theta =$

	x^{-2}	x^{-1}	1	x	x^{2}
y^{-2}	0	0	0	0	0
y^{-1}	0	0	T_3	$-t_0^{-1}$	0
1	0	0	$\boxed{q^{-1}t_0^{2}T_2}$	T_1	0
y	0	t_0	0	0	0
y^{2}	0	0	0	0	0

" " $-$

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ONE-INCH GRAPH PAPER

$y^{-1}\theta =$

	x^{-2}	x^{-1}	1	x	x^2
y^{-2}	0	0	T_3	$-t_0^{-1}$	0
y^{-1}	0	0	$q^{-1}t_0^{2}T_2$	T_1	0
1	0	t_0	$\boxed{0}$	0	0
y	0	0	0	0	0
y^2	0	0	0	0	0

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$y^{-1}\theta(x+x^{-1}) =$

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	x^{-2}	x^{-1}	1	x	x^{2}
y^{-2}	0	T_3	$-t_0^{-1}$	T_3	$-t_0^{-1}$
y^{-1}	0	$q^{-1}t_0^{2}T_2$	T_1	$q^{-1}t_0^{2}T_2$	T_1
1	t_0	0	$\boxed{t_0}$	0	0
y	0	0	0	0	0
y^{2}	0	0	0	0	0

$X\theta =$

ONE-INCH GRAPH PAPER

	x^{-2}	x^{-1}	1	x	x^2
y^{-2}	0	0	0	0	0
y^{-1}	0	$-q^{-2}t_0^{2}T_3$	$t_0(q^{-2}+q^{-2}t_0^{2}+T_3^{2})$	$-q^{-1}t_0T_3(q^{-1}t_0+qt_0^{-1})$	$q^{-2}t_0$
1	0	$q^{-1}t_0^{2}T_2$	$\boxed{-t_0(t_0T_1+qT_2T_3)}$	$\gamma^{4}t_0$	0
y	0	0	$q^{2}t_0$	0	0
y^2	0	0	0	0	0

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$YX\theta =$

	x^{-2}	x^{-1}	1	x	x^{2}
y^{-2}	0	0	0	0	0
y^{-1}	0	0	0	0	0
1	0	$-q^{-2}t_0^{2}T_3$	$\boxed{t_0(q^{-2}+q^{-2}t_0^{-2}+T_3^{-2})}$	$-q^{-1}t_0T_3(q^{-1}t_0+q\,t_0^{-1})$	$q^{-2}t_0$
y	0	$q^{-1}t_0^{2}T_2$	$-t_0(t_0T_1+qT_2T_3)$	$q^{4}t_0$	0
y^{2}	0	0	$q^{2}t_0$	0	0

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$y^{-1} X \theta =$

ONE-INCH GRAPH PAPER

	x^{-2}	x^{-1}	1	x	x^2
y^{-2}	0	$-q^{-2}t_0^2T_3$	$t_0(q^{-2}+q^{-2}t_0^{-2}+T_3^2)$	$-q^{-1}t_0T_3(q^{-1}t_0+qt_0^{-1})$	$q^{-2}t_0$
y^{-1}	0	$q^{-1}t_0^2T_2$	$-t_0(t_0T_1+qT_2T_3)$	q^4t_0	0
1	0	0	$\boxed{q^2t_0}$	0	0
y	0	0	0	0	0
y^2	0	0	0	0	0

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ONE-INCH GRAPH PAPER

$R =$	x^{-2}	x^{-1}	1	x	x^{2}
y^{-2}			q^{-2}		
y^{-1}		$q^{-2}\gamma^{4}$	$-q^{-1}\alpha^{4}$	$q^{-2}\gamma^{4}$	
1	q^{-2}	$-q^{-1}\beta^{4}$	$\boxed{q^{2}+3q^{-2}}$	$-q^{-1}\beta^{4}$	q^{-2}
y		γ^{4}	$-q\alpha^{4}$	γ^{4}	
y^{2}			q^{2}		

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$\mathbb{F}$ arb

$0 \neq q \in \mathbb{F}$ $\quad q^4 \neq 1$

Recall algebra hom $\psi : \Delta_q \to \hat{H}_q$

General goal: show ψ is inj.

Last time we found image under ψ of Casimir element Ω of Δ_q.

$$\Omega^\psi = (q+q^{-1})^2 - (q^{-1}t_0 + q t_0^{-1})^2 - T_1^2 - T_2^2 - T_3^2$$
$$- (q^{-1}t_0 + q t_0^{-1}) T_1 T_2 T_3$$

$T_i = t_i + t_i^{-1}$

Recall also

$$\alpha^\psi = (q^{-1}t_0 + q t_0^{-1}) T_1 + T_2 T_3$$
$$\beta^\psi = (q^{-1}t_0 + q t_0^{-1}) T_3 + T_1 T_2$$
$$\gamma^\psi = (q^{-1}t_0 + q t_0^{-1}) T_2 + T_1 T_3$$

$\Omega^\psi, \alpha^\psi, \beta^\psi, \gamma^\psi$ contained in

$$\Pi = \langle t_0^{\pm 1}, T_1, T_2, T_3 \rangle$$
$$\simeq \mathbb{F}[\lambda_0^{\pm 1}, \lambda_1, \lambda_2, \lambda_3]$$

We will need the fact that $\Omega^\psi, \alpha^\psi, \beta^\psi, \gamma^\psi$ are algebraically indep over $\mathbb{F}$.

Consider mut com indets

$$x_1, x_2, x_3, x_4$$

Define

$$y_1 = x_1 x_2 x_3 x_4 + x_1^2 + x_2^2 + x_3^2 + x_4^2$$

$$y_2 = x_1 x_2 + x_3 x_4$$

$$y_3 = x_1 x_3 + x_2 x_4$$

$$y_4 = x_1 x_4 + x_2 x_3$$

LEM 107 Above elements y_1, y_2, y_3, y_4 are algebraically indep over $\mathbb{F}$.

pf The following is a basis for the $\mathbb{F}$-vector space $\mathbb{F}[x_1, x_2, x_3, x_4]$:

$$x_1^h \, x_2^i \, x_3^j \, x_4^k \qquad h, i, j, k \in \mathbb{N} \qquad *$$

An element $x_1^h x_2^i x_3^j x_4^k$ from $*$ is called a

<u>monomial</u>

the <u>rank</u> of this monomial is

$$2h + i + j + k$$

For example, consider the monomials that make up y_1.
The monomial $x_1 x_2 x_3 x_4$ has rank 5,

x_1^2 " 4

x_2^2, x_3^2, x_4^2 " 2

Mon	Rank
$x_1x_2,\ x_1x_3,\ x_1x_4$	3
$x_3x_4,\ x_2x_4,\ x_2x_3$	2

Suf to show the following are lin indep over $\mathbb{F}$:

$$y_1^r y_2^s y_3^t y_4^u \qquad r, s, t, u \in \mathbb{N} \qquad ☆$$

Given $r, s, t, u \in \mathbb{N}$ write $y_1^r y_2^s y_3^t y_4^u$ as a linear comb of monomial:

$$y_1^r y_2^s y_3^t y_4^u =$$

$$(x_1x_2x_3x_4 + x_1^2 + x_2^2 + x_3^2 + x_4^2)^r (x_1x_2 + x_3x_4)^s (x_1x_3 + x_2x_4)^t (x_1x_4 + x_2x_3)^u$$

$$= \overbrace{(x_1x_2x_3x_4)^r (x_1x_2)^s (x_1x_3)^t (x_1x_4)^u}^{\text{"Leading monomial"}} + \text{sum of monomials with lower rank}$$

Show each monomial is the leading monom for at most one element of ☆:

Given monomal $x_1^h x_2^i x_3^j x_4^k$ from ☆

consider system of lin equations in unknowns r, s, t, u:

$$\begin{aligned} r+s+t+u &= h \\ r+s \quad\quad &= i \\ r+\quad t \quad &= j \\ r+\quad\quad u &= k \end{aligned}$$

Over $\mathbb{Q}$ (rationals) the coef matrix is nonsing

so $\exists$ unique sol for r.a.t.i.u.

Therefore $x_1^h x_2^i x_3^j x_4^k$ is the leading monom for at most one term in ✩

List elements $*$ weakly ordered by rank

List -- ✩ according to order of leading mon.

For each term in ✩ write in basis $*$.

Coef matrix is upper triangular. diagonal entries 1.

Therefore ✩ are lin indep. Result follows. $\square$

COR 108 The following are alg indep over $\mathbb{F}$:

$$\Omega^4, \quad \alpha^4, \quad \beta^4, \quad \gamma^4$$

pf. The following are alg indep over $\mathbb{F}$:

$$t_0, \quad T_1, \quad T_2, \quad T_3$$

Therefore the following are alg indep over $\mathbb{F}$:

$$\underbrace{q^{-1}t_0 + q\, t_0^{-1}}_{X_1}, \quad \underset{X_2}{\overset{\shortparallel}{T_1}}, \quad \underset{X_3}{\overset{\shortparallel}{T_2}}, \quad \underset{X_4}{\overset{\shortparallel}{T_3}}$$

*

By L107 the following are alg indep over $\mathbb{F}$:

$$X_1 X_2 X_3 X_4 + X_1^2 + X_2^2 + X_3^2 + X_4^2$$

$$X_1 X_2 + X_3 X_4$$

$$X_1 X_3 + X_2 X_4$$

$$X_1 X_4 + X_2 X_3$$

Above four elements are

$$(q+q^{-1})^2 - \Omega^4$$

$$\alpha^4$$

$$\beta^4$$

$$\gamma^4$$

Result follows. $\square$

Ready to show ψ is inj.

Recall our basis for Δ_q from Th 93

$$A^i C^j B^k \Omega^l \alpha^r \beta^s \gamma^t \qquad j \in \{0,1\} \qquad \text{✱}$$

$$i,k,l,r,s,t \in \mathbb{N}$$

For each term in ✱ image under ψ is

$$(y+y^{-1})^i (C^\psi)^j (x+x^{-1})^k (\Omega^\psi)^l (\alpha^\psi)^r (\beta^\psi)^s (\gamma^\psi)^t$$

Suf to show these are lin indep.

We will do even better, and show

Prop 109 The following are lin indep

$$(y+y^{-1})^i (C^\psi)^j (x+x^{-1})^k \, t_0^{\overset{\text{type}}{l}} \, T_1^r \, T_2^s \, T_3^t \qquad \text{✩}$$

$$j \in \{0,1\}, \quad l \in \mathbb{Z}, \quad i,k,r,s,t \in \mathbb{N}$$

pf Suppose $\exists$ lin dep subset of ☆

Let S denote a lin dep subset of ☆ that has min cardinality. So $S \neq \phi$.

For $j \in \{0,1\}$ define

S_j = set of elements in S that have type j

[type is exponent of C^4]

So $S = S_0 \cup S_1$ (disj union)

claim $S_1 \neq \phi$

pf cl. Suppose $S_1 = \phi$ then S_0 is lin dep subset of

$$(y+y^{-1})^i \, (x+x^{-1})^k \, t_0^{\ell} \, T_1^{r} \, T_2^{s} \, T_3^{t} \qquad \ell \in \mathbb{Z}, \quad i, k, r, s, t \in \mathbb{N} \qquad ※※$$

But ※※ are lin indep by Th 5.2

claim proved

Lets partition S_1 according to the value of i, k

For $i, k \in \mathbb{N}$ define

$$u(i,k) = \left\{ (\ell, r, s, t) \in \mathbb{Z} \times \mathbb{N}^3 \;\middle|\; (y+y^{-1})^i C^4 (x+x^{-1})^k t_0^{\ell} T_1^{r} T_2^{s} T_3^{t} \in S_1 \right\}$$

Since $S_1 \neq \phi$ $\exists\, i, k \in \mathbb{N}$ s.t.

$$U(i,k) \neq \phi$$

Define

$$M = \max \{ i+k \ / \ U(i,k) \neq \phi \}$$

By constr $\exists\, \mu, \nu \in \mathbb{N}$ s.t.

$$U(\mu,\nu) \neq \phi \qquad \mu+\nu = M$$

For each term

$$(y+y^{-1})^i (C^4)^j (x+x^{-1})^k t_0^{\,l}\, T_1^{\,r}\, T_2^{\,s}\, T_3^{\,t}$$

in ☆ consider coeff in each of

$$y^{\mu+1} X^{\nu+1}, \qquad y^{\mu+1} X^{-\nu-1}$$

Case	$j = 0$	$j = 1$ $i = \mu,\ j = \nu$	$j = 1$ $(i,j) \neq (\mu,\nu)$
$Y^{\mu+1} X^{\nu+1}$	— ↑	0	0
	same		
$Y^{\mu+1} X^{-\nu-1}$	↓ —	$-q^{-1} t_0^{\,l}\, T_1^{\,r}\, T_2^{\,s}\, T_3^{\,t}$	0

***

Above we used the fact that

the coef matrix for C^4 is

$$C^4: \quad \begin{array}{c|ccc} & x^{-1} & 1 & x \\ \hline y^{-1} & 0 & * & * \\ 1 & 0 & * & * \\ y & -q^{-1} & 0 & 0 \end{array}$$

Comparing the rows of $***$ the orig dependency S

yields a linear dependency among

$$t_0^l T_1^r T_2^s T_3^t \qquad (l,r,s,t) \in \mathcal{U}(\mu,\nu)$$

But these are lin indep since they are part of the

basis Th 52. Cont. $\square$

th. 110 The map $\psi : \Delta_q \to \hat{H}_q$

is an injection.

pf. By Prop 109 and Cor 108. $\square$